Abstract polymer models with general pair interactions

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Abstract

A convergence criterion of cluster expansion is presented in the case of an abstract polymer system with general pair interactions (i.e. not necessarily hard core or repulsive). A concrete example is given in which polymers are r_0-connected sets of the cubic lattice in d dimensions interacting via a polynomially decaying potential (possibly attractive), of the type $1/r^{d+\lambda}$ with $\lambda > 0$.

1. Introduction

The abstract polymer gas is an important tool to study the high temperature/low density or low temperature phase of many statistical mechanics models. Generally speaking, the abstract polymer model consists of a collection objects (the polymers) which play the role of the particles of the gas. These polymers have a given activity and they interact via a hard core pair potential suitably defined. Typically, one wants to show that the pressure of this polymer gas can be written in terms of an absolutely convergent series if the activities are taken sufficiently small.

The first example of such a model appeared in [8]. where the polymers were finite non overlapping subsets of the cubic lattice $\mathbb{Z}^d$. The authors proved convergence of the pressure via the method of Kilkwood-Salsburg equations. Subsequently, the same system studied in [8] was treated in [17] and [4] via cluster expansion methods based on tree graphs inequalities.

In [9] the most general version of this system was given. There, polymers were simply a collection of objects with a given activity and interacting through an hard core pair potential introduced via a symmetric and reflexive relation in the polymer space. Polymers belonging to this relation were called incompatible, and compatible otherwise. The hard core condition was simply to forbid configurations of polymers containing pairs of incompatible polymers. Differently from the cases considered previously, in which polymers had a cardinality and a size, the Kotecký-Preiss polymers were characterized only by the activity.

In [5] the convergence condition for the Kotecký-Preiss polymer gas was slightly improved and the proof was greatly simplified being reduced to a simple inductive argument, as it was shown very clearly in [11] and [18]. In particular, in [18] it has been observed that the Dobrsushin proof works for even more general abstract polymer gases, in which polymer may interact through a repulsive soft-core pair interaction. However, the self interaction is still needed to be hard core to develop the induction argument.

Very recently [7] the Kotecký-Preiss and the Dobrushin conditions for convergence of the abstract polymer gas with purely hard core interactions were reobtained via the standard cluster expansion methods and a new improved condition was given by exploiting an old tree graph identity valid for hard core systems due to O. Penrose.

In all these works, the basically hard core character of the interaction seemed to be an essential ingredient to control the convergence. Exceptions can be found in [6], [10]. In [6] a contour model with interaction (exponentially decaying al large distances) is proposed. However the model is rewritten in term of the usual hard core polymer gas where polymers are objects more complicated than the original contours. This philosophy has also been pursued in [10] where a one-dimensional contour model with long range interaction is rewritten in term of new objects with hard core pair interactions.

It would be of interest to treat also cases in which polymers interact via more general pair interactions, e.g., not necessarily repulsive, not necessarily hard core, not necessarily finite range. Such abstract polymer model with interaction could be a useful tool in the study of spin systems at low temperature interacting via infinite range polynomially decaying potential, see e.g. [12].

In this paper we develop a model of abstract polymers (of the type of [9]) with interactions more general than the hard-core. Our polymers interact through a "short distance" repulsive, not necessarily hard core, interaction which is non zero only on pair of incompatible polymers, plus a interaction with no definite signal (hence it can be attractive) acting only on pairs of compatible polymers. We give a condition convergence for the pressure of this gas by using a cluster expansion method similar to the one developed in [7]. However, differently from [7], we could not use here the Penrose identity, since our interaction is not purely hard-core. We rather used another well known tree graph identity originally proposed in [2] and further developed in [1].

The rest of the paper is organized as follows. In section 2 we introduce the model, notations and the main result of the paper. In section 3 we give the proof of our result (theorem 1). Namely, in subsection 3.1. we present the tree graph identity and show how it can be used to bound the Ursell coefficients of the Mayer series of our polymer model. In subsection 3.2 we give the convergence argument based on map iterations developed in [7]. In subsection 3.3 we conclude the proof of our main theorem. Finally in section 4 we give an example in which polymers are r_0-connected sets of the cubic lattice in d dimensions $\mathbb{Z}^d$ interacting via a polynomially decaying potential, of the type $1/r^{d+\lambda}$ with $\lambda > 0$.

2. Polymer gas: notations and results

2.1. The model.

Let $\mathcal{P}$ denotes the set of polymers (i.e. $\mathcal{P}$ is the *single particle state space*). We will assume here that $\mathcal{P}$ is a countable set. We associate to each polymer $\gamma \in \mathcal{P}$ a complex number z_γ (a positive number in physical situations) which is interpreted as the *activity* of the polymer γ. We will denote $z = \{z_\gamma\}_{\gamma\in\mathcal{P}}$. We introduce a symmetric and reflexive relation Γ in $\mathcal{P} \times \mathcal{P}$. If $(\gamma, \gamma') \in \Gamma$ we write $\gamma \not\sim \gamma'$ and, following the tradition, we say that γ and γ' are *incompatible*. Conversely, if $(\gamma, \gamma') \notin \Gamma$, we write $\gamma \sim \gamma'$ and we say that γ and γ' are *compatible*. In most of the concrete realizations of this abstract polymers gas the condition $\gamma_i \not\sim \gamma_j$ means (in some sense depending on the geometry of the realization) that γ_i and γ_j are near (e.g. overlapping) and $\gamma_i \sim \gamma_j$ are far apart.

Polymers interact through a pair potential. Namely, the energy E of a configuration $\gamma_1, \ldots, \gamma_n$ of n polymers is given by

$$E(\gamma_1, \ldots, \gamma_n) = \sum_{1\le i<j\le n} V(\gamma_i, \gamma_j) \tag{2.1}$$

where pair potential $V(\gamma, \gamma')$ is a symmetric function in $\mathcal{P} \times \mathcal{P}$ taking values in $\mathbb{R} \cup \{+\infty\}$.

Fix now a finite set $\Lambda \subset \mathcal{P}$ (the "volume" of the gas). Then the probability to see the configuration $(\gamma_1, \ldots, \gamma_n) \in \Lambda^n$ is given by

$$Prob(\gamma_1, \ldots, \gamma_n) = \frac{1}{\Xi_\Lambda} z_{\gamma_1} z_{\gamma_2} \ldots z_{\gamma_n} e^{-E(\gamma_1,\ldots,\gamma_n)}$$

where the normalization constant Ξ_Λ is the grand-canonical partition function in the volume Λ and is given by

$$\Xi_\Lambda(z) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\dots,\gamma_n)\subset\Lambda^n} z_{\gamma_1} z_{\gamma_2} \dots z_{\gamma_n} e^{-E(\gamma_1,\dots,\gamma_n)} \tag{2.2}$$

Note that the configurations $\gamma_1, \dots, \gamma_n$ for which $E = +\infty$ are those with zero probability to occur, i.e. are forbidden.

We assume that the pair potential $V(\gamma_i, \gamma_j)$ is the sum of a repulsive (positive) interaction $U(\gamma_i, \gamma_j)$ which is non zero only when $\gamma_i \not\sim \gamma_j$ plus an interaction $W(\gamma_i, \gamma_j)$ which is non zero only when $\gamma_i \sim \gamma_j$. The repulsive potential $U(\gamma_i, \gamma_j)$ should be interpreted as the short range interaction, while $W(\gamma_i, \gamma_j)$ plays the role of the long range part of the interaction. We define these two interactions as follows. Let $\mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$. Let $\mathcal{C} = \{(\gamma, \gamma') \in \mathcal{P}^2 : \gamma \sim \gamma'\}$ and $\mathcal{I} = \{(\gamma, \gamma') \in \mathcal{P}^2 : \gamma \not\sim \gamma'\}$ and let us give the functions

$$U_{\not\sim} : \mathcal{I} \to [0, +\infty) \cup \{\infty\} : (\gamma, \gamma') \mapsto U_{\not\sim}(\gamma, \gamma')$$

$$W_\sim : \mathcal{C} \to (-\infty, +\infty) : (\gamma, \gamma') \mapsto W_\sim(\gamma, \gamma')$$

We put

$$V(\gamma_i, \gamma_j) = U(\gamma_i, \gamma_j) + W(\gamma_i, \gamma_j) \tag{2.3}$$

with the short range given by

$$U(\gamma_i, \gamma_j) = \begin{cases} 0 & \text{if } \gamma_i \sim \gamma_j \\ \\ U_{\not\sim}(\gamma_i, \gamma_j) & \text{if } \gamma_i \nsim \gamma_j \end{cases} \tag{2.4}$$

while the long range interaction is

$$W(\gamma_i, \gamma_j) = \begin{cases} W_\sim(\gamma_i, \gamma_j) & \text{if } \gamma_i \sim \gamma_j \\ \\ 0 & \text{if } \gamma_i \not\sim \gamma_j \end{cases} \tag{2.5}$$

Note that we allow $U(\gamma_i, \gamma_j)$ to take the value $+\infty$, while $W(\gamma_i, \gamma_j)$ is always a finite number. Observe also that we don't make any hypothesis on the signal of $W(\gamma_i, \gamma_j)$, so this interaction could be for some pairs attractive and for other pairs repulsive.

Since we are admitting non purely repulsive interaction among polymers, we also need to require that the potential energy E is stable in the classical sense. This can be achieved by imposing that the long range interaction $W(\gamma_i, \gamma_j)$ satisfies a stability property. Namely, we assume that there exists a function $B(\gamma) \geq 0$ such that

$$\sum_{1\leq i<j\leq n} W(\gamma_i, \gamma_j) \geq -\sum_{i=1}^n B(\gamma_i) \tag{2.6}$$

for all $n \in \mathbb{N}$ and all $(\gamma_1, \dots, \gamma_n) \in \mathcal{P}^n$. Such a condition implies that Ξ_Λ is convergent and

$$\Xi_\Lambda \leq 1 + \sum_{n\geq 1} \frac{1}{n!} \left[\sum_{\gamma\subset\Lambda} z_\gamma e^{B(\gamma)} \right]^n \leq \exp\Big\{ \sum_{\gamma\in\Lambda} z_\gamma e^{B(\gamma)} \Big\} \leq |\Lambda| \max_{\gamma\in\Lambda} \exp\{z_\gamma e^{B(\gamma)}\}$$

Actually, (2.6) implies that $\Xi_\Lambda(z)$ is a analytic in the whole $\mathbb{C}^{|\Lambda|}$ ($|\Lambda|$ is the cardinality of Λ).

As we said in the introduction, the usual choice available in the literature is $U_{\not\sim}(\gamma_i,\gamma_j) = +\infty$ for all $(\gamma,\gamma') \in \mathcal{I}$ and $W_{\sim}(\gamma_i,\gamma_j) = 0$ for all $(\gamma,\gamma') \in \mathcal{C}$ (purely hard core pair potential), but it has been also treated [18] the case $U_{\not\sim}(\gamma_i,\gamma_j) \geq 0$ if $\gamma_i \not\sim \gamma_j$ and $\gamma_i \neq \gamma_j$, $U_{\not\sim}(\gamma,\gamma) = +\infty$ for all $\gamma \in \mathcal{P}$ and of course $W(\gamma_i,\gamma_j) = 0$ (soft core repulsive and hard core self repulsive pair potential).

We remark that, even in the case that the long range interaction $W(\gamma_i,\gamma_j) = 0$ for all pairs, in this paper we are still treating cases which are more general than those studied until now. Namely, estimates of section 3 below hold for every repulsive interaction $0 \leq U(\gamma_i,\gamma_j) \leq +\infty$ with no needing of the hard core self repulsion condition (i.e. $U(\gamma,\gamma) = \infty$, for all $\gamma \in \mathcal{P}$) required e.g. in [18]. However, in view of the possible connections with the low temperature phase of spin systems with infinite range interactions, we think that the most interesting situation treated in the present paper is the case $W(\gamma_i,\gamma_j) \neq 0$, i.e. when a large distance potential, possibly attractive and possibly infinite range is acting among polymers.

2.2. Results.

The pressure of this gas, namely $\log \Xi_\Lambda$, can be written as a formal series through a Mayer expansion on the Gibbs factor $\exp - \sum_{1\leq i<j\leq n} V(\gamma_i,\gamma_j)$. Namely, a standard calculations (see e.g. [4]) gives

$$\log \Xi_\Lambda(z) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\dots,\gamma_n)\subset\Lambda^n} \phi^T(\gamma_1,\dots,\gamma_n) z_{\gamma_1}\dots z_{\gamma_n} \tag{2.7}$$

with

$$\phi^T(\gamma_1,\dots,\gamma_n) = \begin{cases} 1 & \text{if } n=1 \\ \sum\limits_{g\in G_n} \prod\limits_{\{i,j\}\in E_g} \left(e^{-V(\gamma_i,\gamma_j)} - 1\right) & \text{if } n \geq 2 \end{cases} \tag{2.8}$$

where G_n is the set all connected graphs with vertex set $\{1,2,\dots,n\}$. We recall that a graph $g \in G_n$ is a pair $g = (V_g, E_g)$ where $V_g = \{1,2,\dots,n\}$ is the set of vertices of g and $E_g \subset \{\{i,j\} \subset \{1,2,\dots,n\}\}$ is the set of edges of g. We also recall that $g = (V_g,E_g)$ is connected if for any A, B such that $A \cup B = V_g$ and $A \cap B = \emptyset$, there exists $\{i,j\} \in E_g$ such That $A \cap \{i,j\} \neq \emptyset$ and $B \cap \{i,j\} \neq \emptyset$.

The equation (2.7) makes sense only for those z for which the formal series in the r.h.s. of (2.7) converges absolutely. To study absolute convergence, we will consider the positive term series

$$|\log \Xi_\Lambda|(\rho) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\dots,\gamma_n)\subset\Lambda^n} |\phi^T(\gamma_1,\dots,\gamma_n)| \rho_{\gamma_1}\cdots\rho_{\gamma_n} \tag{2.9}$$

where now $\rho_\gamma \in [0,+\infty)$, for all $\gamma \in \mathcal{P}$ and $\rho = \{\rho_\gamma\}_{\gamma\in\mathcal{P}}$. Of course $|\log \Xi_\Lambda(z)| \leq |\log \Xi_\Lambda|(\rho)$ for z in the polydisk $\{|z_\gamma| \leq \rho_\gamma\}_{\gamma\in\mathcal{P}}$.

We further define, for each $\gamma_0 \in \mathcal{P}$, a function $\Pi^{\gamma_0}_{\mathcal{P}}(\rho)$ directly related to (2.9) (a "pinned" sum defined in the whole set $\mathcal{P}$) as follows

$$\Pi^{\gamma_0}_{\mathcal{P}}(\rho) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\gamma_2,\dots,\gamma_n)\in\mathcal{P}^n} |\phi^T(\gamma_0,\gamma_1,\dots,\gamma_n)| \rho_{\gamma_1}\dots\rho_{\gamma_n} \tag{2.10}$$

Clearly, if we are able to show that $\Pi^{\gamma_0}_{\mathcal{P}}(\rho)$ converges, then $|\log \Xi_\Lambda|(\rho)$ and hence $|\log \Xi_\Lambda(z)|$ for $|z_\gamma| \le \rho_\gamma$ also converge, since it is easy to check that

$$|\log \Xi_\Lambda|(\rho) \le |\Lambda| \sup_{\gamma_0 \in \Lambda} \rho_{\gamma_0} \Pi^{\gamma_0}_{\mathcal{P}}(\rho) \tag{2.11}$$

To understand the meaning of the series $\Pi^{\gamma_0}_{\mathcal{P}}(z)$ just observe that its finite volume version $\Pi^{\gamma_0}_{\mathcal{P}}(\rho)$, namely

$$\Pi^{\gamma_0}_{\Lambda}(\rho) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\gamma_2,\ldots,\gamma_n)\in\Lambda^n} |\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)| \rho_{\gamma_1} \ldots \rho_{\gamma_n} \tag{2.12}$$

is directly related to $\log \Xi_\Lambda(\rho_\Lambda)$. It is immediate to see that

$$\Pi^{\gamma_0}_{\Lambda}(\rho) = \frac{\partial}{\partial \rho_{\gamma_0}} |\log \Xi_\Lambda|(\rho) \tag{2.13}$$

The main result of the paper is a convergence criterion for the positive series (2.10). Such criterion can be considered as a generalization of the Kotecký-Preiss criterion for polymer system interacting through a pair potential which is not purely hard core. The criterion can be stated as the following theorem

Theorem 1 . *Let Let $\mu : \mathcal{P} \to [0,\infty) : \gamma \mapsto \mu_\gamma$ be a non negative valued function and let, for each $\gamma \in \mathcal{P}$, $\rho_\gamma \in [0,\infty)$ such that*

$$\rho_\gamma e^{B(\gamma)} \le \mu_\gamma \, e^{-\sum_{\tilde\gamma\in\mathcal{P}} F(\gamma,\tilde\gamma)\mu_{\tilde\gamma}}, \qquad \forall \gamma \in \mathcal{P} \tag{2.14}$$

where $B(\gamma)$ is the function defined in (2.6) and

$$F(\gamma_i,\gamma_j) = \begin{cases} |W(\gamma_i,\gamma_j)| & \text{if } \gamma_i \sim \gamma_j \\ \\ \left|e^{-U(\gamma_i,\gamma_j)} - 1\right| & \text{if } \gamma_i \nsim \gamma_j \end{cases} \tag{2.15}$$

Then the series $\Pi_{\gamma_0}(\rho)$ [defined in (2.10)] converges and satisfies $\rho_{\gamma_0} \Pi_{\gamma_0}(\rho) \le \mu_{\gamma_0}$.

Remark. Observe that in the usual case U hard-core and $W = 0$ one obtains from theorem 1 the usual Kotecký-Preiss condition. We recall however that when $W = 0$, i.e. when polymers interact just through a purely repulsive potential, one can do better than (2.14). In particular, for the purely hard core case (i.e. $U_{\nsim}(\gamma,\gamma') = \infty$ for all $(\gamma,\gamma') \in \mathcal{I}$ and $W_{\sim}(\gamma,\gamma') = 0$ for all $(\gamma,\gamma') \in \mathcal{C}$), it has been shown in [7] that the condition (2.14) can be considerably improved by taking advantage of the Penrose tree identity [13], (see also [14], [18] [7]) valid in the case of purely hard core interactions.

3. Proof of theorem 1.

The strategy of the proof is quite similar to the one used in [7]. In particular we use here the very same convergence argument for positive series which has been developed in [7]. On the other hand, in the present case we cannot use the Penrose identity in order to bound the Ursell coefficients $|\phi^T(\gamma_1,\ldots,\gamma_n)|$, since the pair potential (2.3) is not purely hard-core (and also not purely repulsive). We will rather make use of another well known "tree graph identity" originally proved in [2] (see also [3, 1, 16, 15]).

3.1. Tree graph inequality for $|\phi^T(\gamma_0, \gamma_1, \dots, \gamma_n)|$

We state the so called tree graph identity [2],[1], [3] by using the notations of [16] and [15]. We use the short notation $\mathrm{I}_n = \{1, 2, \dots, n\}$. A graph $\tau = (\mathrm{I}_n, E_\tau) \in G_n$ is called a *tree* if and only if its edge set E_τ has cardinality equal to $n-1$. Let us denote by T_n the set of trees with vertex set I_n.

In the following whenever U is a finite set, $|U|$ denotes its cardinality.

Lemma 2 (Tree graph identity). *Let V_{ij}, with $1 \le i < j \le n$ be $n(n-1)/2$ real numbers, then the following identity holds*

$$\sum_{g \in G_n} \prod_{\{i,j\} \in E_g} \left(e^{-V_{ij}} - 1\right) = \sum_{\tau \in T_n} \prod_{\{i,j\} \in E_\tau} (-V_{ij}) \int d\mu_\tau(\mathbf{t}_{n-1}, \mathbf{X}_{n-1}) e^{-K(\mathbf{X}_{n-1}, \mathbf{t}_{n-1})} \tag{3.1}$$

where:

- *$\mathbf{t}_{n-1}$ denote a set on $n-1$ interpolating parameters $\mathbf{t}_{n-1} \equiv (t_1, \dots, t_{n-1}) \in [0,1]^{n-1}$;*
- *$\mathbf{X}_{n-1}$ denote a set of "increasing" sequences of $n-1$ subsets, $\mathbf{X}_{n-1} \equiv X_1, \dots, X_{n-1}$ such as $\forall i$, $X_i \subset \mathrm{I}_n$, we must have $X_i \subset X_{i+1}$, $|X_i| = i$ and $X_1 = \{1\}$.*
- *$K(\mathbf{X}_{n-1}, \mathbf{t}_{n-1})$ is a convex decomposition of the potential, explicitly given by*

$$K(\mathbf{X}_{n-1}, \mathbf{t}_{n-1}) = \sum_{1 \le i < j \le n} t_1(\{i,j\}) \dots t_{n-1}(\{i,j\}) V_{ij} \tag{3.2}$$

  where

$$t_l(\{i,j\}) = \begin{cases} t_l \in [0,1] & \text{if } i \in X_l \text{ and } j \notin X_l \text{ or viceversa} \\ 1 & \text{otherwise} \end{cases}$$

  (a pair $\{i,j\}$ such that $i \in X_l$ and $j \notin X_l$ or viceversa is said to "cross" X_l).

- *The measure*

$$\int d\mu_\tau(\mathbf{t}_{n-1}, \mathbf{X}_{n-1}) \, [\cdots] \doteq \int_0^1 dt_1 \dots \int_0^1 dt_{n-1} \sum_{\substack{\mathbf{X}_{n-1} \\ \text{comp. } \tau}} t_1^{b_1 - 1} \dots t_{n-1}^{b_{n-1} - 1} \, [\cdots] \tag{3.3}$$

  has total mass equal to one (i.e. is a probability measure). In formula above "$\mathbf{X}_{n-1}$ comp. τ" means that for all $i = 1, 2, \dots, n-1$, X_i contains exactly $i-1$ edges of τ and b_i is the numebr of edges in τ which "cross" X_i

We want to use (3.1) to bond $|\phi^T(\gamma_1, \dots, \gamma_n)|$. This formula is useful when the pair potential is not purely repulsive. However, due to the restriction V_{ij} finite (otherwise the r.h.s. of (3.1) is not well defined), one in general can apply (3.1) only if the pair potential is finite and absolutely integrable, see [3], which is a quite restrictive condition. In particular this rules out a pair potential with hard core at short distances which is precisely one of the situations we would like to treat.

We show here that it is possible to give meaning to r.h.s. of (3.1) even when some among the V_{ij}'s take the value ∞ (the l.h.s. of (3.1) makes sense even in this case). This is possible due to our assumption that the pair potential is the sum of a purely repulsive part plus a stable potential. We define

$$U_H(\gamma_i,\gamma_j) = \begin{cases} H & \text{if } U(\gamma_i,\gamma_j)=\infty \\ \\ U(\gamma_i,\gamma_j) & \text{otherwise} \end{cases}$$

and $V_{H,J}(\gamma_i,\gamma_j) = U_H(\gamma_i,\gamma_j) + W(\gamma_i,\gamma_j)$. Then, for any fixed $(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n$

$$\phi^T(\gamma_1,\dots,\gamma_n) = \lim_{H\to\infty}\sum_{g\in G_n}\prod_{\{i,j\}\in E_g}(e^{-V_H(\gamma_i,\gamma_j)}-1)$$

We can now use (3.1) for the *finite* potential V_H and we get

$$\sum_{g\in G_n}\prod_{\{i,j\}\in E_g}(e^{-V_H(\gamma_i,\gamma_j)}-1) =$$

$$= \lim_{H\to\infty}\sum_{\tau\in T_n}\prod_{\{i,j\}\in E_\tau}(-V_{H,J}(\gamma_i,\gamma_j))\int d\mu_\tau(\mathbf{t}_{n-1},\mathbf{X}_{n-1})e^{-K_H(\mathbf{X}_{n-1},\mathbf{t}_{n-1})}$$

where

$$K_H(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) = \sum_{1\le i<j\le n} t_1(\{i,j\})\dots t_{n-1}(\{i,j\})V_H(\gamma_i,\gamma_j) \tag{3.4}$$

Now, for fixed $\tau = (\mathrm{I}_n, E_\tau)\in T_n$ and $(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n$, let us consider the factor

$$w^\tau_H(\gamma_1,\dots,\gamma_n) = \prod_{\{i,j\}\in E_\tau}|V_{H,J}(\gamma_i,\gamma_j)|\int d\mu_\tau(\mathbf{t}_{n-1},\mathbf{X}_{n-1})e^{-K_H(\mathbf{X}_{n-1},\mathbf{t}_{n-1})}$$

By the assumptions (2.3)-(2.5), the edges $\{i,j\}\subset \mathrm{I}_n$ are naturally partitioned into two disjoint sets E^H_n and $\mathrm{E}_n\backslash\mathrm{E}^H_n$ where $\mathrm{E}^H_n = \{\{i,j\}\subset I_n : \gamma_i \not\sim \gamma_j)\}$. Thus also the edges of the tree τ are partitioned into two disjoint sets E^H_τ and $E_\tau\backslash E^H_\tau$ where $E^H_\tau = E_\tau\cap \mathrm{E}^H_n$. So we have

$$w^\tau_H(\gamma_1,\dots,\gamma_n) = \prod_{\{i,j\}\in E_\tau\backslash E^H_\tau}|W(\gamma_i,\gamma_j)|\prod_{\{i,j\}\in E^H_\tau}U_H(\gamma_i,\gamma_j)\int d\mu_\tau(\mathbf{t}_{n-1},\mathbf{X}_{n-1})e^{-K_H(\mathbf{X}_{n-1},\mathbf{t}_{n-1})}$$

Now observe that

$$K_H(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) = K_{U_H}(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) + K_W(\mathbf{X}_{n-1},\mathbf{t}_{n-1})$$

where

$$K_{U_H}(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) = \sum_{1\le i<j\le n} t_1(\{i,j\})\dots t_{n-1}(\{i,j\})U_H(\gamma_i,\gamma_j)$$

and

$$K_W(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) = \sum_{1\le i<j\le n} t_1(\{i,j\})\dots t_{n-1}(\{i,j\})W(\gamma_i,\gamma_j)$$

The potential $K_{U_H}(\mathbf{X}_{n-1},\mathbf{t}_{n-1})$ is non negative and can be bounded, for $\eta>0$, as follows

$$K_{U_H}(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) \geq \sum_{\{i,j\}\subset E^H_\tau} t_1(\{i,j\})\dots t_{n-1}(\{i,j\})U_H(\gamma_i,\gamma_j) =$$

$$= \sum_{\{i,j\}\subset E^H_\tau} t_1(\{i,j\})\dots t_{n-1}(\{i,j\})U_H(\gamma_i,\gamma_j) + \eta \sum_{\{i,j\}\subset E_\tau\backslash E^H_\tau} t_1(\{i,j\})\dots t_{n-1}(\{i,j\}) -$$

$$-\eta \sum_{\{i,j\}\subset E_\tau\backslash E^H_\tau} t_1(\{i,j\})\dots t_{n-1}(\{i,j\}) \quad \geq$$

$$\geq \sum_{\{i,j\}\subset E^H_\tau} t_1(\{i,j\})\dots t_{n-1}(\{i,j\})U_H(\gamma_i,\gamma_j) + \eta \sum_{\{i,j\}\subset E_\tau\backslash E^H_\tau} t_1(\{i,j\})\dots t_{n-1}(\{i,j\}) - \eta|E_\tau\backslash E^H_\tau| =$$

$$= K_{V_\tau}(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) - \eta|E_\tau\backslash E^H_\tau|$$

where

$$K_{V^\tau}(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) = \sum_{1\leq i<j\leq n} t_1(\{i,j\})\dots t_{n-1}(\{i,j\})V^\tau_{ij}$$

with V^τ_{ij} being the positive (H dependent) pair potential given by

$$V^\tau_{ij} = \begin{cases} U_H(\gamma_i,\gamma_j) & \text{if } \{i,j\}\in E^H_\tau \\ \eta & \text{if } \{i,j\}\in E_\tau\backslash E^H_\tau \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, from the fact that W is stable it follows that also $K_W(\mathbf{X}_{n-1},\mathbf{t}_{n-1})$ is stable, see e.g. [3], [15], [16]. Namely, from (2.6) it follows that

$$K_W(\mathbf{X}_{n-1},\mathbf{t}_{n-1}) \geq -\sum_{i=1}^n B(\gamma_i)$$

Hence $w^\tau_H(\gamma_1,\dots,\gamma_n)$ can be bounded by

$$w^\tau_H(\gamma_1,\dots,\gamma_n) \leq e^{+\sum_{i=1}^n B(\gamma_i)+\eta|E_\tau\backslash E^H_\tau|}\left[\prod_{\{i,j\}\in E_\tau\backslash E^H_\tau}|W(\gamma_i,\gamma_j)|\right]\times\left[\frac{1}{\eta}\right]^{|E_\tau\backslash E^H_\tau|}\times$$

$$\times\prod_{\{i,j\}\in E_\tau} V^\tau_{ij}\int d\mu_\tau(\mathbf{t}_{n-1},\mathbf{X}_{n-1})e^{-K_{V_\tau}(\mathbf{X}_{n-1},\mathbf{t}_{n-1})}$$

Applying now the tree graph identity (3.1) to the pair potential V^τ_{ij} one conclude immediately (see e.g. [3]) that, for all $H\in[0,+\infty)$

$$\prod_{\{i,j\}\in E_\tau} V^\tau_{ij}\int d\mu_\tau(\mathbf{t}_{n-1},\mathbf{X}_{n-1})e^{-K_{V_\tau}(\mathbf{X}_{n-1},\mathbf{t}_{n-1})} = \prod_{\{i,j\}\in E_\tau}\left|e^{-V^\tau_{ij}}-1\right| = \tag{3.5}$$

$$= \prod_{\{i,j\}\in E_\tau^H} \left|e^{-U_H(\gamma_i,\gamma_j)}-1\right| \left|e^{-\eta}-1\right|^{|E_\tau \setminus E_\tau^H|}$$

Hence, considering that $e^{\eta|E_\tau\setminus E_\tau^H|}\left|e^{-\eta}-1\right|^{|E_\tau\setminus E_\tau^H|} = (e^{\eta}-1)^{|E_\tau\setminus E_\tau^H|}$ we get

$$w_H^\tau(\gamma_1,\dots,\gamma_n) \le e^{+\sum_{i=1}^n B(\gamma_i)} \prod_{\{i,j\}\in E_\tau^H} \left|e^{-U_H(\gamma_i,\gamma_j)}-1\right| \prod_{\{i,j\}\in E_\tau\setminus E_\tau^H} \left|\frac{(e^\eta-1)}{\eta}W(\gamma_i,\gamma_j)\right|$$

and due to the arbitrarity of η which can be take as small as we please, we obtain

$$w_H^\tau(\gamma_1,\dots,\gamma_n) \le e^{+\sum_{i=1}^n B(\gamma_i)} \prod_{\{i,j\}\in E_\tau^H} \left|e^{-U_H(\gamma_i,\gamma_j)}-1\right| \prod_{\{i,j\}\in E_\tau\setminus E_\tau^H} |W(\gamma_i,\gamma_j)|$$

and so

$$w^\tau(\gamma_1,\dots,\gamma_n) = \lim_{H\to\infty} w_H^\tau(\gamma_1,\dots,\gamma_n) \le e^{+\sum_{i=1}^n B(\gamma_i)} \prod_{\{i,j\}\in E_\tau^H} \left|e^{-U(\gamma_i,\gamma_j)}-1\right| \prod_{\{i,j\}\in E_\tau\setminus E_\tau^H} |W(\gamma_i,\gamma_j)|$$

In conclusion we have that

$$|\phi^T(\gamma_1,\dots,\gamma_n)| \le e^{+\sum_{i=1}^n B(\gamma_i)} \sum_{\tau\in T_n} \prod_{\{i,j\}\in E_\tau} F(\gamma_i,\gamma_j) \tag{3.6}$$

where

$$F(\gamma_i,\gamma_j) = \begin{cases} |W(\gamma_i,\gamma_j)| & \text{if } \gamma_i \sim \gamma_j \\ \\ \left|e^{-U(\gamma_i,\gamma_j)}-1\right| & \text{if } \gamma_i \nsim \gamma_j \end{cases}$$

and hence also, for $n \ge 1$

$$|\phi^T(\gamma_0,\gamma_1,\dots,\gamma_n)| \le e^{+\sum_{i=0}^n B(\gamma_i)} \sum_{\tau\in T_n^0} \prod_{\{i,j\}\in E_\tau} F(\gamma_i,\gamma_j) \tag{3.7}$$

where T_n^0 is the set of all trees with vertex set $\mathrm{I}_n^0 \doteq \{0,1,2,\dots,n\}$. Inserting (3.7) in (2.10) we get

$$\Pi_{\mathcal{P}}^{\gamma_0}(\rho) \le 1 + \sum_{n=1}^\infty \frac{1}{n!} \sum_{(\gamma_1,\gamma_2,\dots,\gamma_n)\in\mathcal{P}^n} e^{+\sum_{i=0}^n B(\gamma_i)} \sum_{\tau\in T_n^0} \prod_{\{i,j\}\in E_\tau} F(\gamma_i,\gamma_j)\rho_{\gamma_1}\dots\rho_{\gamma_n} \le$$

$$\le e^{B(\gamma_0)}\left[1 + \sum_{n=1}^\infty \frac{1}{n!} \sum_{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n} \sum_{\tau\in T_n^0} \prod_{\{i,j\}\in E_\tau} F(\gamma_i,\gamma_j)\rho_{\gamma_1}e^{B(\gamma_1)}\dots\rho_{\gamma_n}e^{B(\gamma_n)}\right] =$$

If we pose

$$\tilde\rho_\gamma = \rho_\gamma e^{B(\gamma)} \tag{3.8}$$

and

$$|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\tau \in T^0_n} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \prod_{\{i,j\}\in E_\tau} F(\gamma_i,\gamma_j)\tilde{\rho}_{\gamma_1}\ldots\tilde{\rho}_{\gamma_n} \tag{3.9}$$

We get

$$|\Pi|^{\gamma_0}_{\mathcal{P}}(\rho) \le e^{B(\gamma_0)}|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho}) \tag{3.10}$$

So the convergence of $|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho})$ implies that of $|\Pi|^{\gamma_0}_{\mathcal{P}}(\rho)$.

3.2. Planar rooted trees and convergence

We think the trees with vertex set $\mathrm{I}^0_n = \{0, 1, 2, \ldots, n\}$ (i.e the elements of T^0_n) as rooted in 0. We define a map $m : \tau \mapsto m(\tau)$ which associate to each labelled tree $\tau \in T^0_n$ a unique drawing $t = m(\tau)$ in the plane, called the *planar rooted tree* associated to τ, as follows.

Given τ in T^0_n, place the vertex 0 (the root) at the leftmost position of the drawing. From 0 there emerge s_0 branches ending at the *first-generation* vertices $i_1, \ldots, i_{s_0}$. Drawn these vertices along a vertical line at the right of the root in such way that the higher has the low label and labels increase as we go down along the vertical line (ordering increasing label vertices "from high to low"). Then iterate this procedure for the descendants of each firs generation vertex (i.e. the *second generation* vertices) $i_1, \ldots i_{s_0}$ and so on... (see figure 1).

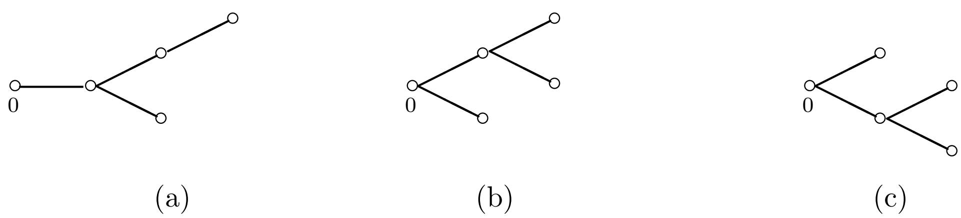


Figure 1: the planar rooted trees associated to the trees (a) with edge set $\{0,3\},\{1,3\},\{2,3\},\{1,4\}$, (b) with edge set $\{0,2\},\{0,3\},\{1,2\},\{2,4\}$ and (c) with edge set $\{0,2\},\{0,4\},\{4,3\},\{1,4\}$. Observe that (b) and (c) are *different* planar rooted tree

There is a natural partial order $\prec$ among the vertices in a rooted tree. For $u, v \in t$, we say that u precedes v and write $u \prec v$ (or $v \succ u$) if the (unique) path from the root to v contains u. If $\{v, u\}$ is an edge of t rooted tree, then either $v \prec u$ or $u \prec v$. If $u \prec v$. u is called the *predecessor* and v is called the *descendant*. The root has no predecessor and it is the extremum respect to the partial order relation $\prec$ in t. For each vertex v of t, we will denote by s_v the *branching factor* of v and we denote by $v^1 \ldots, v^{s_v}$ the s_v descendants of v, (v^1 being the higher and v^{s_v} being the lower in the drawing).

Clearly the map $\tau \mapsto m(\tau) = t$ is many-to-one and the cardinality of the preimage of a planar rooted tree t (=number of ways of labelling the n non-root vertices of the tree with n distinct labels consistently with the rule "from high to low") is given by

$$\left|\{\tau \in T^0_n : m(\tau) = t\}\right| \;=\; \frac{n!}{\prod_{v\succeq 0} s_{v_i}!} \tag{3.11}$$

We denote by $\mathbb{T}^0_n =$ the set of all planar rooted trees with n vertices and by $\mathbb{T}^{0,k}$ the set of planar rooted trees with maximal generation number k; let also $\mathbb{T}^0 = \cup_{n\geq 0}\mathbb{T}^0_n = \cup_{k\geq 0}\mathbb{T}^{0,k}$ be the set of all planar rooted trees.

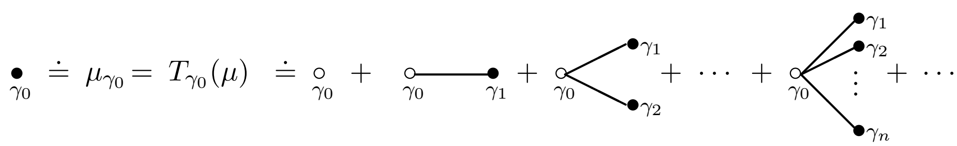
Let now $\mu : \mathcal{P} \to [0,\infty)^{\mathcal{P}} : \gamma \mapsto \mu_\gamma$ be a positive valued function defined in $\mathcal{P}$ and let, for each $\gamma \in \mathcal{P}$, $R_\gamma \in [0,\infty)^{\mathcal{P}}$ be defined by the equations

$$\mu_\gamma = R_\gamma \varphi_\gamma(\mu), \qquad \gamma \in \mathcal{P} \tag{3.12}$$

with

$$\varphi_\gamma(\mu) \;=\; 1 + \sum_{n\geq 1} \sum_{(\gamma_1,\dots,\gamma_n)\in\mathcal{P}^n} b_n(\gamma;\gamma_1,\dots,\gamma_n)\,\mu_{\gamma_1}\dots\mu_{\gamma_n} \tag{3.13}$$

for certain functions $b_n : \mathcal{P}^{n+1} \to [0,\infty)$. Denoting $R_\gamma\varphi_\gamma(\mu) = T_{\gamma_0}(\mu)$ the equation (3.12) can be visualized in the diagrammatic form



The sum is over all single-generation rooted trees. In each tree, vertices with open circles with subscript γ represents a factor R_γ, vertices with bullets with subscript γ a factor μ_γ and vertices other than the root must be summed over all possible polymers γ. At each vertex with n descendants, a "vertex function" b_n acts, having as arguments the $n+1$-tuple formed by the polymer at the vertex and the n polymers associated to the n descendants of that vertex. With this representation, the iteration $T^2(\mu) = T(T(\mu))$ corresponds to replacing each of the bullets by each one of the diagrams of the expansion for T. This leads to planar rooted trees of up to two generations, with open circles at first-generation vertices and bullets at second-generation ones. In particular, all single-generation trees have only open circles. Notice that the two drawings of Figure 1 appear in two different terms of the expansion, and hence should be counted as *different* diagrams. More generally, the k-th iteration of T involves all possible planar rooted trees up to k generations. In each term of the expansion, vertices of generation k are occupied by bullets and all the others by open circles. A straightforward inductive argument shows that

$$T^k_{\gamma_0}(\mu) \;=\; R_{\gamma_0}\Big[\sum_{\ell=0}^{k-1}\Phi^{(\ell)}_{\gamma_0}(R) + \Phi^{(k)}_{\gamma_0}(R,\mu)\Big] \tag{3.14}$$

where we have denoted $R = \{R_\gamma\}_{\gamma\in\mathcal{P}}$ and

$$\Phi^{(\ell)}_{\gamma_0}(R) \;=\; \sum_{t\in\mathbb{T}^{0,\ell}} \prod_{v\succeq 0} \Bigg\{ \sum_{(\gamma_{v^1},\dots,\gamma_{v^{s_v}})\in\mathcal{P}^{s_v}} b_{s_v}(\gamma_v;\gamma_{v^1},\dots,\gamma_{v^{s_v}})\,R_{\gamma_{v^1}}\dots R_{\gamma_{v^{s_v}}} \Bigg\} \tag{3.15}$$

Here the product $\prod_{v\succeq 0}$ over the vertices of t must be done respecting the partial order of the set of vertices in t, i.e. if $v \succ u$ the v must be at the right of u in the product. The factor $\Phi^{(k)}_{\gamma_0}(R,\mu)$ has a similar expression but with the activities of the vertex of the k-th generation weighted by μ. Here we agree that $b_0(\gamma_v) \equiv 1$ and $\prod_{\emptyset} \equiv 1$. We are interested in the $\ell \to \infty$ limit of (3.15).

Proposition 2 *Let $\mu : \mathcal{P} \to [0,\infty)^{\mathcal{P}} : \gamma \mapsto \mu_\gamma$ be a positive valued function and let, for each $\gamma \in \mathcal{P}$, $R_\gamma \in [0,\infty)^{\mathcal{P}}$ be defined by the equations (3.12). Let, $\forall \gamma \in \mathcal{P}$, $\tilde{\rho}_\gamma \in [0,\infty)$ such that $\tilde{\rho}_\gamma \le R_\gamma$. Then the series*

$$\Phi_{\gamma_0}(\tilde{\rho}) \;:=\; \sum_{t\in\mathbb{T}^0}\prod_{v\succeq 0}\left\{ \sum_{(\gamma_{v^1},\dots,\gamma_{v^{s_v}})\in\mathcal{P}^{s_v}} b_{s_v}(\gamma_v;\gamma_{v^1},\dots,\gamma_{v^{s_v}})\,\tilde{\rho}_{\gamma_{v^1}}\dots\tilde{\rho}_{\gamma_{v^{s_v}}} \right\} \tag{3.16}$$

is finite for each $\gamma_0 \in \mathcal{P}$. Furthermore

$$\tilde{\rho}_{\gamma_0}\Phi_{\gamma_0}(\tilde{\rho}) \le \mu_{\gamma_0} \tag{3.17}$$

for each $\gamma_0 \in \mathcal{P}$.

Proof. By definition $\Phi_{\gamma_0}(\tilde{\rho}) = \sum_{\ell=0}^{\infty}\Phi^{(\ell)}_{\gamma_0}(\tilde{\rho})$. By (3.14), the fact that $T^k_{\gamma_0}(\mu) = \mu_{\gamma_0}$ for all $k \in \mathbb{N}$, and the assumption $\tilde{\rho}_\gamma \le R_\gamma$ for all $\gamma \in \mathcal{P}$, we obtain that

$$\tilde{\rho}_{\gamma_0}\sum_{\ell=0}^{n}\Phi^{(\ell)}_{\gamma_0}(\tilde{\rho}) \;\le\; R_{\gamma_0}\sum_{\ell=0}^{n}\Phi^{(\ell)}_{\gamma_0}(R) \;\le\; \mu_{\gamma_0}$$

for all n. Thus, since the sequence of partial sums of the series $\rho_{\gamma_0}\Phi_{\gamma_0}(\tilde{\rho})$ is monotonic increasing and bounded by μ_{γ_0}, $\tilde{\rho}_{\gamma_0}\Phi_{\gamma_0}(\tilde{\rho})$ converges, and $\tilde{\rho}_{\gamma_0}\Phi_{\gamma_0}(\tilde{\rho}) \le \mu_{\gamma_0}$. $\square$

3.3. End of the proof of theorem 1

We first reorganize the sum over labelled trees appearing in formula (3.9) in terms of the called planar rooted trees previously introduced. Namely, recalling that $\mathbb{T}^0_n$ is the set of all planar rooted trees with fixed root 0 and n vertices (different from the root), we can rewrite the r.h.s. of (3.9) as

$$|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho}) \;=\; 1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{t\in\mathbb{T}^0_n}\sum_{\substack{\tau\in T^0_n\\ m(\tau)=t}}\sum_{(\gamma_1,\dots,\gamma_n)\subset\mathcal{P}^n}\prod_{\{i,j\}\in E_\tau}F(\gamma_i,\gamma_j)\tilde{\rho}_{\gamma_1}\dots\tilde{\rho}_{\gamma_n} \tag{3.18}$$

Observe that the factor

$$\sum_{(\gamma_1,\dots,\gamma_n)\subset\mathcal{P}^n}\prod_{\{i,j\}\in E_\tau}F(\gamma_i,\gamma_j)\tilde{\rho}_{\gamma_1}\dots\tilde{\rho}_{\gamma_n}$$

depends only on the planar rooted tree $t = m(\tau)$ associated to τ (labels of τ are dummy indices in the sum), i.e.

$$\sum_{(\gamma_1,\dots,\gamma_n)\subset\mathcal{P}^n}\prod_{\{i,j\}\in E_\tau}F(\gamma_i,\gamma_j)\tilde{\rho}_{\gamma_1}\dots\tilde{\rho}_{\gamma_n} = \prod_{v\succeq v_0}\left\{\prod_{i=1}^{s_v}\sum_{\gamma_{v^i}\in\mathcal{P}}F(\gamma_v,\gamma_{v^i})\tilde{\rho}_{\gamma_{v^i}}\right\} \tag{3.19}$$

with the convention that $\prod_{i=1}^{s_v}\sum_{\gamma_{v^i}\in\mathcal{P}}F(\gamma_v,\gamma_{v^i})\tilde{\rho}_{\gamma_{v^i}} = 1$ when $s_v = 0$.

So in conclusion, inserting (3.19) into (3.18) and using also (3.11), we obtain

$$|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho}) \;=\sum_{t\in\mathbb{T}^0}\prod_{v\succeq v_0}\frac{1}{s_v!}\left\{\prod_{i=1}^{s_v}\sum_{\gamma_{v^i}\in\mathcal{P}}F(\gamma_v,\gamma_{v^i})\tilde{\rho}_{\gamma_{v^i}}\right\} \tag{3.20}$$

Comparing (3.20) with (3.16) we immediately see that $|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho}) = \Phi_{\gamma_0}(\tilde{\rho})$ provided

$$b_n(\gamma;\gamma_1,\ldots,\gamma_n) = \frac{1}{n!}\prod_{i=1}^{n} F(\gamma,\gamma_i) \tag{3.21}$$

so that

$$\varphi_\gamma(\mu) \;=\; 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n}\prod_{i=1}^{n} F(\gamma,\gamma_i)\mu_{\gamma_i} \;=\; e^{\sum_{\tilde{\gamma}\in\mathcal{P}} F(\gamma,\tilde{\gamma})\mu_\gamma} \tag{3.22}$$

Hence proposition 2 yields the criterion (2.14) for the convergence of the series $|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho})$ defined in (2.10). As a matter of fact, by proposition 2, with the identification (3.21), we have immediately that the series $|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho})$ defined in (3.9) is finite for each $\gamma_0 \in \mathcal{P}$ and $\tilde{\rho}_{\gamma_0}|\tilde{\Pi}|^{\gamma_0}_{\mathcal{P}}(\tilde{\rho}) \leq \mu_{\gamma_0}$ for each $\gamma_0 \in \mathcal{P}$. Now recalling (3.10) and (3.8) we obtain $\rho_{\gamma_0}\Pi_{\gamma_0}(\rho) \leq \mu_{\gamma_0}$.

4. Example. Contours in $\mathbb{Z}^d$ with infinite range interactions

Let $\mathbb{Z}^d$ be the cubic unit lattice in d-dimensions equipped with the usual nearest neighbor path distance, i.e., if $x, y \in \mathbb{Z}^d$, their distance $|x-y|$ is the length of the shortest path of nearest neighbors connecting x to y. If W and U are sets of $\mathbb{Z}^d$, their distance $d(U,W)$ is the number $d(U,W) = \min_{x\in W, y\in U}|x-y|$.

Let r_0 be a positive integer; we say that a set $U \subset \mathbb{Z}^d$ is *r_0-connected* if for any partition $\{A,B\}$ of U (i.e. such that $A\cup B = U$ and $A\cap B = \emptyset$), we have that $d(A,B)\leq r_0$.

We take as the set of polymers the set

$$\mathcal{P}_C = \{\gamma\subset\mathbb{Z}^d :\; 1\leq|\gamma|<\infty,\; \gamma \text{ is } r_0 \text{ connected}\}$$

Thus, in this concrete example polymers are finite sets of vertices of $\mathbb{Z}^d$, which are r_0-connected. We call them *contours* from now on. Incompatibility now means the following: two contours γ and γ' are incompatible, i.e. $\gamma\not\sim\gamma'$, if $\gamma\cup\gamma'\in\mathcal{P}_C$, or, equivalently, if $d(\gamma,\gamma')\leq r_0$. So two contours are γ and γ' compatibles, i.e. $\gamma\sim\gamma'$, if $d(\gamma,\gamma')>r_0$.

We suppose that the long range interaction among contours is defined as follows. Given two contours $\{\gamma,\gamma'\}\in\mathcal{P}_C$ such that $\gamma\sim\gamma'$, we put

$$W_\sim(\gamma,\gamma') = \sum_{\substack{x\in\gamma\\ y\in\gamma'}} J_{xy} \tag{4.1}$$

where J_{xy} decays (at least) polynomially in an absolutely summable way. Namely, we suppose that there exist a positive constants J_1 and λ such that

$$|J_{xy}|\leq\frac{2J_1}{|x-y|^{d+\lambda}}, \qquad \forall\{x,y\}\in\mathbb{Z}^d \tag{4.2}$$

An interaction among polymers (i.e. contours) of the form (4.1) is what one can reasonably expect from a low temperature contour expansion of some spin system in $\mathbb{Z}^d$ with infinite range interaction.

The potential J_{xy} is the original pair interaction among the spins. Observe also that $W(\gamma,\gamma')$ is a stable interaction in the sense of (2.6). As a matter of fact,

$$\sum_{1\le i<j\le n} |W(\gamma_i,\gamma_j)| \le J_1 \sum_{i=1}^{n} |\gamma_i| \sup_{x\in\gamma_i} \sum_{\substack{y\in\mathbb{Z}^d \\ |x-y|>r_0}} \frac{1}{|x-y|^{d+\lambda}}$$

Now, for all γ_i

$$\sup_{x\in\gamma_i} \sum_{\substack{y\in\mathbb{Z}^d \\ |x-y|>r_0}} \frac{1}{|x-y|^{d+\lambda}} \le \sum_{\substack{y\in\mathbb{Z}^d \\ |y|>r_0}} \frac{1}{|y|^{d+\varepsilon}} \le \sum_{n>r_0} \frac{1}{n^{d+\lambda}} |S_n|$$

where $S_n = \{y \in \mathbb{Z}^d : |y| = n\}$. An easy calculation show that

$$|S_n| \le \frac{(2d)^d}{d!} n^{d-1}$$

So if we put

$$J_2 = J_1 \frac{(2d)^d}{d!} \sum_{n>r_0} \frac{1}{n^{1+\lambda}} \tag{4.3}$$

we get that $W(\gamma,\gamma')$ defined by (4.1) satisfies (2.6) with

$$B(\gamma) = J_2|\gamma| \tag{4.4}$$

We also suppose, as it is usual in low temperature expansions, that the activity ρ_γ of a contour γ decays exponentially with its size (i.e. its cardinality $|\gamma|$). Namely we will make the following assumption

$$|\rho_\gamma| \le \varepsilon^{|\gamma|} \qquad\qquad \text{for some } 0<\varepsilon<1 \tag{4.5}$$

where ε is a small parameter (typically $\varepsilon = e^{-\beta}$ where β is the inverse temperature). Thus the condition of convergence will be here a condition on the smallness of the physical parameter ε.

Let us check what condition we obtain for ε by using theorem 1. We choose

$$\mu_\gamma = \rho_\gamma e^{\alpha|\gamma|}$$

so, by (2.14), the pressure of such contour gas can be written in terms of an absolutely convergent series if, for some $\alpha > 0$

$$\sum_{\tilde\gamma\in\mathcal{P}_C} F(\gamma,\tilde\gamma)\xi(\tilde\gamma) \le \alpha|\gamma| \tag{4.6}$$

where we have put

$$\xi(\tilde\gamma) = \rho_{\tilde\gamma} e^{J_2|\tilde\gamma|} e^{\alpha|\tilde\gamma|} \tag{4.7}$$

By bounding again $F(\gamma,\tilde\gamma) \le 1$ whenever $\gamma \nsim \gamma$ (i.e. as if U is purely hard core, which is the worst case), we get

$$\sum_{\tilde\gamma\in\mathcal{P}_C} \xi(\tilde\gamma)F(\gamma,\tilde\gamma) = \sum_{\substack{\tilde\gamma\in\mathcal{P}_C \\ d(\gamma,\tilde\gamma)\le r_0}} \xi(\tilde\gamma) + \sum_{\substack{\tilde\gamma\in\mathcal{P}_C \\ d(\gamma,\tilde\gamma)>r_0}} \xi(\tilde\gamma)|W(\gamma,\tilde\gamma)| \le$$

$$|\gamma|\Big[(1+2dr_0^d)\sup_{x\in\mathbb{Z}^d}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ x\in\gamma}}\xi(\tilde\gamma)\Big]+\max_{x\in\gamma}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ d(\tilde\gamma,x)>r_0}}\xi(\tilde\gamma)|W(\gamma,\tilde\gamma)|$$

Observe that, by assumption (4.2), $|W(\gamma,\tilde\gamma)|\leq J_1|\gamma||\tilde\gamma|n^{-(d+\lambda)}$ whenever $d(\gamma,\tilde\gamma)=n$. Therefore

$$\max_{x\in\gamma}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ d(\tilde\gamma,x)>r_0}}\xi(\tilde\gamma)|W(\gamma,\tilde\gamma)|\leq|\gamma|\sum_{n>r_0}\frac{J_1}{n^{d+\lambda}}\max_{x\in\gamma}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ d(\gamma,x)=n}}\xi(\tilde\gamma)|\tilde\gamma|\leq$$

$$\leq|\gamma|\sum_{n>r_0}\frac{J_1}{n^{d+\lambda}}\sup_{x\in\mathbb{Z}^d}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ \tilde\gamma\cap S_n(x)\neq\emptyset}}\xi(\tilde\gamma)|\tilde\gamma|\leq|\gamma|\sum_{n>r_0}\frac{J_1}{n^{d+\lambda}}|S_n|\sup_{x\in\mathbb{Z}^d}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ x\in\gamma}}\xi(\tilde\gamma)|\tilde\gamma|\leq$$

$$\leq|\gamma|\frac{(2d)^dJ_1}{d!}\sum_{n>r_0}\frac{1}{n^{1+\lambda}}\sup_{x\in\mathbb{Z}^d}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ x\in\gamma}}\xi(\tilde\gamma)|\tilde\gamma|=J_2|\gamma|\sup_{x\in\mathbb{Z}^d}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ x\in\gamma}}\xi(\tilde\gamma)|\tilde\gamma|$$

and so

$$\sum_{\tilde\gamma\in\mathcal{P}_C}\xi(\tilde\gamma)F(\gamma,\tilde\gamma)\leq|\gamma|\Bigg[\Big((1+2dr_0^d)\sup_{x\in\mathbb{Z}^d}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ x\in\gamma}}\xi(\tilde\gamma)\Big)+J_2\sup_{x\in\mathbb{Z}^d}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ x\in\gamma}}\xi(\tilde\gamma)|\tilde\gamma|\Bigg]\leq$$

$$\leq|\gamma|\Big[1+2dr_0^d+J_2\Big]\sup_{x\in\mathbb{Z}^d}\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ x\in\tilde\gamma}}\xi(\tilde\gamma)|\tilde\gamma|\leq J_3|\gamma|\sum_{\substack{\tilde\gamma\in\mathcal{P}_C\\ x\in\tilde\gamma}}\xi(\tilde\gamma)|\tilde\gamma|$$

where

$$J_3=1+2dr_0^d+J_2$$

Thus, recalling definition (4.7) and using also assumption (4.5) on the activity, convergence condition (4.6) becomes

$$J_3\sum_{n=1}^{\infty}n(\varepsilon e^{J_2}e^{\alpha})^nC_n\leq\alpha \tag{4.8}$$

where C_n is the number of r_0 connected set of vertices of $\mathbb{Z}^d$ with cardinality n containing the origin. C_n can be easily bounded by C^n for some C, e.g. one can take $C\leq eB(r_0)$ where $B(r_0)$ is the number of vertices of $\mathbb{Z}^d$ at distance less or equal r_0 from the origin. So condition (4.8) becomes

$$\sum_{n=1}^{\infty}n(xe^{\alpha})^n\leq\frac{\alpha}{J_3} \tag{4.9}$$

where

$$x=\varepsilon e^{J_2+1}B(r_0) \tag{4.10}$$

Formulas (4.9) and (4.10) imply that

$$\varepsilon\leq\Big[e^{-\alpha}f(\alpha/J_3)\Big]\frac{1}{e^{J_2+1}B(r_0)}$$

where

$$f(u) = \frac{2u}{2u+1+\sqrt{4u+1}}$$

For example, taking $\alpha = J_3/2$ (which however is not the optimal value), we obtain that the convergence condition (4.6) is satisfied if

$$\varepsilon \leq \frac{e^{-(\frac{J_3}{2}+J_2+1)}}{(2+\sqrt{3})B(r_0)}$$

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