Erratum and Addendum: "Abstract polymer models with general pair interactions"

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Abstract

The purpose of this note is to correct an error present in ref. [5] and to give a generalization of the new criterion given in [2] for the convergence of cluster expansion.

§1. *Correction of the error in ref. [5].*

The mistake in [5] originates from the splitting in l.h.s of (2.3) of ref. [5] of the pair potential $V(\gamma_i, \gamma_j)$ in a purely hard core part $U(\gamma_i, \gamma_j)$ plus a non hard core part $W(\gamma_i, \gamma_j)$. This last potential $W(\gamma_i, \gamma_j)$ is zero for incompatible pairs and must satisfy the stability condition (2.6) of ref. [5]. The problem is that the potential $W(\gamma_i, \gamma_j)$ cannot be negative (i.e. attractive) in order to satisfy (2.6). Hence conditions (2.3)-(2.6) only allow potentials $V(\gamma_i, \gamma_j)$ purely repulsive, which were already treated in [7] and [8]. Moreover, the BEG model example in sec. 4 of ref [5] involves a long range attractive pair potential not satisfying conditions (2.3)-(2.6).

This error can be corrected by simply replacing conditions (2.3)-(2.6) of ref. [5] by the single assumption that $V(\gamma_i, \gamma_j)$ belongs to the class of stable pair potentials (which is "the best possible"' class) according to the following definition.

Definition 1. *A pair interaction $V(\gamma_i, \gamma_j)$ among polymers is said to be stable if there exists a function $B(\gamma) \geq 0$ such that (hereafter $\mathcal{P}$ is the set of abstract polymers)*

$$\sum_{1 \leq i < j \leq n} V(\gamma_i, \gamma_j) \geq -\sum_{i=1}^{n} B(\gamma_i) \qquad \textit{for all } n \in \mathbb{N} \textit{ and all } (\gamma_1, \ldots, \gamma_n) \in \mathcal{P}^n \tag{1}$$

Then theorem 1 in ref. [5] continues to hold with the definition of $F(\gamma_i, \gamma_j)$ replaced by

$$F(\gamma_i, \gamma_j) = \begin{cases} \left|e^{-V(\gamma_i, \gamma_j)} - 1\right| = 1 & \text{if } V(\gamma_i, \gamma_j) = +\infty \\ |V(\gamma_i, \gamma_j)| & \text{otherwise} \end{cases} \tag{2}$$

The proof of theorem 1 has to be modified only in section 3.1 of [5] in order to prove there the inequality (3.5) when $V(\gamma_i, \gamma_j)$ satisfies (1) and $F(\gamma_i, \gamma_j)$ is defined as in (2). This amounts to prove the following proposition.

Proposition 1. *Let $V(\gamma_i, \gamma_j)$ be a stable pair potential (in the sense of (1)), then, for any fixed $(\gamma_1, \ldots, \gamma_n) \in \mathcal{P}^n$, the following inequality holds:*

$$\left| \sum_{g \in G_n} \prod_{\{i,j\} \in E_g} \left(e^{-V(\gamma_i, \gamma_j)} - 1\right) \right| \leq e^{\sum_{i=1}^{n} B(\gamma_i)} \sum_{\tau \in T_n} \prod_{\{i,j\} \in E_\tau} F(\gamma_i, \gamma_j) \tag{3}$$

where $B(\gamma)$ is the function defined in (1), $F(\gamma_i, \gamma_j)$ is the function defined in (2), G_n denotes the set of all connected graphs with vertex set $\{1, 2, \dots, n\}$, T_n denotes the set of all trees with vertex set $\{1, 2, \dots, n\}$ and E_g and E_τ denote the edge sets of $g \in G_n$ and $\tau \in T_n$ respectively.

We will give the proof of proposition 1 in the next section. The rest of the paper [5] is unchanged apart from the following modification in section 4 (BEG model example).
Replace the portion of the text in sec. 4 of ref. [5] from the sentence "*We now extend the definition of $W(\boldsymbol{p}_i, \boldsymbol{p}_j)$ ...*" to the sentence "*This long range pair interaction $W(\boldsymbol{p}_i, \boldsymbol{p}_j)$ is stable in the sense of (2.6)*" with the following text:
"*We now extend the definition of $W(\boldsymbol{p}_i, \boldsymbol{p}_j)$ to all pairs in $\mathcal{P}$ as*

$$W(\boldsymbol{p}_i, \boldsymbol{p}_j) = \begin{cases} -\beta \sum\limits_{\substack{x \in p_i \\ y \in p_j}} [J_{xy} s_x s_y + K_{xy}] & \textit{if } d(p_i, p_j) \geq 2 \\ +\infty & \textit{otherwise} \end{cases}$$

With these definitions it is immediate to see that r.h.s. of (4.10) can be written as

$$Z_\Lambda(\beta) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{(\boldsymbol{p}_1, \dots, \boldsymbol{p}_n) \in \mathcal{P}_\Lambda^n} \rho_{\boldsymbol{p}_1} \dots \rho_{\boldsymbol{p}_n} e^{-\sum_{1 \leq i < j \leq n} W(\boldsymbol{p}_i, \boldsymbol{p}_j)}$$

which is the partition function of a polymer gas of the type (2.2) in which the polymers are elements of the set $\mathcal{P}$ defined by

$$\mathcal{P} = \Big\{ \boldsymbol{p} = (p, s_p) : p \subset \mathbb{Z}^d \textit{ connected and finite, } s_p \textit{ function from } p \textit{ to } \{-1, +1\} \Big\}$$

with activity given in (4.11) and with incompatibility relation $\boldsymbol{p} \not\sim \tilde{\boldsymbol{p}} \Leftrightarrow d(p, \tilde{p}) < 2$. The long range pair interaction $W(\boldsymbol{p}_i, \boldsymbol{p}_j)$ is stable in the sense of (1)".

§2. *Proof of proposition 1*

In this section we are using the same notations of sec. 3 in ref. [5]. In particular, when $V(\gamma_i, \gamma_j) = +\infty$ we write $\gamma_i \not\sim \gamma_j$ and say that γ_i and γ_j are *incompatible*; otherwise, when $V(\gamma_i, \gamma_j) \in \mathbb{R}$, we say that γ_i and γ_j are *compatible* and write $\gamma_i \sim \gamma_j$.
Let us define, for $H > 0$

$$V_H(\gamma_i, \gamma_j) = \begin{cases} H & \text{if } \gamma_i \nsim \gamma_j \\ V(\gamma_i, \gamma_j) & \text{otherwise} \end{cases} \tag{4}$$

By the stability condition (1), for any fixed $n \in \mathbb{N}$ and $(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n$, there is H_0 (which depends on n and $(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n$) such that, for all $H \geq H_0$ and for all $X \subset \{1, 2, \dots, n\}$,

$$\sum_{\{i,j\} \subset X} V_H(\gamma_i, \gamma_j) \geq -\sum_{i \in X} B(\gamma_i)$$

Now, for any fixed $(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n$

$$\Bigg| \sum_{g \in G_n} \prod_{\{i,j\} \in E_g} \Big(e^{-V(\gamma_i, \gamma_j)} - 1 \Big) \Bigg| = \lim_{H \to \infty} \Bigg| \sum_{g \in G_n} \prod_{\{i,j\} \in E_g} (e^{-V_H(\gamma_i, \gamma_j)} - 1) \Bigg|$$

We now use (3.1) of ref. [5] for the *finite* potential V_H in the r.h.s. of the equation above and get

$$\Big|\sum_{g\in G_n}\prod_{\{i,j\}\in E_g}(e^{-V(\gamma_i,\gamma_j)}-1)\Big| \le \lim_{H\to\infty}\sum_{\tau\in T_n} w_H^\tau(\gamma_1,\dots,\gamma_n) \tag{5}$$

where

$$w_H^\tau(\gamma_1,\dots,\gamma_n)=\prod_{\{i,j\}\in E_\tau}|V_{H,J}(\gamma_i,\gamma_j)|\int d\mu_\tau(\mathbf{t}_n)e^{-\sum_{1\le i<j\le n}\mathbf{t}_n(\{i,j\})V_H(\gamma_i,\gamma_j)}$$

Here we wrote shortly $\mathbf{t}_n(\{i,j\})=t_1(\{i,j\})\dots t_{n-1}(\{i,j\})$ and $d\mu_\tau(\mathbf{t}_n)=d\mu_\tau(\mathbf{t}_{n-1},\mathbf{X}_{n-1})$.
Let now $E_\tau^H=\{\{i,j\}\subset E_\tau:\gamma_i\not\sim\gamma_j)\}$ and $E_\tau\backslash E_\tau^H$. Then

$$w_H^\tau(\gamma_1,\dots,\gamma_n)=\prod_{\{i,j\}\in E_\tau\backslash E_\tau^H}|V_H(\gamma_i,\gamma_j)|\prod_{\{i,j\}\in E_\tau^H}|V_H(\gamma_i,\gamma_j)|\int d\mu_\tau(\mathbf{t}_n)e^{-\sum_{1\le i<j\le n}\mathbf{t}_n(\{i,j\})V_H(\gamma_i,\gamma_j)}$$

Now, for $\varepsilon>0$, we can write

$$V_H(\gamma_i,\gamma_j) \;=\; U_{(1-\varepsilon)H}(\gamma_i,\gamma_j)+V_{\varepsilon H}(\gamma_i,\gamma_j)$$

where

$$U_{(1-\varepsilon)H}(\gamma_i,\gamma_j)=\begin{cases}(1-\varepsilon)H & \text{if } \gamma_i\not\sim\gamma_j\\ 0 & \text{otherwise}\end{cases}\quad,\quad V_{\varepsilon H}(\gamma_i,\gamma_j)=\begin{cases}\varepsilon H & \text{if } \gamma_i\not\sim\gamma_j\\ V(\gamma_i,\gamma_j) & \text{otherwise}\end{cases}$$

The potential $V_{\varepsilon H}(\gamma_i,\gamma_j)$, for H larger than $\varepsilon^{-1}H_0$, is stable, i.e. it satisfies for all $X\subset\{1,\dots,n\}$,

$$\sum_{\{i,j\}\subset X}V_{\varepsilon H}(\gamma_i,\gamma_j)\ge-\sum_{i\in X}B(\gamma_i)$$

and hence (interpolating parameters preserve stability, see e.g. [1], [6]) we also have that

$$\sum_{1\le i<j\le n}\mathbf{t}_n(\{i,j\})V_{\varepsilon H}(\gamma_i,\gamma_j)\ge-\sum_{i=1}^n B(\gamma_i)$$

The potential $U_{(1-\varepsilon)H}(\gamma_i,\gamma_j)$ is non negative, so we have, for $\eta>0$,

$$\sum_{1\le i<j\le n}\mathbf{t}_n(\{i,j\})U_{(1-\varepsilon)H}(\gamma_i,\gamma_j)\ge\sum_{\{i,j\}\subset E_\tau^H}\mathbf{t}_n(\{i,j\})U_{(1-\varepsilon)H}(\gamma_i,\gamma_j)=$$

$$=\sum_{\{i,j\}\subset E_\tau^H}\mathbf{t}_n(\{i,j\})(1-\varepsilon)H+\eta\sum_{\{i,j\}\subset E_\tau\backslash E_\tau^H}\mathbf{t}_n(\{i,j\})-\eta\sum_{\{i,j\}\subset E_\tau\backslash E_\tau^H}\mathbf{t}_n(\{i,j\})\quad\ge$$

$$\ge\sum_{\{i,j\}\subset E_\tau^H}\mathbf{t}_n(\{i,j\})(1-\varepsilon)H+\sum_{\{i,j\}\subset E_\tau\backslash E_\tau^H}\mathbf{t}_n(\{i,j\})\eta-|E_\tau\backslash E_\tau^H|\eta$$

Therefore we get

$$\sum_{1\le i<j\le n}\mathbf{t}_n(\{i,j\})U_{(1-\varepsilon)H}(\gamma_i,\gamma_j)\ge\sum_{1\le i<j\le n}\mathbf{t}_n(\{i,j\})V_{ij}^\tau-|E_\tau\backslash E_\tau^H|\eta$$

where V^τ_{ij} is the positive pair potential (H, η, ε dependent) given by

$$V^\tau_{ij} = \begin{cases} (1-\varepsilon)H & \text{if } \{i,j\} \in E^H_\tau \\ \eta & \text{if } \{i,j\} \in E_\tau \backslash E^H_\tau \\ 0 & \text{otherwise} \end{cases}$$

Hence we obtain that $w^\tau_H(\gamma_1, \dots, \gamma_n)$ can be bounded by

$$w^\tau_H(\gamma_1, \dots, \gamma_n) \le e^{+\sum_{i=1}^n B(\gamma_i) + \eta |E_\tau \backslash E^H_\tau|} \left[\prod_{\{i,j\} \in E_\tau \backslash E^H_\tau} |V(\gamma_i, \gamma_j)| \right] \times \left[\frac{1}{\eta} \right]^{|E_\tau \backslash E^H_\tau|} \times$$

$$\times \left[\frac{1}{1-\varepsilon} \right]^{|E^H_\tau|} \prod_{\{i,j\} \in E_\tau} V^\tau_{ij} \int d\mu_\tau(\mathbf{t}_n) e^{-\sum_{1 \le i < j \le n} \mathbf{t}_n(\{i,j\}) V^\tau_{ij}}$$

Applying now the tree graph identity (3.1) of ref [5] to the pair potential V^τ_{ij} one concludes immediately (see e.g. [1]) that, for all $H \in [0, +\infty)$

$$\prod_{\{i,j\} \in E_\tau} V^\tau_{ij} \int d\mu_\tau(\mathbf{t}_n) e^{-\sum_{1 \le i < j \le n} \mathbf{t}_n(\{i,j\}) V^\tau_{ij}} = \prod_{\{i,j\} \in E_\tau} \left| e^{-V^\tau_{ij}} - 1 \right| =$$

$$= \left| e^{-\eta} - 1 \right|^{|E_\tau \backslash E^H_\tau|} \prod_{\{i,j\} \in E^H_\tau} \left| e^{-U_{(1-\varepsilon)H}(\gamma_i, \gamma_j)} - 1 \right|$$

Hence, using also that $U_{(1-\varepsilon)H}(\gamma_i, \gamma_j) < V(\gamma_i, \gamma_j)$ if $\{i,j\} \in E^H_\tau$, we get, for any $H \ge \varepsilon^{-1} H_0$,

$$w^\tau_H(\gamma_1, \dots, \gamma_n) \le e^{+\sum_{i=1}^n B(\gamma_i)} \prod_{\{i,j\} \in E^H_\tau} \left[\frac{1}{1-\varepsilon} \right] \left| e^{-V(\gamma_i, \gamma_j)} - 1 \right| \prod_{\{i,j\} \in E_\tau \backslash E^H_\tau} \left| \frac{(e^\eta - 1)}{\eta} V(\gamma_i, \gamma_j) \right|$$

So, due to the arbitrariness of η and ε which can be taken as small as we please, we obtain,

$$\lim_{H \to \infty} w^\tau_H(\gamma_1, \dots, \gamma_n) \le e^{+\sum_{i=1}^n B(\gamma_i)} \prod_{\{i,j\} \in E^H_\tau} \left| e^{-V(\gamma_i, \gamma_j)} - 1 \right| \prod_{\{i,j\} \in E_\tau \backslash E^H_\tau} |V(\gamma_i, \gamma_j)| \tag{6}$$

Finally, plugging (6) in (5) we get inequality (3). □

§3. *Addendum: a generalizations of theorem 1 in [2]*

In [2] Fernández and Procacci improved the Kotecký-Preiss criterion [3] for the convergence of cluster expansion of the abstract polymer gas by using the Penrose identity [4]. The new criterion (theorem 1 in [2]), as well as the old Kotecký-Preiss criterion, hold only for purely hard core self repulsive pair potentials $V(\gamma_i, \gamma_j)$, i.e. such that $V(\gamma_i, \gamma_j)$ takes values in $\{0, +\infty\}$ and $V(\gamma, \gamma) = +\infty$ for all $\gamma \in \mathcal{P}$. In [5], taking into account the corrections given in sec. 1 and 2 of this note, the Kotecký-Preiss criterion has been generalized for polymers interacting through a stable pair potential (in the sense of (1)). In this section we generalize the new criterion given in [2] for abstract polymers interacting through an "ultra-stable" pair potential according to the following definition.

Definition 2. *A pair potential $V(\gamma_i, \gamma_j)$ among polymers is said to be ultra-stable if there exists $B(\gamma) \geq 0$ such that, for all $\gamma \in \mathcal{P}$, for all $n \in \mathbb{N}$ and for all compatible $(\gamma_1, \ldots, \gamma_n) \in \mathcal{P}^n$ (i.e. $\gamma_i \sim \gamma_j$ for all $\{i, j\} \subset \{1, \ldots, n\}$), it holds*

$$\sum_{j=1}^{n} V(\gamma, \gamma_j) \geq -B(\gamma) \tag{7}$$

Ultra-stable potentials are not new in the literature. In particular, they were considered in the original Penrose's paper [4]. We remark that an ultra-stable potential is stable and it can be attractive. In particular, the potential $W(\boldsymbol{p}_i, \boldsymbol{p}_j)$ given in sec. 4 of [5] is ultra-stable.

The generalization of the Fernadez-Procacci criterion for abstract polymer models interacting via an ultra-stable pair potential is possible because the bound (3) can be improved when $V(\gamma_i, \gamma_j)$ is ultra-stable. The main tool to obtain this improvement is the Penrose identity [4], based on rooted trees. So, in the next few lines we recall some definitions and notations about rooted trees.

A rooted tree is a tree in which one vertex is distinguished from the other vertices and is called *the root.* Given any vertex v of a rooted tree, *the level of* v is the number of edges along the unique path connecting v to the root, the *children* of v are all those vertices that are adjacent to v and are one level farther away from the root than v. If w is a child of v, then v is called *the parent of* w. Two vertices which are children of the same parent are called *siblings.*

From now on, trees $\tau \in T_n$ (set of all trees with vertex set $\{1, 2, \ldots, n\}$) are thought of as rooted, the common root being the vertex 1. We also put shortly $\mathrm{I}_n = \{1, 2, \ldots, n\}$. For fixed $\tau \in T_n$ and $i \in \mathrm{I}_n$, s_i denotes the number of children of the vertex i and $i^1, \ldots, i^{s_i}$ are the vertices which are children of i in τ. We recall that E_τ denotes the edge set of τ. We also denote by E^*_τ the set of all pairs $\{i, j\} \subset \mathrm{I}_n$ which are siblings in τ.

Proposition 2. *Let $V(\gamma_i, \gamma_j)$ be an ultra-stable pair potential among polymers, then, for any fixed $(\gamma_1, \ldots, \gamma_n) \in \mathcal{P}^n$, the following inequality holds:*

$$\left| \sum_{g \in G_n} \prod_{\{i,j\} \in E_g} \left(e^{-V(\gamma_i, \gamma_j)} - 1 \right) \right| \leq e^{\sum_{i=1}^{n} B(\gamma_i)} \sum_{\tau \in T_n} \prod_{\{i,j\} \in E_\tau} |e^{-|V(\gamma_i, \gamma_j)|} - 1| \prod_{\{i,j\} \in E^*_\tau} \mathbb{1}_{\gamma_i \sim \gamma_j} \tag{8}$$

Remark. The improvement obtained in (8) with respect to the bound given by (3) is twofold. First, the factors $|e^{-|V(\gamma_i, \gamma_j)|} - 1|$ in the r.h.s. of (8) are always smaller than the factors $F(\gamma_i, \gamma_j)$ in the r.h.s. of (3). Second, the term $\prod_{\{i,j\} \in E^*_\tau} \mathbb{1}_{\gamma_i \sim \gamma_j}$ present in the r.h.s. of (8) is non-vanishing only for those trees τ such that any sibling pair $\{i, j\}$ in τ is *compatible* (i.e. $\gamma_i \sim \gamma_j$). Note also that, by the definitions and notations given above on rooted trees, we can rewrite

$$\prod_{\{i,j\} \in E^*_\tau} \mathbb{1}_{\gamma_i \sim \gamma_j} = \prod_{i=1}^{n} \Big[\prod_{1 \leq j < k \leq s_i} \mathbb{1}_{\gamma_{i^j} \sim \gamma_{i^k}} \Big] \tag{9}$$

I.e. the term $\prod_{\{i,j\} \in E^*_\tau} \mathbb{1}_{\gamma_i \sim \gamma_j}$ is non-vanishing only for those trees τ such that, for any vertex i of τ, the polymers $\gamma_{i^1}, \ldots, \gamma_{i^{s_i}}$ associated to the children of i are pairwise compatible.

Proposition 2, whose proof is postponed at the end of this section, immediately implies the following generalization of theorem 1 in [2] to abstract polymer systems interacting through an ultra-stable potential.

Theorem 2. *Let $\mathcal{P}$ a polymer space with polymers interacting via an ultra-stable pair potential $V(\gamma,\gamma')$ satisfying (7). Let $\mu : \mathcal{P} \to [0,\infty) : \gamma \mapsto \mu_\gamma$ be a non negative valued function and let, for each $\gamma \in \mathcal{P}$, $\rho_\gamma \in [0,\infty)$ such that*

$$\rho_\gamma e^{B(\gamma)} \leq \frac{\mu_\gamma}{\Xi_\gamma(\mu)} \qquad \forall \gamma \in \mathcal{P} \tag{10}$$

where $B(\gamma)$ is the function defined in (7) and

$$\Xi_\gamma(\mu) \; = \; 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} (|e^{-|V(\gamma,\gamma_1)|} - 1|\mu_{\gamma_1}) \ldots (|e^{-|V(\gamma,\gamma_n)|} - 1|\mu_{\gamma_n}) \prod_{1\leq i<j\leq n} \mathbb{1}_{\gamma_i\sim\gamma_j} \tag{11}$$

Then the series $\Pi_{\gamma_0}(\rho)$ [defined in (2.10) of ref. [5]] converges and satisfies $\rho_{\gamma_0}\,\Pi_{\gamma_0}(\rho) \leq \mu_{\gamma_0}$.

Indeed, if $V(\gamma_i,\gamma_j)$ is ultra stable, by (8) and (9) the series $\Pi_{\gamma_0}(\rho)$ is bounded above by

$$\Pi_{\gamma_0}(\rho) \leq e^{B(\gamma_0)} \bar{\Pi}_{\gamma_0}(\tilde{\rho})$$

where $\bar{\Pi}_{\gamma_0}(\tilde{\rho})$ is the series (notations here are the same as sec. 3.1 of ref. [5])

$$\bar{\Pi}_{\gamma_0}(\tilde{\rho}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\tau\in T_n^0} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \prod_{\{i,j\}\in E_\tau} |e^{-|V(\gamma_i,\gamma_j)|} - 1| \prod_{i=0}^{n} \Big[\prod_{1\leq j<k\leq s_i} \mathbb{1}_{\gamma_{ij}\sim\gamma_{ik}} \Big] \tilde{\rho}_{\gamma_1} \ldots \tilde{\rho}_{\gamma_n} \tag{12}$$

So, following the reasoning of section 3.2 in ref. [5] or sec. 4.2 in [2], condition (10) implies that the series (12) converges.

Proof of proposition 2. We follow closely [4]. For $i \in \mathrm{I}_n$ and $\tau \in T_n$, let $d_\tau(i)$ be the level of the vertex i in τ and let i'_τ is the parent of i in τ. Let p be the map that to each tree $\tau \in T_n$ associates the graph $p(\tau) \in G_n$ formed by adding to τ all edges $\{i,j\}$ such that either $d_\tau(i) = d_\tau(j)$, or $d_\tau(j) = d_\tau(i) - 1$ and $j > i'_\tau$. Then the Penrose identity (formula (6) in [4], see also formula (4.2) in [7] or formula (4.4) in [2]) states that

$$\sum_{g\in G_n} \prod_{\{i,j\}\in E_g} \left(e^{-V(\gamma_i,\gamma_j)} - 1\right) \; = \; \sum_{\tau\in T_n} e^{-\sum_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} V(\gamma_i,\gamma_j)} \prod_{\{i,j\}\in E_\tau} \left(e^{-V(\gamma_i,\gamma_j)} - 1\right)$$

which immediately yields the bound

$$\Big| \sum_{g\in G_n} \prod_{\{i,j\}\in E_g} \left(e^{-V(\gamma_i,\gamma_j)} - 1\right) \Big| \; \leq \; \sum_{\tau\in T_n} e^{-\sum_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} V(\gamma_i,\gamma_j)} \prod_{\{i,j\}\in E_\tau} \left| e^{-V(\gamma_i,\gamma_j)} - 1 \right| \tag{13}$$

Observe now that if, for $\tau \in T_n$, we let $E_\tau^- = \{\{i,j\} \in E_\tau : V(\gamma_i,\gamma_j) < 0\}$, then we can write

$$\prod_{\{i,j\}\in E_\tau} \left| e^{-V(\gamma_i,\gamma_j)} - 1 \right| = e^{-\sum_{\{i,j\}\in E_\tau^-} V(\gamma_i,\gamma_j)} \prod_{\{i,j\}\in E_\tau} \left| e^{-|V(\gamma_i,\gamma_j)|} - 1 \right| \tag{14}$$

Moreover, for any fixed $(\gamma_1,\ldots,\gamma_n) \in \mathcal{P}^n$, the factor $e^{-\sum_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} V(\gamma_i,\gamma_j)}$ is zero whenever $E_{p(\tau)}\setminus E_\tau$ contains a pair $\{i,j\}$ which is *incompatible*, i.e. such that $\gamma_i \not\sim \gamma_j$. So

$$e^{-\sum_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} V(\gamma_i,\gamma_j)} = e^{-\sum_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} V(\gamma_i,\gamma_j)} \prod_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} \mathbb{1}_{\gamma_i\sim\gamma_j} \tag{15}$$

Therefore, in view of (14) and (15), we can rewrite the r.h.s. of (13) as

$$\text{r.h.s of (13)} \;=\; \sum_{\tau\in T_n} e^{-\sum_{\{i,j\}\in[E_{p(\tau)}\setminus E_\tau]\cup E^-_\tau} V(\gamma_i,\gamma_j)} \prod_{\{i,j\}\in E_\tau} \left|e^{-|V(\gamma_i,\gamma_j)|}-1\right| \prod_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} \mathbb{1}_{\gamma_i\sim\gamma_j} \tag{16}$$

Now, following Penrose, for $\tau\in T_n$ fixed, we associate to each $i\in \mathrm{I}_n$ a set of vertices S_i^τ formed by all those vertices of τ connected to i by an edge in $[E_{p(\tau)}\setminus E_\tau]\cup E^-_\tau$ and satisfying:

1 either $d_\tau(j)=d_\tau(i)+1$

2 or $d_\tau(j)=d_\tau(i)$ *and* $j>i$.

Then we define

$$W(i;S_i^\tau)=\sum_{j\in S_i^\tau} V(\gamma_i,\gamma_j)$$

so that,

$$\sum_{\{i,j\}\in[E_{p(\tau)}\setminus E_\tau]\cup E^-_\tau} V(\gamma_i,\gamma_j)=\sum_{i=1}^n W(i;S_i^\tau) \tag{17}$$

Therefore, plugging (17) into (16) the r.h.s. of (13) is rewritten as

$$\text{r.h.s of (13)} \;=\; \sum_{\tau\in T_n} e^{-\sum_{i=1}^n W(i;S_i^\tau)} \prod_{\{i,j\}\in\tau}\prod_{\{i,j\}\in E_\tau} \left|e^{-|V(\gamma_i,\gamma_j)|}-1\right| \prod_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} \mathbb{1}_{\gamma_i\sim\gamma_j} \tag{18}$$

It is now crucial to observe that when the term $e^{-\sum_{i=1}^n W(i;S_i)}$ is different from zero then, for any $i\in \mathrm{I}_n$, *all elements in S_i^τ are pairwise compatible.* As a matter of fact, if a pair $\{j,k\}\subset S_i^\tau$ then, either $d_\tau(j)=d_\tau(k)$ which implies $\{j,k\}\in E_{p(\tau)}\setminus E_\tau$, or $d_\tau(j)=d_\tau(k)-1$ with $j\ge k'_\tau$ which implies $\{j,k\}\in E_{p(\tau)}\setminus E_\tau$ when $j>k'_\tau$ and $\{j,k\}\in E^-_\tau$ when $j=k'_\tau$.
So, since $V(\gamma_i,\gamma_j)$ is assumed to be ultra-stable, we can use (7) to obtain the bound

$$W(i;S_i^\tau)=\sum_{j\in S_i^\tau} V(\gamma_i,\gamma_j)\ge -B(\gamma_i) \qquad\qquad \text{for all } i\in \mathrm{I}_n$$

and hence in the r.h.s. of (18) we have the bound

$$e^{-\sum_{i=1}^n W(i;S_i^\tau)}\le e^{+\sum_{i=1}^n B(\gamma_i)} \tag{19}$$

Finally, by the definition of the map p, we have that all siblings pairs of τ are in $E_{p(\tau)}\setminus E_\tau$, i.e. $E^*_\tau\subset E_{p(\tau)}\setminus E_\tau$. So

$$\prod_{\{i,j\}\in E_{p(\tau)}\setminus E_\tau} \mathbb{1}_{\gamma_i\sim\gamma_j}\le \prod_{\{i,j\}\in E^*_\tau} \mathbb{1}_{\gamma_i\sim\gamma_j} \tag{20}$$

Plugging now (19) and (20) into (18) we get the inequality (8). $\square$

Acknowledgments. I am extremely grateful to Alan Sokal for discovering the error in [5]. I also thank David Brydges, Benedetto Scoppola and Simone Severini for useful discussions. I thank the Isaac Newton Institute for Mathematical Sciences, University of Cambridge, for generous support during the programme on Combinatorics and Statistical Mechanics (JanuaryJune 2008), where this work was begun. This research was supported in part by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and by the Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG).

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