

Potts model on infinite graphs and the limit of chromatic polynomials

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Abstract

Given an infinite graph $\mathbb{G}$ quasi-transitive and amenable with maximum degree Δ , we show that reduced ground state degeneracy per site $W_r(\mathbb{G}, q)$ of the q -state antiferromagnetic Potts model at zero temperature on $\mathbb{G}$ is analytic in the variable $1/q$, whenever $|2\Delta e^3/q| < 1$. This result proves, in an even stronger formulation, a conjecture originally sketched in [12] and explicitly formulated in [16] and [19], based on which a sufficient condition for $W_r(\mathbb{G}, q)$ to be analytic at $1/q = 0$ is that $\mathbb{G}$ is a regular lattice.

Keywords: Potts model, chromatic polynomials, cluster expansion

§1 Introduction

The Potts model with q states (or q “colors”) is a system of random variables (spins) σ_x sitting in the vertices $x \in \mathbb{V}$ of a locally finite graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ with vertex set $\mathbb{V}$ and edge set $\mathbb{E}$, and taking values in the set of integers $\{1, 2, \dots, q\}$. Usually the graph $\mathbb{G}$ is a regular lattice, such as $\mathbb{Z}^d$ with the set of edges $\mathbb{E}$ being the set on nearest neighbor pairs, but of course more general situations can be considered. A *configuration* $\sigma_{\mathbb{V}}$ of the system is a function $\sigma_{\mathbb{V}} : \mathbb{V} \rightarrow \{1, 2, \dots, q\}$ with σ_x representing the value of the *spin* at the site x . We denote by $\Gamma_{\mathbb{V}}$ the set of all spin configurations in $\mathbb{V}$. If $V \subset \mathbb{V}$ we denote σ_V the restriction of $\sigma_{\mathbb{V}}$ to V and by Γ_V the set of all spin configurations in V .

Let $V \subset \mathbb{V}$ and let $\mathbb{G}|_V = (V, \mathbb{E}_V)$, where $\mathbb{E}|_V = \{\{x, y\} \in \mathbb{E} : x \in V, y \in V\}$. Then for $V \subset \mathbb{V}$ finite, the *energy of the spin configuration σ_V in $\mathbb{G}|_V$* is defined as

$$H_{\mathbb{G}|_V}(\sigma_V) = -J \sum_{\{x, y\} \subset \mathbb{E}|_V} \delta_{\sigma_x \sigma_y} \quad (1.1)$$

where $\delta_{\sigma_x \sigma_y}$ is the Kronecker symbol which is equal to one when $\sigma_x = \sigma_y$ and zero otherwise. The coupling J is called *ferromagnetic* if $J > 0$ and *anti-ferromagnetic* if $J < 0$.

The *statistical mechanics* of the system can be done by introducing the *Boltzmann weight* of a configuration σ_V , defined as $\exp\{-\beta H_{\mathbb{G}|_V}(\sigma_V)\}$ where $\beta \geq 0$ is the inverse temperature. Then the *probability* to find the system in the configuration σ_V is given by

$$\text{Prob}(\sigma_V) = \frac{e^{-\beta H_{\mathbb{G}|_V}(\sigma_V)}}{Z_{\mathbb{G}|_V}(q)} \quad (1.2)$$

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The normalization constant in the denominator is called the *partition function* and is given by

$$Z_{\mathbb{G}|V}(q, \beta) = \sum_{\sigma_V \in \Gamma_V} e^{-\beta H_{\mathbb{G}|V}(\sigma_V)} \quad (1.3)$$

The case $\beta J = -\infty$ is the *anti-ferromagnetic* and *zero temperature* Potts model with q -states. In this case configurations with non zero probability are only those in which adjacent spins have different values (or colors) and $Z_{\mathbb{G}|V}(q)$ becomes simply the number of all allowed configurations.

The *thermodynamics* of the system at inverse temperature β and “volume” V is recovered through the *free energy per unit volume* given by

$$f_{\mathbb{G}|V}(q, \beta) = \frac{1}{|V|} \ln Z_{\mathbb{G}|V}(q) \quad (1.4)$$

where $|V|$ denotes the cardinality of V . All thermodynamic functions of the system can be obtained via derivative of the free energy. In the zero temperature anti-ferromagnetic case the function $f_{\mathbb{G}|V}(q, \beta)$ is usually called *the ground state entropy* of the system.

The Potts model, despite its simple formulation, is a intensely investigated subject. Besides its own interest as a statistical mechanics model, it has deep connections with several areas in theoretical physics, probability and combinatorics.

In particular, Potts models on general graphs are strictly related to a typical combinatorial problem. As a matter of fact, the partition function of the Potts model with q state on a finite graph G , is equal, in the zero temperature antiferromagnetic case, to the number of proper coloring with q colors of the graph G , where proper coloring means that adjacent vertices of the graph must have different colors. This number viewed as a function of the number of colors q is actually a polynomial function in the variable q which is known as *chromatic polynomial*. On the other hand, the same partition function in the general case can be related to more general chromatic type polynomials, known as *Tutte polynomials* [21]. This beautiful connection between statistical mechanics and graph coloring problems, first discussed by Fortuin and Kasteleyn [9], has been extensively studied and continues to attract many researchers till nowadays (see e.g. [1], [8], [15], [18], [19], [20], [22] and reference therein).

One of the main interests in statistical physics is to establish whether or not a given system exhibits *phase transitions*. This means to search for points in the interval $\beta \in [0, \infty]$ where some thermodynamic function (like e.g. the free energy defined above) is non analytic. Now, functions as (1.3) and (1.4) are manifestly analytic as long as V is a finite set. Hence phase transition (i.e. non-analyticity) can arise only in the so called *infinite volume limit* or *thermodynamic limit*. That is, the graph $\mathbb{G}$ is some countably infinite graph, usually a regular lattice, and the infinite volume limit

$$f_{\mathbb{G}}(q, \beta) = \lim_{N \rightarrow \infty} \frac{1}{|V_N|} \ln Z_{\mathbb{G}|V_N}(q, \beta) \quad (1.5)$$

is taken along a sequence V_N of finite subsets of $\mathbb{V}$ such that, roughly speaking, $\mathbb{G}|_{V_N}$ increase in size equally in all directions. Typically, when $\mathbb{V}$ is $\mathbb{Z}^d$, V_N are usually cubes of increasing size L_N . There is a considerable amount of rigorous results about thermodynamic limit and phase transitions for the Potts model on $\mathbb{Z}^d$ and other regular lattices, see e.g. the reviews [23] and, more recently, [20].

On the other hand, the study of thermodynamic limits of spin systems on infinite graphs which are not usual lattices has recently drew the attention of many researchers (e.g. [3], [10], [11], [13] and references therein).

Concerning specifically the antiferromagnetic Potts model and/or chromatic polynomial on graphs, Sokal [19] has shown recently that for any *finite graph* G with maximum degree Δ , the zeros of the chromatic polynomial lie in a disk $q \leq C\Delta$, and he optimally estimated such constant, using the Dobrushin inductive approach in the cluster expansion framework and numerical methods, giving the value $C = 7.963907$. An important extension of this result would be to prove the existence and analyticity of the limiting free energy per unit volume (1.5) for a suitable class, as wide as possible, of infinite graphs. Such a generalization would be relevant from the statistical mechanics point of view, since it would imply that anti-ferromagnetic Potts model on such class of infinite graphs, if q is sufficiently large, does not present a phase transition at zero temperature (and hence at any temperature).

To this respect in various papers, see e.g. [12], [16] and [19] for a more explicit mathematical formulation, has been conjectured that the ground state entropy per unit volume of the antiferromagnetic Potts model at zero temperature on an infinite graph $\mathbb{G}$ should be analytic in the neighbor of $1/q = 0$ whenever $\mathbb{G}$ is a regular lattice.

In this paper we actually prove that this conjecture is true not only for regular lattices, but even for a wide class of graphs. In particular we prove that the ground state zero entropy is analytic near $1/q = 0$ for all infinite graphs which are quasi transitive and amenable, and the limit may be evaluated along *any* Følner sequence in $\mathbb{V}$. We stress that this result proves the conjecture above in a considerably stronger formulation, since *all regular lattices*, either with the elementary cell made by one single vertex or by more than one vertex, are indeed quasi-transitive amenable graphs but actually the class of quasi-transitive amenable graphs is much wider than that of regular lattices.

In order to obtain this result we perform a traditional polymer expansion. In the proof of analyticity we use quite rough analytical bounds to improve readability. Therefore our value of the constant appearing in the estimate of the convergence radius is not optimized as in [19]. However it should be possible to enlarge our radius of convergence with the same numerical techniques used in [19]. The constant could be greater than the quoted value since in order to prove the existence of the thermodynamic limit we have to show the absolute convergence not only of the pressure of the polymer gas, but also of more general quantities (see (3.12) and Lemma 4 below).

We would also like to stress that the polymer expansion introduced here can be used without changes to prove the same result for finite temperature. We restrict ourselves to the zero temperature case since we were interested in the proof of the conjecture cited above.

The paper is organized as follows. In section 2 we introduce the notations used along the paper, and we state the main result (theorem 2). In section 3 we rephrase the problem in term of polymer expansion and prove a main technical result (lemma 4). In section 4 we prove a graph theory property (lemma 6) concerning quasi-transitive amenable graphs. Finally in section 5 we give the proof of the main result of the paper, i.e. theorem 2.

§2. Some further notations and statement of the main result

In general, if V is any finite set, we denote by $|V|$ the number of elements of V . The set $\{1, 2, \dots, n\}$ will be denoted shortly I_n . We denote by $P_2(V)$ the set of all subsets $U \subset V$ such that $|U| = 2$ and by $P_{\geq 2}(V)$ the set of all *finite* subsets $U \subset V$ such that $|U| \geq 2$.

Given a countable set V , and given $E \subset P_2(V)$, the pair $G = (V, E)$ is called a *graph* in V . The elements of V are called *vertices* of G and the elements of E are called *edges* of G . Given two graphs $G = (V, E)$ and $G' = (V', E')$ in V , we say that $G' \subset G$ if $E' \subset E$ and $V' \subset V$.

Given a graph $G = (V, E)$, two vertices x and y in V are said to be *adjacent* if $\{x, y\} \in E$. The *degree* d_x of a vertex $x \in V$ in G is the number of vertices y adjacent to x . A graph $G = (V, E)$ is said *locally finite* if $d_x < +\infty$ for all $x \in V$, and it is said *bounded degree* if $\max_{x \in V} \{d_x\} \leq \Delta < \infty$. A graph $G = (V, E)$ is said to be *connected* if for any pair B, C of subsets of V such that $B \cup C = V$ and $B \cap C = \emptyset$, there is an edge $e \in E$ such that $e \cap B \neq \emptyset$ and $e \cap C \neq \emptyset$.

We denote by $\mathcal{G}_V$ the set of all connected graphs with vertex set V . If $V = I_n$ we use the notation $\mathcal{G}_n$ in place of $\mathcal{G}_{I_n}$. A *tree* graph τ on V is a connected graph $\tau \in \mathcal{G}_V$ such that $|\tau| = |V| - 1$. We denote by $\mathcal{T}_V$ the set of all tree graphs of V and shortly $\mathcal{T}_n$ in place of $\mathcal{T}_{I_n}$.

Let $\mathbf{R}_n \equiv (R_1, \dots, R_n)$ an ordered n -ple of non empty sets, then we denote by $E(\mathbf{R}_n)$ the set $\subset P_2(I_n)$ defined as $E(\mathbf{R}_n) = \{\{i, j\} \in P_2(I_n) : R_i \cap R_j \neq \emptyset\}$. We denote $G(\mathbf{R}_n)$ the graph $(I_n, E(\mathbf{R}_n))$.

Given two distinct vertices x and y of $G = (V, E)$, a *path* $\tau(x, y)$ joining x to y is a *tree* sub-graph of G with $d_x = d_y = 1$ and $d_z = 2$ for any vertex z in $\tau(x, y)$ distinct from x and y . We define the *distance* between x and y as $|x - y| = \min\{|\tau(x, y)| : \tau(x, y) \text{ path in } G\}$. Remark that $|x - y| = 1 \Leftrightarrow \{x, y\} \in E$.

Given $G = (V, E)$ connected and $R \subset V$, let $E|_R = \{\{x, y\} \in E : x \in R, y \in R\}$ and define the graph $G|_R = (R, E|_R)$. Note that $G|_R$ is a sub-graph of G . We call $G|_R$ *the restriction of G to R* . We say that $R \subset V$ is *connected* if $G|_R$ is connected. For any non void $R \subset V$, we further denote by ∂R the *external boundary* of R which is the subset of $V \setminus R$ given by

$$\partial R = \{y \in V \setminus R : \exists x \in R : |x - y| = 1\} \quad (2.1)$$

An *automorphism* of a graph $G = (V, E)$ is a bijective map $\gamma : V \rightarrow V$ such that $\{x, y\} \in E \Rightarrow \{\gamma x, \gamma y\} \in E$.

A graph $G = (V, E)$ is called *transitive* if, for any x, y in V , it exists an automorphism γ on G which maps x to y . The graph G is called *quasi-transitive* if V can be partitioned in finitely many sets $O_1, \dots, O_s$ (vertex orbits) such that for $\{x, y\} \in O_i$ it exists an automorphism γ on G which maps x to y and this holds for all $i = 1, \dots, s$. If $x \in O_i$ and $y \in O_i$ we say that x and y are equivalent. Remark that a locally finite quasi-transitive graph is necessarily bounded degree.

Roughly speaking in a transitive graph any vertex of the graph is equivalent; in other words G “looks the same” by observers sitting in different vertices. In a quasi-transitive graph there is a finite number of different type of vertices and G “looks the same” by observers sitting in vertices of the same type.

As a immediate example all periodic lattices with the elementary cell made by one site (e.g. square lattice, triangular lattice, hexagonal lattice, etc.) are transitive infinite graphs, while periodic lattices with the elementary cell made by more than one site are quasi-transitive infinite graphs.

Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a connected infinite graph. $\mathbb{G}$ is said to be *amenable* if

$$\inf \left\{ \frac{|\partial W|}{|W|} : W \subset \mathbb{V}, 0 < |W| < +\infty \right\} = 0$$

A sequence $\{V_N\}_{N \in \mathbb{N}}$ of finite sub-sets of $\mathbb{V}$ is called a *Følner sequence* if

$$\lim_{N \rightarrow \infty} \frac{|\partial V_N|}{|V_N|} = 0 \quad (2.2)$$

Note that such definition reminds the notion of Van Hove sequence in statistical mechanics. From now on $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ will denote a connected locally finite infinite graph and $V_N \subset \mathbb{V}$ a *finite* subset.

The partition function of the *antiferromagnetic* Potts model with q colours on $\mathbb{G}|_{V_N}$ at zero temperature can be rewritten (in a slightly different notation respect (1.1)) as

$$Z_{\mathbb{G}|_{V_N}}(q) = \sum_{\sigma_{V_N}} \exp \left\{ - \sum_{\{x,y\} \in P_2(V_N)} J_{xy} \delta_{\sigma_x \sigma_y} \right\} \quad (2.3)$$

where

$$J_{xy} = \begin{cases} +\infty & \text{if } |x - y| = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.4)$$

and when in (2.3) it appears a product $0 \cdot \infty$ it must be interpreted as zero. We stress again that, due to assumption (2.4) (i.e. antiferromagnetic interaction + zero temperature), the function $Z_{\mathbb{G}|_{V_N}}(q)$ represents the number of ways that the vertices $x \in V_N$ of $\mathbb{G}|_{V_N}$ can be assigned “colors” from the set $\{1, 2, \dots, q\}$ in such way that adjacent vertices always receive different colors. We also recall that the function $Z_{\mathbb{G}|_{V_N}}(q)$ is called, in the graph theory language, the *chromatic polynomial* of $\mathbb{G}|_{V_N}$.

Definition 1. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ connected and locally finite infinite graph and let $\{V_N\}_{N \in \mathbb{N}}$ be a Følner sequence of subsets of $\mathbb{V}$. Then we define, if it exists, the ground state specific entropy of the antiferromagnetic Potts model at zero temperature on $\mathbb{G}$ as

$$S_{\mathbb{G}}(q) = \lim_{N \rightarrow \infty} \frac{1}{|V_N|} \ln Z_{\mathbb{G}|_{V_N}}(q) \quad (2.5)$$

We also define the reduced ground state degeneracy per site as

$$W_r(\mathbb{G}, q) = \frac{1}{q} \lim_{N \rightarrow \infty} \left[Z_{\mathbb{G}|_{V_N}}(q) \right]^{\frac{1}{|V_N|}} \quad (2.6)$$

The ground state specific entropy $S_{\mathbb{G}}(q)$ and the reduced ground state degeneracy $W_r(\mathbb{G}, q)$ are directly related by the identity

$$S_{\mathbb{G}}(q) = \ln W_r(\mathbb{G}, q) + \ln q \quad (2.7)$$

We can now state our main result.

Theorem 2. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ a locally finite connected quasi-transitive amenable infinite graph with maximum degree Δ , and let $\{V_N\}_{N \in \mathbb{N}}$ a Følner sequence in $\mathbb{G}$. Then, $W_r(\mathbb{G}, q)$ exists, is finite, is independent on the choice of the sequence $\{V_N\}_{N \in \mathbb{N}}$, and is analytic in the variable $1/q$ whenever $|1/q| < 1/2e^3\Delta$ (e being the basis of natural logarithm).

Again we stress that this result proves the Schrock conjecture in a considerably stronger formulation, since any regular lattice is a quasi-transitive amenable graph but the class of quasi-transitive amenable graphs is actually much wider than that of regular lattices.

§3. Polymer expansion and analyticity

We first rewrite the partition function of the Potts model on a generic *finite* graph $G = (V, E)$ as a hard core Polymer gas grand canonical partition function. This is a standard procedure (see e.g. [19]). Without loss in generality, we will assume in this section that G is a sub-graph of a bounded degree infinite graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ with maximum degree Δ . Denote by $\pi(V)$ the set of all unordered partitions of V , i.e. an element of $\pi(V)$ is an unordered n -ple $\{R_1, R_2, \dots, R_n\}$, with $1 \leq n \leq |V|$, such that, for $i, j \in \mathbb{I}_n$, $R_i \subset V$, $R_i \neq \emptyset$, $R_i \cap R_j = \emptyset$, and $\cup_{i=1}^n R_i = V$. Then, by writing the factor $\exp\{-\sum_{\{x,y\} \subset V} \delta_{\sigma_x \sigma_y} J_{xy}\}$ in (2.3) as $\prod_{\{x,y\} \subset V} [(\exp\{-\delta_{\sigma_x \sigma_y} J_{xy}\} - 1) + 1]$ and developing the product (a standard Mayer expansion procedure, see e.g [7] or [6] and references therein) we can rewrite the partition function on G (2.3) as

$$Z_G(q) = q^{|V|} \Xi_G(q) \quad (3.1)$$

where

$$\Xi_G(q) = \sum_{n \geq 1} \sum_{\{R_1, \dots, R_n\} \in \pi(V)} \rho(R_1) \dots \rho(R_n) \quad (3.2)$$

with

$$\rho(R) = \begin{cases} 1 & \text{if } |R| = 1 \\ q^{-|R|} \sum_{\sigma_R \in \Gamma_R} \sum_{\substack{E' \subset P_2(R) \\ (R, E') \in \mathcal{G}_R}} \prod_{\{x,y\} \in E'} [e^{-\delta_{\sigma_x \sigma_y} J_{xy}} - 1] & \text{if } |R| \geq 2 \text{ and } \mathbb{G}|_R \in \mathcal{G}_R \\ 0 & \text{if } |R| \geq 2 \text{ and } \mathbb{G}|_R \notin \mathcal{G}_R \end{cases} \quad (3.3)$$

Observe that the sum in l.h.s. of (3.3) runs over all possible connected graphs with vertex set R . The r.h.s. of (3.2) can be written in a more compact way, by using the short notations

$$\mathbf{R}_n \equiv (R_1, \dots, R_n) \quad ; \quad \rho(\mathbf{R}_n) \equiv \rho(R_1) \dots \rho(R_n)$$

as

$$\Xi_G(q) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [P_{\geq 2}(V)]^n \\ R_i \cap R_j = \emptyset \quad \forall \{i,j\} \subset \mathbb{I}_n}} \rho(\mathbf{R}_n) \quad (3.4)$$

where $[P_{\geq 2}(V)]^n$ denote the n -times Cartesian product of $P_{\geq 2}(V)$ (which, we recall, denotes the set of all finite subsets of V with cardinality greater than 2).

It is also convenient to simplify the expression for the activity (3.3) by performing the sum over σ_R . As a matter of fact

$$\begin{aligned} q^{-|R|} \sum_{\sigma_R \in \Gamma_R} \sum_{\substack{E' \subset P_2(R) \\ (R, E') \in \mathcal{G}_R}} \prod_{\{x,y\} \in E'} [e^{-\delta_{\sigma_x \sigma_y} J_{xy}} - 1] &= q^{-|R|} \sum_{\sigma_R \in \Gamma_R} \sum_{\substack{E' \subset P_2(R) \\ (R, E') \in \mathcal{G}_R}} \prod_{\{x,y\} \in E'} \delta_{\sigma_x \sigma_y} [e^{-J_{xy}} - 1] = \\ &= q^{-|R|} \sum_{\substack{E' \subset P_2(R) \\ (R, E') \in \mathcal{G}_R}} \left[\sum_{\sigma_R \in \Gamma_R} \prod_{\{x,y\} \in E'} \delta_{\sigma_x \sigma_y} \right] \prod_{\{x,y\} \in E'} [e^{-J_{xy}} - 1] \end{aligned}$$

But now, for any connected graph $(R, E') \in \mathcal{G}_R$

$$\sum_{\sigma_R \in \Gamma_R} \prod_{\{x,y\} \in E'} \delta_{\sigma_x \sigma_y} = q$$

Hence we get, for $|R| > 1$

$$\rho(R) = \begin{cases} q^{-(|R|-1)} \sum_{\substack{E' \subset \mathbb{P}_2(R) \\ (R, E') \in \mathcal{G}_R}} \prod_{\{x, y\} \in E'} [e^{-J_{xy}} - 1] & \text{if } \mathbb{G}|_R \in \mathcal{G}_R \\ 0 & \text{otherwise} \end{cases} \quad (3.5)$$

By definitions (3.5) or (3.3), the polymer activity $\rho(R)$ can be viewed as a real valued function defined on any finite subset R of $\mathbb{V}$. Of course this function depends on the “topological structure” of $\mathbb{G}$. We remark that if γ is an automorphism of $\mathbb{G}$, then (3.5) clearly implies that $\rho(\gamma R) = \rho(R)$. In other words the activity $\rho(R)$ is invariant under automorphism of $\mathbb{G}$.

The function $\Xi_G(q)$ is the standard grand canonical partition function of an *hard core polymer gas* in which the polymers are finite subsets $R \subset V$ with cardinality greater than 2, with *activity* $\rho(R)$, and submitted to an *hard core* condition ($R_i \cap R_j = \emptyset$ for any pair $\{i, j\} \in \mathbb{I}_n$).

Note that by (3.1) and definitions (2.6)-(2.7) we have

$$W_r(\mathbb{G}, q) = \exp \left\{ \lim_{N \rightarrow \infty} \frac{1}{|V_N|} \ln \Xi_{\mathbb{G}|_{V_N}}(q) \right\} \quad (3.6)$$

It is a well known fact in statistical mechanics that the natural logarithm of Ξ_G can be rewritten as formal series, called the *Mayer series* (see e.g. [7]) as

$$\ln \Xi_G(q) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(V)]^n} \phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n) \quad (3.7)$$

where

$$\phi^T(\mathbf{R}_n) = \begin{cases} \sum_{\substack{E' \subset E(\mathbf{R}_n) \\ (\mathbb{I}_n, E') \in \mathcal{G}_n}} \prod_{\{i, j\} \in E'} (-1)^{|E'|} & \text{if } G(\mathbf{R}_n) \in \mathcal{G}_n \\ 0 & \text{otherwise} \end{cases} \quad (3.8)$$

and $G(\mathbf{R}_n) \equiv G(R_1, \dots, R_n)$ defined at the beginning of section 2. The reader should note that the summation in the l.h.s. of (3.4) is actually a *finite sum*. On the contrary, the summation in the r.h.s. of (3.7) is an *infinite series*.

We conclude this section showing two important technical lemmas concerning precisely the convergence of the series (3.7). In the proof of both lemmas we will use a well known combinatorial inequality due to Rota [14], which states that if $G = (V, E)$ is a connected graph, i.e. $G \in \mathcal{G}_V$, then

$$\left| \sum_{\substack{E' \subset E: \\ (V, E') \in \mathcal{G}_V}} (-1)^{|E'|} \right| \leq \sum_{\substack{E' \subset E: \\ (V, E') \in \mathcal{T}_V}} 1 = N_{\mathcal{T}_V}[G] \quad (3.9)$$

where $N_{\mathcal{T}_V}[G]$ is the number of tree graphs with vertex set V which are sub-graphs of G .

We want to outline that a similar inequality can be used to treat the finite temperature case. Actually in the latter case the quantity -1 in the l.h.s. of (3.9) is replaced by $e^{-\beta J} - 1$ which is, in the antiferromagnetic case, a quantity between 0 and 1. Then one can use, instead of the Rota inequality, the more general Penrose inequality (see e.g. [6] and [19]) and get a bound similar to (3.10) with $e\Delta/q$ replaced by $e(e^{-\beta J} - 1)\Delta/q$.

Lemma 3. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ a bounded degree infinite graph with maximum degree Δ , and let, for any $R \in \mathbb{V}$ such that $|R| \geq 2$, the activity $\rho(R)$ be given as in (3.5). Then, for any $n \geq 2$

$$\sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} |\rho(R)| \leq \left[\frac{e\Delta}{|q|} \right]^{n-1} \quad (3.10)$$

Proof. By definition

$$\sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} |\rho(R)| = |q|^{-(n-1)} \sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n, \mathbb{G}|_R \in \mathcal{G}_R}} \left| \sum_{\substack{E' \subset P_2(R) \\ (R, E') \in \mathcal{G}_R}} \prod_{\{x, y\} \in E'} [e^{-J_{xy}} - 1] \right| \quad (3.11)$$

Using thus the Rota inequality (3.9), recalling that $\mathbb{E}|_R = \{\{x, y\} \in \mathbb{E} : x \in R, y \in R\}$, and observing that $e^{-J_{xy}} - 1 = -1$ if $|x - y| = 1$ and $e^{-J_{xy}} - 1 = 0$ otherwise, we get

$$\left| \sum_{\substack{E' \subset P_2(R) \\ (R, E') \in \mathcal{G}_R}} \prod_{\{x, y\} \in E'} [e^{-J_{xy}} - 1] \right| = \left| \sum_{\substack{E' \subset \mathbb{E}|_R \\ (R, E') \in \mathcal{G}_R}} (-1)^{|E'|} \right| \leq \sum_{\substack{E' \subset \mathbb{E}|_R \\ (R, E') \in \mathcal{T}_R}} 1 = \sum_{\substack{E' \subset P_2(R) \\ (R, E') \in \mathcal{T}_R}} \prod_{\{x, y\} \in E'} \delta_{|x-y|=1}$$

where $\delta_{|x-y|=1} = 1$ if $|x - y| = 1$ and $\delta_{|x-y|=1} = 0$ otherwise. Hence

$$\begin{aligned} \sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n, \mathbb{G}|_R \in \mathcal{G}_R}} |\rho(R)| &\leq |q|^{-(n-1)} \sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} \sum_{\substack{E' \subset P_2(R) \\ (R, E') \in \mathcal{T}_R}} \prod_{\{x, y\} \in E'} \delta_{|x-y|=1} \leq \\ &\leq \frac{|q|^{-(n-1)}}{(n-1)!} \sum_{\substack{E' \subset P_2(I_n) \\ (I_n, E') \in \mathcal{T}_n}} \left[\sup_{x \in \mathbb{V}} \sum_{\substack{x_1=x, (x_2, \dots, x_n) \in \mathbb{V}^{n-1} \\ x_i \neq x_j \ \forall \{i, j\} \in I_n}} \prod_{\{i, j\} \in E'} \delta_{|x_i-x_j|=1} \right] \end{aligned}$$

It is now easy to check that, for any $E' \subset P_2(I_n)$ such that (I_n, E') is a tree, it holds

$$\sup_{x \in \mathbb{V}} \sum_{\substack{x_1=x, (x_2, \dots, x_n) \in \mathbb{V}^{n-1} \\ x_i \neq x_j \ \forall \{i, j\} \in I_n}} \prod_{\{i, j\} \in E'} \delta_{|x_i-x_j|=1} \leq \frac{\Delta^{n-1}}{(n-1)!}$$

and since, by Cayley formula, $\sum_{\substack{E' \subset P_2(I_n) \\ (I_n, E') \in \mathcal{T}_n}} 1 = n^{n-2}$, we get

$$\sup_{x \in \mathbb{V}} \sum_{\substack{R \subset \mathbb{V}: x \in R \\ |R|=n}} |\rho(R)| \leq \left(\frac{\Delta}{|q|} \right)^{n-1} \frac{n^{n-2}}{(n-1)!} \leq \left[\frac{e\Delta}{|q|} \right]^{n-1}$$

□

To state the second lemma we need to introduce a formal series more general than the r.h.s. of (3.7). Let thus $U \subset \mathbb{V}$ finite and let m a positive integer. We define

$$\mathcal{S}_U^m(\mathbb{G}, q) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ |\mathbf{R}_n| \geq m, R_1 \cap U \neq \emptyset}} \phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n) \quad (3.12)$$

where $|\mathbf{R}_n| = \sum_{i=1}^n |R_i|$ and recall that $\mathbb{P}_{\geq 2}(\mathbb{V})$ denotes the set of all finite subsets of $\mathbb{V}$ with cardinality greater or equal than 2 and $[\mathbb{P}_{\geq 2}(\mathbb{V})]^n$ denote the n -times Cartesian product. We will now prove the following:

Lemma 4. *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ a locally finite infinite graph with maximum degree Δ . Let $U \subset \mathbb{V}$ finite and let m a positive integer. Then $\mathcal{S}_U^m(\mathbb{G}, q)$ defined in (3.12) exists and is analytic as a function of $1/q$ in the disk $|2\Delta e^3/q| < 1$. Moreover it satisfies the following bound*

$$|\mathcal{S}_U^m(\mathbb{G}, q)| \leq |U| \frac{1}{1 - \sqrt{2e^3|\Delta/q|}} \left| 2e^3 \frac{\Delta}{q} \right|^{m/2}$$

Proof.

We will prove the theorem by showing directly that the r.h.s. of (3.12) converge absolutely when $|1/q|$ is sufficiently small. Let us define

$$|\mathcal{S}_U^m(\mathbb{G})| = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ |\mathbf{R}_n| \geq m, R_1 \cap U \neq \emptyset}} |\phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n)| \quad (3.13)$$

then $|\mathcal{S}_U^m(\mathbb{G})| \leq |\mathcal{S}_U^m(\mathbb{G})|$. We now bound $|\mathcal{S}_U^m(\mathbb{G})|$. We have:

$$|\mathcal{S}_U^m(\mathbb{G})| \leq \sum_{s=m}^{\infty} \sum_{n=1}^{\lfloor s/2 \rfloor} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap U \neq \emptyset, |\mathbf{R}_n| = s}} |\phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n)| = \sum_{s=m}^{\infty} \sum_{n=1}^{\lfloor s/2 \rfloor} \frac{1}{n!} \sum_{\substack{\mathbf{k}_n \in \mathbb{N}^n: k_i \geq 2 \\ k_1 + \dots + k_n = s}} B_n(\mathbf{k}_n)$$

where $\mathbf{k}_n \equiv (k_1, \dots, k_n)$, $\mathbb{N}^n$ denotes the n -times Cartesian product of $\mathbb{N}$, $\lfloor s/2 \rfloor = \max\{\ell \in \mathbb{N} : \ell \leq s/2\}$, and

$$B_n(\mathbf{k}_n) = \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap U \neq \emptyset \\ |R_1| = k_1, \dots, |R_n| = k_n}} |\phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n)|$$

recalling now (3.8) and using again the Rota bound (3.9) we get

$$|\phi^T(\mathbf{R}_n)| \begin{cases} \leq N_{\mathcal{T}_n}[G(\mathbf{R}_n)] & \text{if } G(\mathbf{R}_n) \in \mathcal{G}_n \\ = 0 & \text{otherwise} \end{cases}$$

Hence

$$B_n(\mathbf{k}_n) \leq \sum_{G \in \mathcal{G}_n} N_{\mathcal{T}_n}[G] \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap U \neq \emptyset, G(\mathbf{R}_n) = G \\ |R_1| = k_1, \dots, |R_n| = k_n}} |\rho(\mathbf{R}_n)| \quad (3.14)$$

Observing now that

$$\sum_{G \in \mathcal{G}_n} N_{\mathcal{T}_n}[G](\cdots) = \sum_{\tau \in \mathcal{T}_n} \sum_{G \in \mathcal{G}_n: G \supset \tau} (\cdots)$$

We can rewrite

$$B_n(\mathbf{k}_n) \leq \sum_{\tau \in \mathcal{T}_n} B_n(\tau, \mathbf{k}_n) \quad (3.15)$$

where

$$B_n(\tau, \mathbf{k}_n) = \sum_{\substack{\mathbf{R}_n \in [P_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap U \neq \emptyset, \quad G(\mathbf{R}_n) \supset \tau \\ |R_1| = k_1, \dots, |R_n| = k_n}} |\rho(\mathbf{R}_n)|$$

Note now that for any non negative function $F(R)$ it holds

$$\sum_{\substack{R \in \mathbb{V}: R \cap R' \neq \emptyset \\ |R| = k}} F(R) \leq |R'| \sup_{x \in \mathbb{V}} \sum_{\substack{R \in \mathbb{V} \\ x \in R, |R| = k}} F(R) \quad (3.16)$$

Hence we can now estimates $B_n(\tau, \mathbf{k}_n)$ for any fixed τ by explicitly perform the sum over polymers $\mathbf{R}_n$ submitted to the constraint that $g(\mathbf{R}_n) \supset \tau$, summing first over the “outermost polymers”, i.e. those polymers R_i such that i is a vertex of degree 1 in τ , and using repetively the bounds (3.16). Then one can easily check that

$$B_n(\tau, \mathbf{k}_n) \leq |U| \sup_{x \in \mathbb{V}} \sum_{\substack{R_1 \in \mathbb{V} \\ x \in R_1, |R_1| = k_1}} \cdots \sup_{x \in \mathbb{V}} \sum_{\substack{R_n \in \mathbb{V} \\ x \in R_n, |R_n| = k_n}} |\rho(R_1)| |R_1|^{d_1} \prod_{i=2}^k \left[|R_i|^{d_i-1} |\rho(R_i)| \right] \quad (3.17)$$

where d_i is the degree of the vertex i of τ . Recall that, for any tree $\tau \in \mathcal{T}_n$, it holds $1 \leq d_i \leq n-1$ and $d_1 + \dots + d_n = 2n-2$. Now, by lemma 3, (3.10), we can bound

$$B_n(\tau, \mathbf{k}_n) \leq |U| \varepsilon^{k_1-1} k_1^{d_1} \prod_{i=2}^k \left[k_i^{d_i-1} \varepsilon^{k_n-1} \right] \quad (3.18)$$

where we have put for simplicity $\varepsilon = e\Delta/|q|$. Noting that estimates in l.h.s. of (3.17) depends only on the degrees $d_1, \dots, d_n$ of the vertices in τ , we can now easily sum over all connected tree graphs in $\mathcal{T}_n$ and obtain

$$\begin{aligned} B_n(\mathbf{k}_n) &\leq \sum_{\tau \in \mathcal{T}_n} B_n(\tau, \mathbf{k}_n) = \sum_{\substack{r_1, \dots, r_n \\ r_1 + \dots + r_n = 2n-2 \\ 1 \leq r_i \leq n-1}} \sum_{\tau \in \mathcal{T}_n} B_n(\tau, \mathbf{k}_n) \leq \\ &\leq |U| \sum_{\substack{r_1, \dots, r_n \\ r_1 + \dots + r_n = 2n-2 \\ 1 \leq r_i \leq n-1}} (n-2)! k_1 \prod_{i=1}^n \left[\frac{k_i^{r_i-1}}{(r_i-1)!} \varepsilon^{k_i-1} \right] \end{aligned}$$

where in the second line we used the bound (3.17) and Cayley formula

$$\sum_{\substack{\tau \in \mathcal{T}_n \\ d_1, \dots, d_n \text{ fixed}}} 1 = \frac{(n-2)!}{\prod_{i=1}^n (d_i-1)!} \quad (3.19)$$

Now, recalling that $k_1 + \dots + k_n = s$ and using the Newton multinomial formula, we get

$$B_n(\mathbf{k}_n) \leq |U| k_1 s^{n-2} \varepsilon^{s-n} \leq |U| s^n \varepsilon^{s-n}$$

thus, since $\sum_{\substack{k_1, \dots, k_n: k_i \geq 2 \\ k_1 + \dots + k_n = s}} 1 \leq 2^{s-n}$, we obtain

$$|\mathcal{S}_U^m(\mathbb{G})| \leq |U| \sum_{s=m}^{\infty} \sum_{n=1}^{\lfloor s/2 \rfloor} \frac{s^n}{n!} \varepsilon^{s-n} \sum_{\substack{k_1, \dots, k_n: k_i \geq 2 \\ k_1 + \dots + k_n = s}} 1 \leq |U| \sum_{s=m}^{\infty} \sum_{n=1}^{\lfloor s/2 \rfloor} \frac{s^n}{n!} [2\varepsilon]^{s-n}$$

The series above converge if $\varepsilon < \frac{1}{2e}$ and we get the bound

$$\begin{aligned} |\mathcal{S}_U^m(\mathbb{G})| &\leq |U| \sum_{s=m}^{\infty} \sum_{n=1}^{\lfloor s/2 \rfloor} \frac{s^n}{n!} [2\varepsilon]^{s-n} \leq \sum_{s=m}^{\infty} [2\varepsilon]^{s-\lfloor s/2 \rfloor} \sum_{n=1}^{\infty} \frac{s^n}{n!} \leq \sum_{s=m}^{\infty} [2e^2 \varepsilon]^{s/2} \leq \\ &\leq |U| \frac{[2e^2 \varepsilon]^{m/2}}{1 - e\sqrt{2\varepsilon}} \end{aligned}$$

provided

$$2e^2 \varepsilon < 1$$

Hence, recalling that $\varepsilon = e\Delta/|q|$, the lemma is proved. $\square$

The following corollary is now a trivial consequence of the two lemmas above.

Corollary 5. *Let $G = (V, E)$ any finite connected sub-graph of an infinite connected bounded degree graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ with maximum degree Δ . Then the function $|V|^{-1} \log \Xi_G(q)$ is analytic in the variable $1/q$ for $|1/q| < 1/2e^3 \Delta$ and it admits the following bound uniformly in $|V|$:*

$$\left| \frac{1}{|V|} \log \Xi_G(q) \right| \leq \frac{1}{1 - \sqrt{2e^3 |\Delta/q|}} \left| 2e^3 \frac{\Delta}{q} \right| \quad (3.20)$$

Proof. For any $G = (V, E) \subset \mathbb{G} = (\mathbb{V}, \mathbb{E})$ with V finite, by definition (3.7) and (3.12), it holds that $|\ln \Xi_G(q)| \leq |\mathcal{S}_V^2(\mathbb{G}, q)|$ and one can thus apply lemma 4. $\square$

§4. A graph theory lemma

Lemma 6. *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a locally finite quasi-transitive infinite graph and let $\{V_N\}_{N \in \mathbb{N}}$ be a Følner sequence of finite subsets of $\mathbb{V}$. Then, for every vertex orbit $O \subset \mathbb{V}$ of $\text{Aut}(G)$, there exists a non-zero finite limit*

$$\lim_{N \rightarrow \infty} \frac{|O \cap V_N|}{|V_N|} \quad (4.1)$$

and it is independent on the choice of the sequence $\{V_N\}_{N \in \mathbb{N}}$.

Remark. A proof of this lemma can be found in [2] (Proposition 3.6). We give an alternative and self-contained proof for completeness

Proof. For a natural r and a finite set $F \subset \mathbb{V}$ denote by $B_r F$ the set

$$B_r(F) = \{x \in \mathbb{V} : \exists y \in F \ |x - y| \leq r\} \quad (4.2)$$

Thus, for a single-point set $\{y\}$, $B_r(\{y\})$ is the ball of radius r centered at y . Moreover we have the bound

$$|B_r(F)| \leq |F|(1 + \Delta + \dots + \Delta^r) \leq \Delta^{r+1}|F| \quad (4.3)$$

Let $O_1, \dots, O_s$ be the complete list of vertex orbits of $\text{Aut}(\mathbb{G})$ in the set $\mathbb{V}$ and let $A_0 \subset \mathbb{V}$ be a set with exactly one element in common with every orbit. Denote by d the diameter of A_0 . Consider the orbit $\mathcal{A} = \{gA_0 : g \in \text{Aut}(\mathbb{G})\}$ of A_0 . A set $A \subset \mathbb{V}$ is therefore an element of $\mathcal{A}$ if it exists a $g \in \text{Aut}(\mathbb{G})$ such that $A = gA_0$. For any set $U \subset V$ we denote $\mathcal{A}_U = \{A \in \mathcal{A} : A \subset U\}$. Note that for any set $A \in \mathcal{A}$ and any vertex orbit O , we have that $|A \cap O| = 1$, hence for a fixed vertex orbit O_i we can define the function φ_i as follows.

$$\varphi_i : \mathcal{A} \rightarrow O_i : A \mapsto A \cap O_i$$

The function φ_i is a surjection and for $x \in O_i$ the number $k_i = |\varphi_i^{-1}(x)|$ is finite and does not depend on the choice of $x \in O_i$. To see that k_i is finite simply observe that the automorphisms of $\mathbb{G}$ preserve the graph distance. Hence $\varphi_i^{-1}(x)$ is contained in the ball of radius d centered at x . Such ball has finite cardinality, bounded by Δ^{d+1} , and contains therefore a finite number of subsets. To see that $|\varphi_i^{-1}(x)|$ does not depend on x when x varies in O_i , observe that if $x, y \in O_i$, we can find $g \in \text{Aut}(\mathbb{G})$ such that $y = gx$. Then observe that $\varphi_i^{-1}(y) \supset g\varphi_i^{-1}(x)$ and thus $|\varphi_i^{-1}(y)| \geq |g\varphi_i^{-1}(x)|$. On the other side $\varphi_i^{-1}(x) \supset g^{-1}\varphi_i^{-1}(y)$. Hence $|\varphi_i^{-1}(x)| = |\varphi_i^{-1}(y)|$.

Define now the sets

$$V_N^- = V_N \setminus B_d(\partial V_N), \quad V_N^+ = V_N \cup B_d(\partial V_N) \quad (4.4)$$

By construction $V_N^- \subset V_N \subset V_N^+$ and

$$\varphi_i^{-1}(V_N^- \cap O_i) \subset \mathcal{A}_{V_N} \subset \varphi_i^{-1}(V_N \cap O_i) \subset \mathcal{A}_{V_N^+} \quad (4.5)$$

Indeed, suppose that $A \in \varphi_i^{-1}(V_N^- \cap O_i)$ and $A \not\subset V_N$. Let $a_1 = \varphi_i(A) \in V_N^- \subset V_N$ and let $a_2 \in A \setminus V_N$. There exists a path $\tau(a_1, a_2)$ in $\mathbb{G}$ of length $\leq d$. This chain must have at least one point in ∂V_N . This implies $a_1 \in B_d(\partial V_N)$ contradicting the assumption $a_1 \in V_N^-$. The second inclusion of (4.5) is obvious and the third one is true by the same reason as the first one. (4.5) implies

$$k_i |V_N^- \cap O_i| \leq |\mathcal{A}_{V_N}| \leq k_i |V_N \cap O_i| \leq |\mathcal{A}_{V_N^+}| \quad (4.6)$$

and

$$|V_N^- \cap O_i| \leq \frac{1}{k_i} |\mathcal{A}_{V_N}| \leq |V_N \cap O_i| \leq \frac{1}{k_i} |\mathcal{A}_{V_N^+}| \quad (4.7)$$

By taking sum over i we get

$$|V_N^-| \leq \alpha |\mathcal{A}_{V_N}| \leq |V_N| \leq \alpha |\mathcal{A}_{V_N^+}| \quad (4.8)$$

where $\alpha = \sum_{i=1}^s \frac{1}{k_i}$. On the other hand $\mathcal{A}_{V_N^+} \setminus \mathcal{A}_{V_N} \subset \mathcal{A}_{B_d(\partial V_N)}$ and, by (4.3),

$$|\mathcal{A}_{V_N^+}| \leq |\mathcal{A}_{V_N}| + |\mathcal{A}_{B_d(\partial V_N)}| \leq |\mathcal{A}_{V_N}| + k \Delta^{d+1} |\partial V_N| \quad (4.9)$$

where $k = \max\{k_i : i \in \{1, \dots, s\}\}$. From the first inequality of (4.8) we have

$$|\mathcal{A}_{V_N}| \geq \frac{1}{\alpha} |V_N^-| \geq \frac{1}{\alpha} (|V_N| - |B_d(\partial V_N)|) \geq \frac{1}{\alpha} (|V_N| - \Delta^{d+1} |\partial V_N|) \quad (4.10)$$

If $\frac{|\partial V_N|}{|V_N|} \leq \varepsilon$ then, by (4.9) and (4.10),

$$1 \leq \frac{|\mathcal{A}_{V_N^+}|}{|\mathcal{A}_{V_N}|} \leq 1 + \frac{\Delta^{d+1} |\partial V_N|}{\frac{1}{\alpha} (|V_N| - \Delta^{d+1} |\partial V_N|)} \leq 1 + \frac{\Delta^{d+1} \varepsilon}{\frac{1}{\alpha} (1 - \Delta^{d+1} \varepsilon)}$$

This proves that

$$\lim_{N \rightarrow \infty} \frac{|\mathcal{A}_{V_N^+}|}{|\mathcal{A}_{V_N}|} = 1 \quad (4.11)$$

By (4.8) and (4.10) we also have

$$\lim_{N \rightarrow \infty} \frac{|\mathcal{A}_{V_N}|}{|V_N|} = \frac{1}{\alpha} \quad (4.12)$$

Dividing (4.7) by $|V_N|$ and using (4.12) we obtain

$$\lim_{N \rightarrow \infty} \frac{|O_i \cap V_N|}{|V_N|} = \frac{1}{k_i \alpha}$$

and the lemma is proved. $\square$

§5. Potts model on infinite graphs: proof of theorem 2.

Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ infinite bounded degree and let $x \in \mathbb{V}$. Then we define

$$f_{\mathbb{G}}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ x \in R_1}} \phi^T(\mathbf{R}_n) \frac{\rho(\mathbf{R}_n)}{|R_1|} \quad (5.1)$$

We stress that, by construction, $f_{\mathbb{G}}(x)$ is invariant under automorphism. I.e. if $x \in \mathbb{V}$ and $y \in \mathbb{V}$ are equivalent (i.e. it exists γ automorphism of $\mathbb{G}$ such that $y = \gamma x$) then $f_{\mathbb{G}}(x) = f_{\mathbb{G}}(y)$.

Given now a *finite* set $V_N \subset \mathbb{V}$, we define

$$F(V_N) = \frac{1}{|V_N|} \sum_{x \in V_N} f_{\mathbb{G}}(x) \quad (5.2)$$

The numbers $F(V_N)$ are actually functions of q . As a trivial corollary of lemma 4 we can state the following

Lemma 7. *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ infinite bounded degree. Then for any $V_N \subset V$ finite, the functions $f_{\mathbb{G}}(x)$ and $F(V_N)$ defined in (5.1) and (5.2) are analytic in the variable $1/q$ for $|1/q| < 1/2e^3\Delta$ and bounded by $|2e^3\Delta/q| / (1 - \sqrt{2e^3|\Delta/q|})$ uniformly in N .*

Proof. Comparing l.h.s. of (3.12) with l.h.s. of (5.1) we have that $|f_{\mathbb{G}}(x)| \leq |\mathcal{S}|_{\{x\}}^2(\mathbb{G})$, hence one can again use lemma 4 and get immediately the proof. $\square$.

From lemma 6 and lemma 7 it follows:

Proposition 8. *Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a locally finite quasi-transitive infinite graph and let $\{V_N\}_{N \in \mathbb{N}}$ be a sequence of finite subsets of $\mathbb{V}$ such that $|\partial V_N|/|V_N| \rightarrow 0$ as $N \rightarrow \infty$. Let Δ be the maximum degree of $\mathbb{G}$, then the limits*

$$\lim_{N \rightarrow \infty} F(V_N) \doteq F_{\mathbb{G}}(q) \quad (5.3)$$

exists, is finite, is independent on the sequence $\{V_N\}_{N \in \mathbb{N}}$, and is analytic as a function of $1/q$ for $|1/q| < 1/2e^3\Delta$.

Proof. If the limit (5.3) exists, then by lemma 7 it is clearly bounded by $|2e^3\Delta/q|/(1 - \sqrt{2e^3|\Delta/q|})$ and it analytic in $1/q$ for $|1/q| < 1/2e^3\Delta$. To prove the existence of the limit (5.3) we proceed as follows.

Since $\mathbb{G}$ is quasi-transitive then $\mathbb{V}$ can be partitioned into orbits $O_1, \dots, O_s$ of $\text{Aut}(\mathbb{G})$ such that for two any vertices x, y in the same orbit O_i there is an automorphism of $\mathbb{G}$ which maps x to y . Hence for such a pair we have $f_{\mathbb{G}}(x) = f_{\mathbb{G}}(y)$ and we can conclude that $f_{\mathbb{G}}(x)$ has value in a finite set $\{f_1, \dots, f_s\}$ with $f_i = f_{\mathbb{G}}(x)$ where x is any vertex $x \in O_i$.

Thus for any finite connected V_N and any $j \in \{1, 2, \dots, s\}$ we have

$$\frac{1}{|V_N|} \sum_{x \in V_N} f_{\mathbb{G}}(x) = \left[\frac{|V_N \cap O_1|}{|V_N|} f_1 + \dots + \frac{|V_N \cap O_s|}{|V_N|} f_s \right]$$

hence

$$\lim_{N \rightarrow \infty} F(V_N) = f_1 \lim_{N \rightarrow \infty} \frac{|V_N \cap O_1|}{|V_N|} + \dots + f_s \lim_{N \rightarrow \infty} \frac{|V_N \cap O_s|}{|V_N|}$$

and by lemma 6 the limit above exists. $\square$

We are at last in the position to prove the main results of the paper, namely the theorem 2 enunciated at the end of section 2.

Proof of theorem 2. We will prove that $\lim_{N \rightarrow \infty} |V_N|^{-1} \log \Xi_{\mathbb{G}|V_N}(q) = F_{\mathbb{G}}(q)$ where $F_{\mathbb{G}}(q)$ is the function defined in (5.3) and then use definition (3.6).

$$\log \Xi_{\mathbb{G}|V_N} - \sum_{x \in V_N} f_{\mathbb{G}}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(V_N)]^n} \phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n) - \sum_{x \in V_N} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ x \in R_1}} \phi^T(\mathbf{R}_n) \frac{\rho(\mathbf{R}_n)}{|R_1|} \right]$$

Now note that

$$\sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ x \in R_1}} (\dots) = \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(V_N)]^n \\ x \in R_1}} (\dots) + \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ x \in R_1 \\ \exists R_i: R_i \not\subset V_N}} (\dots)$$

moreover

$$\sum_{x \in V_N} \sum_{\substack{R_1 \in V_N \\ x \in R_1}} (\dots) = \sum_{R_1 \in V_N} |R_1| (\dots) \quad , \quad \sum_{x \in V_N} \sum_{\substack{R_1 \in \mathbb{V} \\ x \in R_1}} (\dots) = \sum_{R_1 \in \mathbb{V}} |R_1 \cap V_N| (\dots)$$

hence, using also that $|R_1 \cap V_N|/|R_1| \leq 1$ we get

$$\left| \log \Xi_{\mathbb{G}|V_N} - \sum_{x \in V_N} f_{\mathbb{G}}(x) \right| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap V_N \neq \emptyset \\ \exists R_i: R_i \not\subset V_N}} |\phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n)|$$

Let now choose $p > \ln \Delta$ and define

$$m_N^p = \frac{1}{p} \ln \left[\frac{|V_N|}{|\partial V_N|} \right] \quad (5.4)$$

remark that, since by the hypothesis the sequence V_N is Følner and hence (2.2) holds, then $\lim_{N \rightarrow \infty} m_N^p = \infty$, for any integer p . We now can rewrite

$$\sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap V_N \neq \emptyset \\ \exists R_i: R_i \not\subset V_N}} (\dots) = \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap V_N \neq \emptyset, |\mathbf{R}_n| \geq m_N^p \\ \exists R_i: R_i \not\subset V_N,}} (\dots) + \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap V_N \neq \emptyset, |\mathbf{R}_n| < m_N^p \\ \exists R_i: R_i \not\subset V_N,}} (\dots)$$

Hence

$$\begin{aligned} \left| \log \Xi_{\mathbb{G}|V_N} - \sum_{x \in V_N} f_{\mathbb{G}}(x) \right| &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap V_N \neq \emptyset, |\mathbf{R}_n| \geq m_N^p \\ \exists R_i: R_i \not\subset V_N,}} |\phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n)| + \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap V_N \neq \emptyset, |\mathbf{R}_n| < m_N^p \\ \exists R_i: R_i \not\subset V_N,}} |\phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n)| \end{aligned} \quad (5.5)$$

but, concerning the first sum, recalling definition (3.13), we have

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}(\mathbb{V})]^n \\ R_1 \cap V_N \neq \emptyset, |\mathbf{R}_n| \geq m_N^p \\ \exists R_i: R_i \not\subset V_N,}} |\phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n)| \leq |\mathcal{S}|_{V_N}^{m_N^p}(\mathbb{G}, q) \leq \text{Const.} |V_N| \varepsilon^{m_N^p/2}$$

where $\varepsilon = q/2e^3\Delta < 1$ by hypothesis. which, divided by $|V_N|$, converge to zero as $N \rightarrow \infty$ because by hypothesis $m_N^p \rightarrow \infty$ as $N \rightarrow \infty$.

On the other hand, recalling that due to the factor $\phi^T(\mathbf{R}_n)$ the sets R_i must be pair-wise connected, we have that $|\cup_i R_i| < \sum_i |R_i|$. So, since $|\cup_i R_i| < m_N^p$ and at least one among R_i intersects ∂V_N , this means that all polymers R_i must lie in the set

$$B_{m_N^p}(\partial V_N) = \{x \in \mathbb{V} : \exists v \in \partial V_N : |x - v| \leq m_N^p\}$$

Recalling (4.2) we have

$$|B_{m_N^p}(\partial V_N)| \leq |\partial V_N| \Delta^{m_N^p+1}$$

Hence we have, again recalling (3.13), that second sum in l.h.s. of (5.5) is bounded by

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{\mathbf{R}_n \in [\mathbb{P}_{\geq 2}^{(\mathbb{V})}]^n \\ R_1 \cap V_N \neq \emptyset, |\mathbf{R}_n| < m_N^p \\ \exists R_i: R_i \not\subset V_N,}} |\phi^T(\mathbf{R}_n) \rho(\mathbf{R}_n)| &\leq |\mathcal{S}|_{\mathbf{B}_{m_N^p}(\partial V_N)}^2(\mathbb{G}, q) \leq \\
&\leq \text{Cost.} |\mathbf{B}_{m_N^p}(\partial V_N)| \varepsilon \leq \text{Const.} \Delta |\partial V_N| \Delta^{m_N^p} \varepsilon
\end{aligned}$$

Thus recalling definition (5.4), we have

$$\begin{aligned}
\left| \frac{1}{|V_N|} \log \Xi_{\mathbb{G}|V_N} - \frac{1}{|V_N|} \sum_{x \in V_N} f_{\mathbb{G}}(x) \right| &= \left| \frac{1}{|V_N|} \log \Xi_{\mathbb{G}|V_N} - F_{\mathbb{G}}(q) \right| \leq \\
&\leq \text{Const.} \left[\frac{|\partial V_N|}{|V_N|} \right]^{\frac{|\ln \varepsilon|}{p}} + \text{Const.} \varepsilon \left[\frac{|\partial V_N|}{|V_N|} \right]^{1 - \frac{\ln \Delta}{p}}
\end{aligned}$$

Since by hypothesis $|\partial V_N|/|V_N| \rightarrow 0$ as $N \rightarrow \infty$, we conclude that the quantity above is as small as we please for N large enough. This ends the proof of the theorem. $\square$

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