A Cluster Expansion for the Decay of Correlations of Light-Mass Quantum Crystals and Some Stochastic Models Under Intense Noise

Rafael S. Thebaldi[1], Emmanuel Pereira[1] and Aldo Procacci[2]

[1]Dep. Física-ICEx, UFMG, C.P. 702, 30.161-970 Belo Horizonte MG, Brazil
[2]Dep. Matemática - ICEx, UFMG, C.P. 702, 30.161-970 Belo Horizonte MG, Brazil

e-mail: rafaelst@fisica.ufmg.br; emmanuel@fisica.ufmg.br; aldo@mat.ufmg.br

Abstract

We analyze the two and four-point truncated function of a lattice system of unbounded continuous spin variables describing a large class of light-mass quantum anharmonic crystals and some stochastic Ginzburg-Landau type models under intense noise. We develop a cluster expansion and use it to obtain the decay of the two-point truncated function (which gives information about the one-particle excitations), for the interactions with finite range or with polynomial decay. Moreover, using a Bethe-Salpeter equation, we investigate the four-point truncated in a perturbative approach (whose reliability is supported by the convergence of the cluster expansion), and establish a condition for the existence of a two-particle bound state in the low-lying spectrum of the system.

Short Title: Cluster Expansion for Quantum Crystals and Stochastic Models

1 Introduction

In this paper we develop a cluster expansion and investigate the low-lying spectrum of some systems related to commonly used physical models, namely, to light-mass quantum crystals and to stochastic Ginzburg-Landau (GL) type models submitted to intense noise.

The interest of such problems is well known. The study of the basic properties of the quantum crystals, for example, is an old problem in physics. For these crystal models we mean a system of unbounded continuous spin variables $\varphi_i \in \mathbb{R}$, in a lattice space with Hamiltonian given by

$$H = \sum_{i\in\mathbb{Z}^d} -\frac{1}{2m}\frac{\partial^2}{\partial\varphi_i^2} + \frac{1}{2}\sum_{i,j\in\mathbb{Z}^d}\varphi_i J_{ij}\varphi_j + \sum_{i\in\mathbb{Z}^d} V(\varphi_i), \tag{1}$$

where $J_{i,j}$ is usually taken as a positive operator (e.g., the lattice Laplacian; i.e., in some recent rigorous works the pair interaction is restricted to the ferromagnetic case), and V is given, e.g., by $V(\varphi_i) = a\varphi_i^2 + b\varphi_i^4$. In the present paper, we consider more general cases: we do not restrict the quadratic interaction to ferromagnetic cases and neither the anharmonic potential to a local interaction. We give details later. Recently, several works have been devoted to a rigorous analysis of the small mass behavior of such lattice models (light-mass means that the system is strongly quantum). E.g., in [1], [2], the fact that small mass implies uniqueness of Gibbs states of a quantum crystal is rigorously established (there, for the ferromagnetic case and with local anharmonicity), an important result about the influence of quantum effects on structural phase transitions. See also [3] and references there in for more recent results. Effects of nonlocality in the anharmonicity on the spectral properties of quantum crystals (in the regime of large mass) are presented in [4]. In the present paper, besides other results, such as the uniqueness of a phase given by the convergence of the cluster expansion, we will establish conditions for the existence of a two-particle bound state in the excitation spectrum of these light-mass quantum crystals, showing that the system, although in a single phase region, may have nontrivial spectral excitations.

The stochastic GL type models are also extensively investigated in physics, in particular, in the study of dynamical critical phenomena. The spectrum of the generator of the dynamics gives information about, e.g., the relaxation rates of the correlation functions, and so, it is of direct physical interest. Recently, several works have been devoted to detailed investigations on the spectral properties of such systems. For the weakly coupled GL model, the existence of an isolated one-particle and two-particle bound states have been established perturbatively in [5], and rigorously in [6]. Effects on the spectrum due to changes in the noise strength are presented in [7], [8], [9], for the small noise regime, and in [10] for intense noise. In [11], the one-particle state is rigorously established for a particular case of these GL type models under intense noise.

In order to make clear the relation between the stochastic GL system and the anharmonic quantum crystal, we present below a brief technical description of the dynamics generator of the stochastic model.

Consider a system of unbounded continuous spin variables $\phi_{\vec{x},t} \in \mathbb{R}$, in a lattice space box $\vec{x} \in V \subset \mathbb{Z}^d$ (the thermodynamic limit is considered when $V \to \mathbb{Z}^d$), with time evolution (Langevin dynamics) given by

$$\frac{\partial}{\partial t}\phi_{\vec{x},t} = -\frac{1}{2}\nabla S(\phi_{\vec{x},t}) + \eta_{\vec{x},t}, \qquad \phi_{\vec{x},t=0} = \psi_{\vec{x}}, \tag{2}$$

where $\nabla S = \delta S/\delta\phi$, and the GL (spatial) interaction is

$$S(\phi) = \frac{1}{2}\sum_{\vec{x},\vec{y}\in V} J_{\vec{x},\vec{y}}\,\phi_{\vec{x}}\,\phi_{\vec{y}} + \lambda\sum_{\vec{x}\in V}\mathcal{P}(\phi_{\vec{x}}), \tag{3}$$

$\lambda > 0$, $\mathcal{P}$ is a polynomial bounded from below; η is a family of Gaussian white-noise processes with expectations

$$E(\eta_{\vec{x},t}) = 0, \qquad E(\eta_{\vec{x},t}\eta_{\vec{y},t'}) = \gamma\delta_{\vec{x},\vec{y}}\,\delta(t-t'), \tag{4}$$

$\vec{x} \in V$, $t \in [0,\infty)$ (in principle; we extend the time interaction to $-t$ and introduce a time regularizer that makes t discrete later); $\gamma > 0$ is the noise strength: we will treat the regime $\gamma \gg 1$. $J_{\vec{x},\vec{y}}$ is an arbitrary summable spatial interaction, i.e., we assume that $J_{\vec{x},\vec{y}}$ is symmetric, translational invariant with

$$\sup_{\vec{x}\in\mathbb{Z}^d}\sum_{\vec{y}\in\mathbb{Z}^d}|J_{\vec{x},\vec{y}}| < \infty. \tag{5}$$

Note that the pair potential $J(\vec{x},\vec{y})$ does not have to be positive.

The time evolution of functions f of the spin configuration ϕ for such systems is defined by

$$f_t(\psi) = E(f(\phi_t)),$$

where $\psi = \phi_{t=0}$ is some initial condition. ¿From Itô calculus, it follows that

$$f_t(\psi) = e^{-t\mathcal{H}_V^0}f(\psi), \qquad \mathcal{H}_V^0 f = \left\{\sum_{\vec{x}\in V} -\frac{1}{2}\gamma\frac{\partial^2}{\partial\phi_{\vec{x}}^2} + \frac{1}{2}\frac{\partial S}{\partial\phi_{\vec{x}}}\frac{\partial}{\partial\phi_{\vec{x}}}\right\}f. \tag{6}$$

The generator $\mathcal{H}_V^0$ of the strongly continuous semi-group describing the Markov dynamics above is Hermitian and positive in the Hilbert space $L^2(d\sigma_V^0)$, where $d\sigma_V^0$ is the invariant measure given by the Gibbs probability distribution

$$d\sigma_V^0 = e^{-S(\phi)/\gamma} d\phi_V/normalization, \qquad \text{where } \; d\phi_V = \prod_{\vec{x}\in V} d\phi_{\vec{x}}. \tag{7}$$

The ground-state of $\mathcal{H}_V^0$ is the eigenfunction $f \equiv 1$ with eigenvalue zero. We note that $\mathcal{H}_V^0$ is unitarily equivalent to a Schrödinger operator with a potential going to infinity: defining the unitary operator U_V^0 from $L^2(d\sigma_V^0) \to L^2(d\phi_V)$ as

$$(U_V^0 f)(\phi) = Z_V^{-1/2} e^{-\frac{1}{2\gamma}S} f(\phi),$$

where Z_V is the normalization in $d\sigma_V^0$, a straightforward calculation gives us

$$H_V^0 = U_V^0 \mathcal{H}_V^0 (U_V^0)^{-1} = \sum_{\vec{x}\in V} -\frac{1}{2}\gamma \frac{\partial^2}{\partial\phi_{\vec{x}}^2} + \frac{1}{4}\sum_{\vec{x}\in V}\left[\frac{1}{2\gamma}\left(\frac{\partial S}{\partial\phi_{\vec{x}}}\right)^2 - \frac{\partial^2 S}{\partial \phi_{\vec{x}}^2}\right]. \tag{8}$$

Performing the derivatives, we get

$$\begin{aligned} H_V^0 \;\; = \;\; & -\frac{1}{2}\gamma \sum_{\vec{x}\in V} \frac{\partial^2}{\partial\phi_{\vec{x}}^2} + \frac{1}{8\gamma}\sum_{\vec{x},\vec{y}\in V} [J*J]_{\vec{x},\vec{y}}\, \phi_{\vec{x}}\, \phi_{\vec{y}} \; + \\ & + \frac{\lambda}{4\gamma}\sum_{\vec{x},\vec{y}\in V} J_{\vec{x},\vec{y}}\, \mathcal{P}'(\phi_{\vec{x}})\, \phi_{\vec{y}} + \sum_{\vec{x}\in V}\left[\frac{\lambda^2}{8\gamma}\mathcal{P}'(\phi_{\vec{x}})^2 - \frac{\lambda}{4}\mathcal{P}''(\phi_{\vec{x}}) + c\right], \end{aligned} \tag{9}$$

where c is a constant (depending of J and d), and $[J*J]_{\vec{x},\vec{y}} = \sum_{\vec{z}\in V} J_{\vec{x},\vec{z}} J_{\vec{z},\vec{y}}$ is the spatial convolution.

The relation between the spectral properties of the generator of the stochastic dynamics and the study of the excitation spectrum of the quantum anharmonic crystals becomes evident from the expression for H_V^0 above, as we have recalled: note that the large noise intensity γ above represents the inverse of small mass m of the particles for the anharmonic oscillators in (1); note also that the stochastic "potential" is more intricate.

It is useful to construct a Feynman-Kac integral formalism associated to H_V^0 (or to H (1)). It follows from standard procedures [12]. If $f_1, \ldots, f_n$ are functions of the spin configuration in V, $\Omega(\phi) = 1$ is the ground state of $\mathcal{H}_V^0$ and for $t_1 \le t_2 \le \ldots t_n \in \mathbb{R}$, now $t_i \in [-T, T]$, then we have

$$\begin{aligned} & (\Omega, f_1 e^{-(t_2-t_1)\mathcal{H}_V^0} f_2 \ldots e^{-(t_n-t_{n-1})\mathcal{H}_V^0} f_n \Omega)_{L^2(d\sigma_V^0)} = \\ = \;\; & (U_V^0\Omega, f_1 e^{-(t_2-t_1)H_V^0} f_2 \ldots e^{-(t_n-t_{n-1})H_V^0} f_n U_V^0\Omega)_{L^2(d\phi_V)} = \int f_1(\phi(t_1))\ldots f_n(\phi(t_n)) d\rho_V^0, \end{aligned} \tag{10}$$

where the path space measure $d\rho_V^0$ is the weak limit $T \to \infty$ of

$$d\rho_\Lambda^0 = \frac{e^{-W_\Lambda^0(\phi)} d\omega_\Lambda^0}{\int e^{-W_\Lambda^0(\phi)} d\omega_\Lambda^0}, \tag{11}$$

with $\Lambda \equiv ([-T, T], V)$ and

$$W_\Lambda^0(\phi) = \int_{-T}^{T} dt \;\; \left\{ \frac{\lambda}{4\gamma}\sum_{\vec{x},\vec{y}\in V} J_{\vec{x},\vec{y}}\, \mathcal{P}'(\phi_{\vec{x}})\, \phi_{\vec{y}} + \sum_{\vec{x}\in V}\left[\frac{\lambda^2}{8\gamma}\mathcal{P}'(\phi_{\vec{x}})^2 - \frac{\lambda}{4}\mathcal{P}''(\phi_{\vec{x}})\right]\right\}, \tag{12}$$

and $d\omega_\Lambda^0$ is a Gaussian measure with mean zero and variance given by

$$\gamma C_\Lambda\left(t,\vec{x};t',\vec{y}\right) \;=\; \int \phi_{t,\vec{x}}\phi_{t',\vec{y}}d\omega_\Lambda^0 \;=\; \frac{\gamma}{2\pi\,|V|}\int_{-\infty}^{\infty} dp_0 \sum_{\vec{p}\in\tilde{V}} \frac{e^{ip_0(t-t')}\;e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{p_0^2+[\tilde{J}(\vec{p})]^2/4}, \tag{13}$$

where $\tilde{J}(\vec{p})$ is the Fourier transform of $J_{\vec{x},\vec{y}}$; $|V|$ is the number of points in V; $\tilde{V}$ is the Fourier dual lattice; $\vec{p}=(p_1,\ldots,p_d)\in\tilde{V}$ and $\vec{p}\cdot(\vec{x}-\vec{y})=\sum_{i=1}^d p_i(x_i-y_i)$.

The Feynman-Kac formula above and the spectral theorem for Hermitian operators give us the connection between the spectrum of the generator of the dynamics and the behavior of the correlation functions. For example: for the truncated two-point function $S_2(x;y)\equiv\langle\phi_x\phi_y\rangle-\langle\phi_x\rangle\langle\phi_y\rangle\equiv\int\phi_x\phi_y d\rho_V^0-\int\phi_x d\rho_V^0\int\phi_y d\rho_V^0$, $x\equiv(x_0,\vec{x})$, $x_0\equiv t\in\mathbb{R}$, $\vec{x}\in\mathbb{Z}^d$ (let us assume the thermodynamic limit in this short comment), direct calculations lead to

$$\tilde{S}_2(p) \;=\; \int_0^\infty\int_{\mathbf{T}^d}\frac{2E}{E^2+(p_0)^2}(2\pi)^d\delta\left(\vec{q}-\vec{p}\right)\,d\left(\Omega,\phi\mathcal{E}(E,\vec{q})\phi\Omega\right), \tag{14}$$

where $\tilde{S}_2$ is the Fourier transform of S_2, $\mathcal{E}(E,\vec{p})$ is the spectral projection associated with the operators $(H,\vec{P})$ (the momentum operator $\vec{P}$ is the space translation generator), the integral over E runs from 0 to ∞ and that over $\vec{q}$ runs in T^d. Hence, a singularity in $\tilde{S}_2$ for imaginary $p_0=ik_0$ gives a point in the spectrum (one-particle sector) of the generator of the dynamics, and, by the Paley-Wiener theorem, the singularity in $\tilde{S}_2$ is related to the decay of S_2. Similarly, the decay of the truncated four-point function gives us information about, e.g., the existence of two-particle bound states. More details and comments are presented in [6] and references there in.

The rest of the paper is organized as follows.

In section 2 we introduce the model to be treated here, describe a formalism for the correlations (a Feynman-Kac type integral formalism) and list the main results of the paper. In section 3 we present the cluster expansion, whose convergence is proved in section 4. Section 5 is devoted to the study of the decay of the two-point truncated correlation, and section 6 to the analysis of the four-point truncated function.

2 The model and main results

In the present paper we will analyze in detail the model of unbounded continuous scalar field variables with (say) Hamiltonian

$$H_V=\sum_{\vec{x}\in V}-\frac{1}{2}\gamma\frac{\partial^2}{\partial\phi_{\vec{x}}^2}+\frac{1}{2}\sum_{\vec{x},\vec{y}\in V}\phi_{\vec{x}}J_{\vec{x},\vec{y}}\phi_{\vec{y}}+\lambda\sum_{\vec{x}\in V}Q(\phi_{\vec{x}}), \tag{15}$$

where

$$Q(\phi_{\vec{x}})=P(\phi_{\vec{x}})+\sum_{i=1}^{c}\sum_{\vec{y}\in V}\phi_{\vec{x}}^{a_i}D_{\vec{x},\vec{y}}^{(i)}\phi_{\vec{y}}^{b_i}, \tag{16}$$

$\phi_{\vec{x}}\in\mathbb{R}$, $V\subset\mathbb{Z}^d$, $\gamma>0$, $\lambda>0$, P is a polynomial bounded from below of degree $2m$, $c>0$ is an integer, a_i,b_i with $i=1,2,\ldots,c$, are integers such that $1\le a_i,b_i\le m$. $J_{\vec{x},\vec{y}}$ and $D_{\vec{x},\vec{y}}^{(i)}$ are symmetric spatial interactions, translational invariant and such that, $\forall i=1,2,\ldots,c$,

$$\sup_{\vec{x}\in\mathbb{Z}^d}\sum_{\vec{y}\in\mathbb{Z}^d}|\,J_{\vec{x},\vec{y}}|\equiv J_M<\infty, \qquad \sup_{\vec{x}\in\mathbb{Z}^d}\sum_{\vec{y}\in\mathbb{Z}^d}|\,D_{\vec{x},\vec{y}}^{(i)}|\equiv D_M^{(i)}<\infty. \tag{17}$$

Analyzing these specific models, we will get properties of a large class of anharmonic quantum crystals as well as of some stochastic GL models.

The spectral results to be presented are supported by a cluster expansion which we develop here. To obtain these results, in a rough resume, we start from the dynamics generator (or Hamiltonian), establish a lattice Feynman-Kac integral formula for the correlation functions, relating the initial problem to a spatially non-local imaginary time quantum field theory. Then, we study the decay of some truncated correlation functions using our "high temperature" cluster expansion for unbounded spin systems, which is related to that recently proposed in [13]. Information on the two-particle spectrum (related to the four-point correlation functions) is obtained in a perturbative analysis only: we use the Bethe-Salpeter (BS) equation with the BS kernel in a ladder approximation (details ahead). The reliability of the perturbative result is supported (again) by the convergence of the cluster expansion.

The Feynman-Kac functional integral associated to H_V follows, as said, from standard procedures. If $f_1, \ldots, f_n$ are functions of the spin configuration in V, $t_1 \leq t_2 \leq \ldots t_n \in \mathbb{R}$, as described in the introduction, we have an integral representation for the correlations

$$\int f_1(\phi(t_1)) \ldots f_n(\phi(t_n)) d\rho_V, \tag{18}$$

where, now, the path space measure $d\rho_V$ is the weak limit $T \to \infty$ of

$$d\rho_\Lambda = \frac{e^{-W_\Lambda(\phi)} d\omega_\Lambda}{\int e^{-W_\Lambda(\phi)} d\omega_\Lambda}, \tag{19}$$

with $\Lambda \equiv ([-T,T], V)$ and

$$W_\Lambda(\phi) = \int_{-T}^{T} dt \quad \lambda \sum_{\vec{x} \in V} Q(\phi_{t,\vec{x}}) = \int_{-T}^{T} dt \quad \lambda \left[\sum_{\vec{x} \in V} P(\phi_{t,\vec{x}}) + \sum_{i=1}^{c} \sum_{\vec{x},\vec{y} \in V} \phi^{a_i}_{t,\vec{x}} D^{(i)}_{\vec{x},\vec{y}} \phi^{b_i}_{t,\vec{y}} \right], \tag{20}$$

and $d\omega_\Lambda$ is a Gaussian measure with mean zero and variance given by

$$\gamma C_\Lambda\left(t, \vec{x}; t', \vec{y}\right) \;=\; \int \phi_{t,\vec{x}} \phi_{t',\vec{y}} d\omega_\Lambda \;=\; \frac{\gamma}{2\pi\,|V|} \int_{-\infty}^{\infty} dp_0 \sum_{\vec{p} \in \tilde{V}} \frac{e^{ip_0(t-t')}\ e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{p_0^2 + \gamma \tilde{J}(\vec{p})}, \tag{21}$$

where $\tilde{J}(\vec{p})$ is the Fourier transform of $J_{\vec{x},\vec{y}}$; $|V|$ is the number of points in V; $\tilde{V}$ is the Fourier dual lattice; $\vec{p} = (p_1, \ldots, p_d) \in \tilde{V}$ and $\vec{p} \cdot (\vec{x} - \vec{y}) = \sum_{i=1}^{d} p_i(x_i - y_i)$.

For technical reasons, we will consider a time regularized version of the measure $d\rho_\Lambda$. Precisely, from now on, we consider only the values of t in $\mathbb{Z} \cap [-T,T]$, so that our Feynman-Kac integral formula (18) becomes, say, the description of continuous spin system in a finite box $\Lambda \equiv (\mathbb{Z} \cap [-T,T], V)$ in $\mathbb{Z}^{d+1}$ (with periodic boundary conditions in space and free boundary conditions in time). Recall from field theory that such a (ultraviolet) cutoff shall not change the low-lying spectrum. We also will use the notation $x = (x_0, \vec{x}) \in \mathbb{Z}^{d+1}$ with $x_0 \in \mathbb{Z} \cap [-T,T]$ being the discrete time coordinate and $\vec{x} \in V$ the space coordinate. We define $\beta \equiv 1/\gamma$, and so, intense noise regime $\gamma \gg 1$ corresponds to the small β regime. Hence, from the path space measure $d\rho_\Lambda$ with discrete time, we introduce the "Gibbs measure" for such system:

$$\begin{aligned} \mu_\Lambda(\cdot) \;&=\; \left[\int e^{-W_\Lambda(\phi)} d\omega_\Lambda \right]^{-1} \int (\cdot) e^{-W_\Lambda(\phi)} d\omega_\Lambda \\ &=\; \frac{1}{Z_\Lambda} \int (\cdot) \exp\{-\lambda \sum_{x \in \Lambda} Q(\phi_x)\} \exp\{-\frac{1}{2\gamma} \sum_{x,y \in \Lambda} \phi_x C_\Lambda^{-1}(x,y) \phi_y\} \prod_{x \in \Lambda} d\phi_x, \end{aligned} \tag{22}$$

where

$$\begin{aligned}\sum_{x\in\Lambda} Q(\phi_x) &= \sum_{x\in\Lambda} P(\phi_x) + \sum_{i=1}^{c}\sum_{x,y\in\Lambda} \phi_x^{a_i} D^{(i)}_{\vec{x},\vec{y}} \phi_y^{b_i} \delta_{x_0,y_0}[\delta_{\vec{x},\vec{y}} + (1-\delta_{\vec{x},\vec{y}})] \\ &= \sum_{x\in\Lambda}[P(\phi_x) + \sum_{i=1}^{c} D^{(i)}_{\vec{x},\vec{x}}\phi^{a_i+b_i}] + \sum_{i=1}^{c}\sum_{x,y\in\Lambda} \phi_x^{a_i} D^{(i)}_{\vec{x},\vec{y}} \phi_y^{b_i} \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}), \end{aligned} \tag{23}$$

$$\begin{aligned}&\sum_{x,y\in\Lambda} \phi_x C_\Lambda^{-1}(x,y)\phi_y = \sum_{x,y\in\Lambda} \phi_x \left(-\delta_{\vec{x},\vec{y}}\triangle_t + \gamma J_{\vec{x},\vec{y}}\delta_{x_0,y_0}\right)\phi_y \\ &= \sum_{x,y\in\Lambda} \phi_x \left\{\delta_{\vec{x},\vec{y}}(2\delta_{x_0,y_0} - \delta_{|x_0-y_0|,1}) + \gamma J_{\vec{x},\vec{y}}\delta_{x_0,y_0}[\delta_{\vec{x},\vec{y}} + (1-\delta_{\vec{x},\vec{y}})]\right\}\phi_y \\ &= \sum_{x\in\Lambda}[2\phi_x^2 + \gamma J_{\vec{x},\vec{x}}\phi_x^2] + \sum_{x,y\in\Lambda} \phi_x\left[-\delta_{\vec{x},\vec{y}}\delta_{|x_0-y_0|,1} + \gamma J_{\vec{x},\vec{y}}\delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}})\right]\phi_y. \end{aligned} \tag{24}$$

We used above that, due to the discrete time, $-\triangle_t = 2\delta_{x_0,y_0} - \delta_{|x_0-y_0|,1}$. In short, we have

$$\mu_\Lambda(\cdot) = \frac{1}{Z_\Lambda}\int \prod_{x\in\Lambda} d\phi_x e^{-\beta U(\phi_x)} \prod_{\{x,y\}\subset\Lambda} e^{\beta G_{xy}(\phi_x,\phi_y)} \quad (\ \cdot\ ), \tag{25}$$

where $d\phi_x$ is the Lebesgue measure in $\mathbb{R}$ and the partition function Z_Λ is defined by

$$Z_\Lambda \equiv \int \prod_{\{x,y\}\subset\Lambda} e^{\beta G_{xy}(\phi_x,\phi_y)} \prod_{x\in\Lambda} d\phi_x e^{-\beta U(\phi_x)}. \tag{26}$$

Here, $U(\phi)$ is a local polynomial function (on site potential) bounded from below and given by

$$U(\phi) = \frac{\lambda}{\beta}P(\phi) + \frac{J(0)}{2\beta}\phi^2 + \phi^2 + \frac{\lambda}{\beta}\sum_{i=1}^{c}\phi^{a_i+b_i}D_i(0), \tag{27}$$

with $J(0) \equiv J(\vec{x},\vec{x})$ and $D_i(0) \equiv D^{(i)}(\vec{x},\vec{x})$; we assume that $J(\vec{x},\vec{y})$ and $D^{(i)}(\vec{x},\vec{y})$ are translational invariant and $J(\vec{x},\vec{y}) = J(|\vec{y}-\vec{x}|)$, $D^{(i)}(\vec{x},\vec{y}) = D^{(i)}(|\vec{y}-\vec{x}|)$. The coupling term is given by

$$\begin{aligned} G_{xy}(\phi_x,\phi_y) &= -\left[\frac{1}{\beta}J_{\vec{x},\vec{y}}\delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) - \delta_{\vec{x},\vec{y}}\delta_{|x_0-y_0|,1}\right]\frac{\phi_x\phi_y}{2} \\ &\quad -\frac{\lambda}{\beta}\sum_{i=1}^{c} D^{(i)}_{\vec{x},\vec{y}}\delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}})\phi_x^{a_i}\phi_y^{b_i} \\ &= A^{(1)}_{xy}\phi_x\phi_y + \sum_{i=1}^{c} A^{(i+1)}_{xy}\phi_x^{a_i}\phi_y^{b_i} \quad = \quad \sum_{s=1}^{c+1} A^{(s)}_{xy}\phi_x^{a_{s-1}^{(1-\delta_{s,1})}}\phi_y^{b_{s-1}^{(1-\delta_{s,1})}}, \end{aligned} \tag{28}$$

where we define $a_0 \equiv 1$, $b_0 \equiv 1$, and for $s = 2, 3, \ldots, (c+1)$.

$$A^{(1)}_{xy} \equiv -\frac{1}{2}\left[\frac{1}{\beta}J_{\vec{x},\vec{y}}\delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) - \delta_{\vec{x},\vec{y}}\delta_{|x_0-y_0|,1}\right], \quad A^{(s)}_{xy} \equiv -\frac{\lambda}{\beta}D^{(s-1)}_{\vec{x},\vec{y}}\delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}). \tag{29}$$

Note that our problem involves, say, an unusual high temperature spin system: we have a nonlocal anharmonic potential and the interaction decay depends on the direction (polynomial in space, and short range in "time").

The two point truncated function is given by

$$S_2(x;y) = \mu_\Lambda(\phi_x\phi_y) - \mu_\Lambda(\phi_x)\mu_\Lambda(\phi_y). \tag{30}$$

In the next section we will develop a polymer expansion for such model, which will permit us to show that $S_2(x;y)$ is analytic in β and admits upper and lower bounds uniform in Λ.

Now, let us state our main results.

Theorem 1 *The two-point truncated function $S_2(x;y)$, written as series in β, converges absolutely, uniformly in the volume $|\Lambda|$, for β, J_M and λ small enough. Moreover, $S_2(x;y)$ has the following bounds:*

$$|S_2(x;y)| \le C' e^{-m|x_0-y_0|-m_s|\vec{x}-\vec{y}|},$$

if $J_{\vec{x},\vec{y}}$ and $D^{(i)}_{\vec{x},\vec{y}}$ are finite range (i.e., if $\exists\, r > 0$ such that $J_{\vec{x},\vec{y}} = 0$ for $|\vec{x}-\vec{y}| > r$, and the same for $D^{(i)}_{\vec{x},\vec{y}}$), where C' is a constant uniform in β, J_M and $\lambda D^{(i)}_M$; or

$$|S_2(x;y)| \le C'' e^{-m'|x_0-y_0|}\left(\frac{1-\delta_{\vec{x},\vec{y}}}{|\vec{x}-\vec{y}|^p} + \delta_{\vec{x},\vec{y}}\right),$$

if $J_{\vec{x},\vec{y}}$ and $D^{(i)}_{\vec{x},\vec{y}}$ have polynomial decay, i.e., for $\vec{x} \ne \vec{y}$, $J_{\vec{x},\vec{y}} \le J/|\vec{x}-\vec{y}|^p$ and $D^{(i)}_{\vec{x},\vec{y}} \le D_i/|\vec{x}-\vec{y}|^p$; where C'' is a constant uniform in β, J_M and $\lambda D^{(i)}_M$.

Remarks.

1. The upper bound described in theorem 1 shall give, essentially, the asymptotic behavior of the two-point truncated correlation. Namely, assuming that we still have lower bounds for the interactions, e.g. $J'/|\vec{x}-\vec{y}|^p \le |J_{\vec{x},\vec{y}}| \le J/|\vec{x}-\vec{y}|^p$ and $D'_i/|\vec{x}-\vec{y}|^p \le |D^{(i)}_{\vec{x},\vec{y}}| \le D_i/|\vec{x}-\vec{y}|^p$, $\vec{x} \ne \vec{y}$, it follows that we shall also obtain a lower bound for $S_2(x;y)$ involving $C'''/|\vec{x}-\vec{y}|^p$ (see e.g. [13]).

2. The one-particle mass M is directly related to the "time" decay of S_2, i.e., $m' < M < \tilde{m}$, where m' is the term in upper bound above, and $\tilde{m}$ the similar one in a possible lower bound. A more precise estimate of M is possible by adopting standard techniques of constructive field theory, developed to study analyticity properties of one-particle irreducible Green's functions [12], taking as input the cluster expansion developed here. We do not carry out these calculations here, but we go further on the mass spectrum and investigate the existence of two-particle bound states. To get the results, we analyze the "time" decay of the truncated four-point function in terms of a BS equation, and carry out the computation considering the BS kernel up to dominant order in β only (ladder approximation). The convergence of the cluster expansion (leading, e.g., to a convergent series in β for S_2, as described in theorem 1) supports such a perturbative analysis. We emphasize that in some related spectral problems (e.g. [6], [14], [15]), where a complete study has been established, the rigorous results show that the spectral properties (e.g. one-particle and two-particle bound state masses) are given as small corrections of those obtained by similar perturbative approximations. For clearness, we summarize these results on the two-particle bound states in the proposition below.

Proposition 1 *Concerning the low-lying energy-momentum spectrum of the model, for the case of the interactions restricted to even functions of the spin variable ϕ, for β, J_M, and λ small enough, considering the ladder approximation for the Bethe-Salpeter kernel, there is an isolated two-particle bound state with mass $M^* = 2M + \log(1-\zeta)$, if $\zeta > 0$, where M is the one-particle mass and $\zeta = [\langle\phi^4\rangle - 3\langle\phi^2\rangle]/[\langle\phi^4\rangle - \langle\phi^2\rangle]$, $\langle\cdot\rangle$ is the expectation with respect to the single spin distribution $d\nu(\phi)$ (31).*

3 The polymer expansion

We now rewrite the expression for the partition function in terms of a polymer expansion. First we define the single spin distribution (s.s.d.)

$$d\nu(\phi) = \frac{e^{-\beta U(\phi)}}{C(\lambda,\beta)} d\phi, \qquad C(\lambda,\beta) = \int e^{-\beta U(\phi)} d\phi, \tag{31}$$

$d\nu(\phi)$ is a probability measure and (as usual) the partition function (26) can be rewritten as

$$Z_\Lambda = C^{|\Lambda|} \prod_{x\in\Lambda} \int d\nu(\phi_x) \prod_{\{x,y\}\subset\Lambda} \left(e^{\beta G_{xy}(\phi_x,\phi_y)} - 1 + 1 \right) = C^{|\Lambda|} \Xi_\Lambda, \tag{32}$$

with

$$\Xi_\Lambda = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{R_1,\ldots,R_n\subset\Lambda \\ R_i\cap R_j=\emptyset,\ |R_i|\geq 2}} \rho(R_1)\ldots\rho(R_n), \tag{33}$$

where $R_1, \ldots R_n \subset \Lambda$ is a collection of subsets of Λ with cardinality greater than 1, with associated activities $\rho(R)$ given by

$$\rho(R) = \prod_{x\in R} \int d\nu(\phi_x) \sum_{g\in G_R} \prod_{\{x,y\}\subset\Lambda} (e^{\beta G_{xy}(\phi_x,\phi_y)} - 1), \tag{34}$$

where $\sum_{g\in G_R}$ is the sum over the connected graphs on the set R. Given a finite set A, we define a graph g in A as a collection $\{\gamma_1, \ldots, \gamma_m\}$ of distinct pairs of A, i.e., $\gamma_i = \{x_i, y_i\} \subset A$ with $x_i \neq y_i$. A graph $g = \{\gamma_1, \ldots, \gamma_m\}$ in A is connected if for any B,C of subsets of A such that $B \cup C = A$ and $B \cap C = \emptyset$, there is a $\gamma_i \in g$ such that $\gamma_i \cap B \neq \emptyset$ and $\gamma_i \cap C \neq \emptyset$. The pairs γ_i are called links of the graph. We denote by $|g|$ the number of links in g.

By standard arguments one can expand $\log \Xi_\Lambda$ as

$$\log \Xi_\Lambda = \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{R_1,\ldots R_n\subset\Lambda \\ |R_i|\geq 2}} \phi^T(R_1, \ldots R_n)\rho(R_1)\ldots\rho(R_n), \tag{35}$$

with

$$\phi^T(R_1,\ldots,R_n) = \begin{cases} 1, & \text{if } n=1, \\ \sum_{\substack{f\in G_n \\ f\subset g(R_1,\ldots,R_n)}} (-1)^{|f|}, & \text{if } n\geq 2 \text{ and } g(R_1,\ldots,R_n)\in G_n, \\ 0, & \text{if } g(R_1,\ldots,R_n)\notin G_n, \end{cases} \tag{36}$$

where G_n above denotes the set of the connected graphs on $\{1, \ldots, n\}$, and $g(R_1, \ldots, R_n)$ denotes the graph in $\{1, 2, \ldots, n\}$ which has the link $\{i, j\}$ if and only if $R_i \cap Rj \neq \emptyset$. The series (35) is known to converge absolutely, uniformly in Λ, if the activity $\rho(R)$ is sufficiently small [16]: e.g., if

$$\sup_{x\in\Lambda}\sum_{R\ni x}|\rho(R)|e^{a|R|}<a, \tag{37}$$

for some $a>0$. The two-point truncated correlation (30) can now be rewritten as

$$S_2(x_1;x_2)=\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\log\tilde{\Xi}_\Lambda(\alpha_1,\alpha_2)\Big|_{\alpha=0}, \tag{38}$$

where

$$\tilde{\Xi}_\Lambda(\alpha_1,\alpha_2)=\prod_{x\in\Lambda}\int d\nu(\phi_x)e^{\beta G_{xy}(\phi_x,\phi_y)}(1+\alpha_1\phi_{x_1})(1+\alpha_2\phi_{x_2}). \tag{39}$$

Note that $\tilde{\Xi}_\Lambda(\alpha_1=0,\alpha_2=0)=\Xi_\Lambda$. Let us expand $\tilde{\Xi}_\Lambda(\alpha_1,\alpha_2)$ in terms of polymers. For any $R\subset\Lambda$, let us denote by I_R the subset (possibly empty) of $\{1,2\}$ such that $i\in I_R$ iff $x_i\in R$, where $i=1,2$. We get

$$\Xi_\Lambda(\alpha_1,\alpha_2)=1+\sum_{n\geq1}\frac{1}{n!}\sum_{\substack{R_1,\ldots R_n\subset\Lambda\\ R_i\cap R_j=\emptyset\ |R_i|\geq1}}\tilde{\rho}(R_1,\alpha)\ldots\tilde{\rho}(R_n,\alpha), \tag{40}$$

where

$$\tilde{\rho}(R,\alpha)=\begin{cases}\prod_{x\in R}\int d\nu(\phi_x)\prod_{i\in I_R}(1+\alpha_i\phi_{x_i})\sum\limits_{g\in G_R}\prod\limits_{\{x,y\}\in g}(e^{\beta G_{xy}(\phi_x,\phi_y)}-1), & \text{for } |R|\geq2,\\ \prod_{x\in R}\int d\nu(\phi_x)\prod\limits_{i\in I_R}\alpha_i\phi_{x_i}, & \text{for } I_R\neq\emptyset,\ |R|=1,\\ 0, & \text{for } I_R=\emptyset,\ |R|=1.\end{cases} \tag{41}$$

Note that the one-body polymers $R=\{x\}$ can also contribute to the partition function (40), but only if $x=x_i$ for some $i\in\{1,2\}$.

Taking the log of (40) and noting, from (38), that only the terms proportional to $\alpha_1\alpha_2$ give a non vanishing contribution to the two-point truncated correlation function, we get

$$S_2(x_1;x_2)=\sum_{n\geq1}\frac{1}{n!}\sum_{i_1,i_2=1}^{n}\sum_{\substack{R_1,\ldots R_n\subset\Lambda,\ |R_j|\geq2\\ R_{i_1}\ni x_1 R_{i_2}\ni x_2}}\phi^T(R_1,\ldots R_n)\tilde{\rho}(R_1)\ldots\tilde{\rho}(R_n), \tag{42}$$

where

$$\tilde{\rho}(R_i)=\prod_{x\in R_i}\int d\nu(\phi_x)(\phi_{x_1}^{\beta_i^1}+\beta_i^1 l)(\phi_{x_2}^{\beta_i^2}+\beta_i^2 l)\sum_{g\in G_R}\prod_{\{x,y\}\in g}(e^{\beta G_{xy}(\phi_x,\phi_y)}-1), \tag{43}$$

with $\beta_i^j=0$ if $i\neq i_j$, or 1 if $i=i_j$ and $l=\int d\nu(\phi)\phi$.

Note that the one-body polymers are absorbed in the activity of the many body polymers (in the terms proportional to l), due to the fact that $R_1,\ldots,R_n$ must be connected, and so each 1-body polymer (if any) is always contained in (at least) one many-body polymer.

4 Convergence of the polymer expansion

We will treat in detail here the case where the pair potential $J_{\vec{x},\vec{y}}$ and $D^{(i)}_{\vec{x},\vec{y}}$ decays polynomially (the more complicated one). I.e. we will assume that, $\forall i = 1, 2, \ldots, c$, $\vec{x} \neq \vec{y}$,

$$J_{\vec{x},\vec{y}} \leq \frac{J}{|\vec{x}-\vec{y}|^p}, \qquad D^{(i)}_{\vec{x},\vec{y}} \leq \frac{D_i}{|\vec{x}-\vec{y}|^p}, \tag{44}$$

$J, D_i \in \mathbb{R}$ are constants. We will treat the cases where $p \geq d + \varepsilon$ with $\varepsilon > 0$.

The cases of $J_{\vec{x},\vec{y}}$ and $D^{(i)}_{\vec{x},\vec{y}}$ with finite range or with exponential decay are easier and can be treated in a similar way.

In order to ensure the convergence condition (37) we will bound the factor

$$\varepsilon_n(z, z') = \sum_{\substack{R\subset\Lambda:|R|=n\\ z,z'\in R}} |\rho(R)|, \tag{45}$$

and show that $\varepsilon_n(z, z') \leq [f(\beta, J, \lambda D_1, \ldots, \lambda D_c)]^n$, where $f(\beta, J, \lambda D_1, \ldots, \lambda D_c) \to 0$ as $\beta, J, \lambda D_1, \ldots, \lambda D_c \to 0$.

In order to get an efficient bound for the factor

$$\sum_{g\in G_R} \prod_{\{x,y\}\subset\Lambda} (e^{\beta G_{xy}(\phi_x,\phi_y)} - 1) \tag{46}$$

that appears in $\rho(R)$, we use a standard cluster expansion result, namely, the Brydges-Battle-Federbush tree graph inequality, which can be resumed in the following lemma.

Lemma 1 *Let R be a finite set with cardinality $|R|$ and let $\{V_{xy} : \{x, y\} \in \mathbb{R}\}$ a set with $|R|(|R|-1)/2$ real numbers (with $\{x, y\}$ unordered pair in R). Suppose now that there exist $|R|$ positive numbers V_x (with $x \in R$) such that, for any subset $S \subset R$,*

$$\sum_{x\in S} V_x + \sum_{\{x,y\}\in S} V_{xy} \geq 0. \tag{47}$$

Then

$$\left| \sum_{g\in G_R} \prod_{\{x,y\}\in g} \left(e^{-V_{xy}} - 1\right) \right| \leq e^{\sum_{x\in R} V_x} \sum_{\tau\in T_R} \prod_{\{x,y\}\in\tau} |V_{xy}|, \tag{48}$$

where T_R denotes the set of the tree graphs on R.

For the proof of this lemma see e.g. [17], [18].

For any $\varphi, \phi \in \mathbb{R}$, we have $2|\varphi\phi| \leq |\varphi|^2 + |\phi|^2$ and $|\varphi|^a|\phi|^b \leq |\varphi|^{a+b} + |\phi|^{a+b}$. Then, for any $R \subset \Lambda$, it follows that

$$\begin{aligned} &\sum_{\{x,y\}\subset R} |G_{xy}(\phi_x, \phi_y)| \\ \leq \quad &\sum_{\{x,y\}\subset R} \left| \frac{1}{\beta} J_{\vec{x},\vec{y}} \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) + \delta_{\vec{x},\vec{y}}\delta_{|x_0-y_0|,1} \right| \frac{|\phi_x\phi_y|}{2} + \sum_{i=1}^{c} \left| \frac{\lambda}{\beta} D^{(i)}_{\vec{x},\vec{y}} \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) \right| |\phi_x^{a_i}\phi_y^{b_i}| \end{aligned}$$

$$
\begin{aligned}
\leq \quad & \sum_{\{x,y\}\subset R} \left| \frac{1}{\beta} J_{\vec{x},\vec{y}} \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) + \delta_{\vec{x},\vec{y}} \delta_{|x_0-y_0|,1} \right| \frac{1}{4}(|\phi_x|^2 + |\phi_y|^2) \\
& + \sum_{i=1}^{c} \left| \frac{\lambda}{\beta} D^{(i)}_{\vec{x},\vec{y}} \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) \right| (|\phi_x|^{a_i+b_i} + |\phi_y|^{a_i+b_i}) \\
\leq \quad & \sum_{x\in R} \frac{|\phi_x|^2}{2} \sum_{y\in R} \left| \frac{1}{\beta} J_{\vec{x},\vec{y}} \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) + \delta_{\vec{x},\vec{y}} \delta_{|x_0-y_0|,1} \right| \\
& + \sum_{i=1}^{c} \sum_{x\in R} |\phi_x|^{a_i+b_i} \sum_{y\in R} \left| \frac{2\lambda}{\beta} D^{(i)}_{\vec{x},\vec{y}} \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) \right| .
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{y\in R} \delta_{\vec{x},\vec{y}} \delta_{|x_0-y_0|,1} \quad &\leq \quad \sum_{y_0\in\mathbb{Z}} \delta_{|x_0-y_0|,1} \quad \leq \quad 2, \\
\sum_{y\in R} |J_{\vec{x},\vec{y}}| \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) \quad &\leq \quad \sup_{\vec{x}\in\mathbb{Z}^d} \sum_{\vec{y}\in\mathbb{Z}^d} |\, J_{\vec{x},\vec{y}}| \quad \leq \quad J_M, \\
\sum_{y\in R} |D^{(i)}_{\vec{x},\vec{y}}| \delta_{x_0,y_0}(1-\delta_{\vec{x},\vec{y}}) \quad &\leq \quad \sup_{\vec{x}\in\mathbb{Z}^d} \sum_{\vec{y}\in\mathbb{Z}^d} |\, D^{(i)}_{\vec{x},\vec{y}}| \quad \leq \quad D^{(i)}_M .
\end{aligned}
$$

Hence,

$$
\sum_{\{x,y\}\subset R} |G_{xy}(\phi_x,\phi_y)| \quad \leq \quad \sum_{x\in R} \left[\left(\frac{J_M}{2\beta} + 1 \right) |\phi_x|^2 + \sum_{i=1}^{c} \frac{2\lambda D^{(i)}_M}{\beta} |\phi_x|^{a_i+b_i} \right].
$$

As $|\phi|^2 \leq |\phi|^{a_i+b_i} \leq |\phi|^{a_i+b_i}+1$ if $|\phi|\geq 1$ and $|\phi|^2 \leq 1 \leq |\phi|^{a_i+b_i}+1$ if $|\phi|\leq 1$ and, defining $(a,b)\equiv(a_j,b_j)$ for j such that $a_j+b_j = \max\{a_1+b_1,\ldots,a_c+b_c\}$, it follows that

$$
\sum_{\{x,y\}\subset R} |G_{xy}(\phi_x,\phi_y)| \quad \leq \quad \sum_{x\in R} \left(C_1 \phi_x^{a+b} + C_2 \right),
$$

where

$$
C_1 \equiv \frac{J_M}{2\beta} + 1 + \sum_{i=1}^{c} \frac{2\lambda D^{(i)}_M}{\beta}, \qquad C_2 \equiv \frac{J_M}{2\beta} + 1. \tag{49}
$$

Using the lemma 1 we get

$$
\left| \sum_{g\in G_R} \prod_{\{x,y\}\in g} (e^{\beta G_{xy}(\phi_x,\phi_y)} - 1) \right| \leq \prod_{x\in R} e^{\beta(C_1\phi_x^{a+b}+C_2)} \sum_{\tau\in T_R} \prod_{\{x,y\}\in\tau} |\beta G_{xy}(\phi_x,\phi_y)|, \tag{50}
$$

and so,

$$
\begin{aligned}
\sum_{\substack{R\subset\Lambda:|R|\geq 2\\ z,z'\in R}} |\rho(R)| e^{|R|} &= \sum_{n\geq 2} e^n \sum_{\substack{R\subset\Lambda:|R|=n\\ z,z'\in R}} |\rho(R)| = \sum_{n\geq 2} \frac{e^n}{(n-2)!} \sum_{\substack{x_1,\ldots,x_n\in\Lambda:\\ x_1=z,x_2=z',x_i\neq x_j}} |\rho(R=\{x_1,\ldots,x_n\})| \\
&\leq \sum_{n\geq 2} \frac{e^n}{(n-2)!} \sum_{\substack{x_1,\ldots,x_n\in\Lambda:\\ x_1=z,x_2=z',x_i\neq x_j}} \int \prod_{i=1}^{n} d\nu(\phi_{x_i}) e^{\beta(C_1\phi_{x_i}^{a+b}+C_2)} \sum_{\tau\in T_n} \prod_{\{i,j\}\in\tau} |\beta G_{x_iy_j}(\phi_{x_i}\phi_{x_j})|.
\end{aligned}
$$

Recall now (28), (29), and that $|\tau| = n-1$. Hence, fixing $\tau \in T_n$, we have

$$\begin{aligned}
\prod_{\{i,j\}\in\tau} |G_{x_i y_j}(\phi_{x_i}\phi_{x_j})| &\leq \prod_{\{i,j\}\in\tau} \sum_{s=1}^{c+1} |A^{(s)}_{x_i x_j}| |\phi_{x_i}^{a_{s-1}^{(1-\delta_{s,1})}}| |\phi_{x_j}^{b_{s-1}^{(1-\delta_{s,1})}}| \\
&\leq \sum_{\{i,j\}\in\tau} \sum_{s_{ij}=1}^{c+1} \prod_{k=1}^{n} |\phi_{x_k}|^{n_k(s)} \prod_{\{i,j\}\in\tau} |A^{(s_{ij})}_{x_i x_j}|,
\end{aligned}$$

where $n_k(s)$ depends on $a_1, \ldots, a_c$, $b_1, \ldots, b_c$ and on the sequence $\{s_{ij}\}_{\{i,j\}\in\tau}$. In any case, we have $d_k \leq n_k(s) \leq d_k \cdot \max\{a_1, \ldots, a_c, b_1, \ldots, b_c\} \leq m d_k$, where $2m$ is the degree of the polynomial $P(\phi)$, and $\{d_k\}_{k=1}^n$ are the incidence indices of the tree $\tau \in T_n$, with $1 \leq d_k \leq n-1$ and $\sum_{k=1}^n d_k = 2n-2$. Then

$$\begin{aligned}
\sum_{\substack{R\subset\Lambda:|R|\geq 2\\ z,z'\in R}} |\rho(R)| e^{|R|} &\leq \sum_{n\geq 2} \frac{e^n e^{n\beta C_2} \beta^{n-1}}{(n-2)!} \sum_{\substack{x_1,\ldots,x_n\in\Lambda:\\ x_1=z, x_2=z', x_i\neq x_j}} \int \prod_{i=1}^{n} d\nu(\phi_{x_i}) e^{\beta C_1 \phi_{x_i}^{a+b}} \\
&\quad \times \sum_{\tau\in T_n} \sum_{\{i,j\}\in\tau} \sum_{s_{ij}=1}^{c+1} \prod_{k=1}^{n} |\phi_{x_k}|^{n_k(s)} \prod_{\{i,j\}\in\tau} |A^{(s_{ij})}_{x_i x_j}| \\
&\leq \sum_{n\geq 2} \frac{e^n e^{n\beta C_2} \beta^{n-1}}{(n-2)!} \sum_{\substack{x_1,\ldots,x_n\in\Lambda:\\ x_1=z, x_2=z', x_i\neq x_j}} \\
&\quad \times \sum_{\tau\in T_n} \sum_{\{i,j\}\in\tau} \sum_{s_{ij}=1}^{c+1} \prod_{k=1}^{n} \left(\int d\nu(\phi_{x_k}) e^{\beta C_1 \phi_{x_k}^{a+b}} |\phi_{x_k}|^{n_k(s)} \right) \prod_{\{i,j\}\in\tau} |A^{(s_{ij})}_{x_i x_j}|.
\end{aligned}$$

Lemma 2 *If $a \leq m$ and $b \leq m$, then $\forall \alpha > 0$, $\alpha \in \mathbb{R}$, we have*

$$\int d\nu(\phi) e^{\beta C_1 \phi^{a+b}} |\phi|^\alpha \leq \frac{e^{C_4 - C_6} C_3^{1/2m}}{C_5^{(\alpha+1)/2m}} \left[\Gamma\left(\frac{1}{2m}\right)\right]^{-1} \Gamma\left(\frac{\alpha+1}{2m}\right),$$

where C_3, C_4, C_5 e C_6 are constants such that, $\forall \phi \in \mathbb{R}$,

$$\beta U(\phi) \leq C_3 \phi^{2m} + C_4, \qquad \beta U(\phi) - \beta C_1 \phi^{a+b} \geq C_5 \phi^{2m} + C_6.$$

Proof. As $P(\phi)$ is a polynomial of degree $2m$ bounded from below, there exist contants A_1, A_2, A_3 e A_4 (with A_1 and $A_3 > 0$) such that

$$A_1 \phi^{2m} + A_2 \leq P(\phi) \leq A_3 \phi^{2m} + A_4.$$

Then, as $\phi^2 \leq \phi^{a+b} + 1$ and $\phi^{a_i+b_i} \leq \phi^{a+b} \leq \phi^{2m}$ since $a_i + b_i \leq a + b \leq 2m$, we obtain

$$\begin{aligned}
&\beta U(\phi) = \lambda P(\phi) + \frac{J(0)}{2}\phi^2 + \beta\phi^2 + \lambda \sum_{i=1}^{c} \phi^{a_i+b_i} D_i(0) \\
&\leq \lambda(A_3\phi^{2m} + A_4) + \frac{J(0)}{2}(\phi^{2m}+1) + \beta(\phi^{2m}+1) + \lambda\phi^{2m} \sum_{i=1}^{c} D_i(0) \leq C_3\phi^{2m} + C_4,
\end{aligned}$$

where

$$C_3 \equiv \lambda A_3 + \frac{J(0)}{2} + \beta + \lambda \sum_{i=1}^{c} D_i(0), \qquad C_4 \equiv \lambda A_4 + \frac{J(0)}{2} + \beta + \lambda \sum_{i=1}^{c} D_i(0).$$

We also have

$$\begin{aligned} \beta U(\phi) - \beta C_1 \phi^{a+b} &= \lambda P(\phi) + \frac{J(0)}{2}\phi^2 + \beta\phi^2 + \lambda \sum_{i=1}^{c} \phi^{a_i+b_i} D_i(0) - \beta C_1 \phi^{a+b} \\ &\geq \lambda(A_1\phi^{2m} + A_2) + \lambda \sum_{i=1}^{c} \phi^{a_i+b_i} D_i(0) - \beta C_1 \phi^{a+b}. \end{aligned}$$

If $a+b=2m$, we have

$$\beta U(\phi) - \beta C_1 \phi^{a+b} \geq \left[\lambda A_1 - \beta C_1 + \lambda \sum_{i:a_i+b_i=2m}^{c} D_i(0) \right] \phi^{2m} + \lambda A_2 \;\geq\; C_5 \phi^{2m} + C_6.$$

If $a+b < 2m$, then $\phi^{a+b} \leq \phi^{2m} + 1$ and it follows that

$$\beta U(\phi) - \beta C_1 \phi^{a+b} \geq (\lambda A_1 - \beta C_1)\phi^{2m} + \lambda A_2 - \beta C_1 \;\geq\; C_5 \phi^{2m} + C_6.$$

Hence, $C_5 = \lambda A_1 - \beta C_1 + \lambda \sum_{i:a_i+b_i=2m}^{c} D_i(0)$ if $a+b=2m$, or $C_5 = \lambda A_1 - \beta C_1$ if $a+b<2m$. $C_6 = \lambda A_2$ if $a+b=2m$, or $C_6 = \lambda A_2 - \beta C_1$ if $a+b<2m$. ¿From the definition of the s.s.d. $d\nu(\phi)$, we have

$$\int d\nu(\phi) e^{\beta C_1 \phi^{a+b}} |\phi|^\alpha = \frac{1}{C(\lambda,\beta)} \int e^{-\beta U(\phi)} e^{\beta C_1 \phi^{a+b}} |\phi|^\alpha d\phi.$$

Hence,

$$C(\lambda,\beta) = \int_{\mathbb{R}} e^{-\beta U(\phi)} d\phi \geq e^{-C_4} \int_{\mathbb{R}} e^{-C_3 \phi^{2m}} d\phi \geq e^{-C_4} \frac{1}{m C_3^{1/2m}} \Gamma\left(\frac{1}{2m}\right),$$

$$\int e^{-\beta U(\phi)} e^{\beta C_1 \phi^{a+b}} |\phi|^\alpha d\phi \leq e^{-C_6} \int_{\mathbb{R}} e^{-C_5 \phi^{2m}} |\phi|^\alpha d\phi \leq e^{-C_6} \frac{1}{m C_5^{(\alpha+1)/2m}} \Gamma\left(\frac{\alpha+1}{2m}\right),$$

where we used that $C_5 > 0$ for β small enough. The lemma 2 follows from these two bounds. ■

Recalling that $1 \leq d_k \leq n_k(s) \leq m d_k$, we get $\frac{n_k(s)+1}{2m} \leq \frac{2n_k(s)}{2m} \leq d_k$, and so, using the lemma 2, we obtain

$$\begin{aligned} \sum_{\substack{R\subset\Lambda:|R|\geq 2\\ z,z'\in R}} |\rho(R)| e^{|R|} \;\leq\; & \sum_{n\geq 2} \frac{e^n e^{n\beta C_2} \beta^{n-1}}{(n-2)!} \sum_{\substack{x_1,\ldots,x_n\in\Lambda:\\ x_1=z, x_2=z', x_i\neq x_j}} \sum_{\tau\in T_n} \sum_{\{i,j\}\in\tau} \sum_{s_{ij}=1}^{c+1} \\ & \times \prod_{k=1}^{n} \left\{ \frac{e^{C_4-C_6} C_3^{1/2m}}{C_5^{d_k}} \left[\Gamma\left(\frac{1}{2m}\right)\right]^{-1} \Gamma(d_k) \right\} \prod_{\{i,j\}\in\tau} |A_{x_i x_j}^{(s_{ij})}| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n\geq 2}\frac{e^n e^{n\beta C_2}\beta^{n-1}}{(n-2)!}\frac{(e^{C_4-C_6}C_3^{1/2m})^n}{C_5^{2(n-1)}}\left[\Gamma\left(\frac{1}{2m}\right)\right]^{-n}\sum_{\tau\in T_n}\sum_{\{i,j\}\in\tau}\sum_{s_{ij}=1}^{c+1}\\
&\quad\times\prod_{k=1}^{n}\Gamma(d_k)\sum_{\substack{x_1,\ldots,x_n\in\Lambda:\\ x_1=z,x_2=z',x_i\neq x_j}}\prod_{\{i,j\}\in\tau}|A^{(s_{ij})}_{x_ix_j}|, \qquad (51)
\end{aligned}
$$

where we use that $\prod_{k=1}^{n}C_5^{d_k}=C_5^{(d_1+d_2+\ldots+d_n)}=C_5^{2(n-1)}$.

We note that for any $\tau\in T_n$, there is a unique path $\bar\tau$ in τ which joins vertex 1 to vertex 2. Fixing $\tau\in T_n$, let be $\bar\tau\equiv\{1,i_1\},\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_{k-1},i_k\},\{i_k,2\}$ and $I_\tau\equiv\{1,i_1,i_2,\ldots,i_k,2\}$ the subset of $\{1,2,3,\ldots,n\}$ whose elements are the vertices of the path $\bar\tau$. Hence, $|\tau|=n-1$, $|\bar\tau|=k+1$ and $|\tau\setminus\bar\tau|=n-k-2$.

¿From the definitions (29) of $|A^{(s)}_{xy}|\,:\,s=1,\ldots,(c+1)$, we see that all the terms vanish if $|\,x_0-y_0\,|>1$. Hence, fixing $\tau\in T_n$, if $\exists\,\{i,j\}\in\tau$ such that $|\,(x_i)_0-(x_j)_0\,|>1$ we have $|A^{(s)}_{x_ix_j}|=0\ \forall s=1,\ldots,(c+1)$, and so, this tree τ does not contribute to the sum (51). Then, given $|\,(x_1)_0-(x_2)_0\,|$, as $|\bar\tau|=k+1$, if $|\,(x_1)_0-(x_2)_0\,|>k+1$ then $\exists\,\{i,j\}\in\bar\tau$ such that $|\,(x_i)_0-(x_j)_0\,|>1$, and so, τ do not contribute to (51). As $\bar\tau\subset\tau$ we have $n-1\geq k+1$. Therefore, any tree $\tau\in T_n$ such that $|\,(x_1)_0-(x_2)_0|>n-1\geq k+1$ do not contribute to (51), in other words, $\rho(R)$ vanishes if $|\,(x_1)_0-(x_2)_0|>|R|-1$, i.e., if $|R|<|\,(x_1)_0-(x_2)_0\,|+1$.

So, we define $N\equiv|\,z_0-{z'}_0\,|+1$, and we have

$$
\sum_{\substack{R\subset\Lambda:|R|\geq 2\\ z,z'\in R}}|\rho(R)|e^{|R|}\leq\sum_{\substack{R\subset\Lambda:|R|\geq N\\ z,z'\in R}}|\rho(R)|e^{|R|}. \qquad (52)
$$

Now, we note that

$$
\delta_{\vec x,\vec y}\delta_{|x_0-y_0|,1}\leq\delta_{\vec x,\vec y}\;e^{-|x_0-y_0|+1},\qquad \delta_{x_0,y_0}\leq e^{-|x_0-y_0|}.
$$

Then,

$$
\begin{aligned}
|A^{(1)}_{xy}| &= \frac{1}{2}\left|\frac{1}{\beta}J_{\vec x,\vec y}\delta_{x_0,y_0}(1-\delta_{\vec x,\vec y})-\delta_{\vec x,\vec y}\delta_{|x_0-y_0|,1}\right|\leq\frac{e^{-|x_0-y_0|}}{2}\left[\frac{1}{\beta}\frac{J}{|\vec x-\vec y|^p}(1-\delta_{\vec x,\vec y})+e\delta_{\vec x,\vec y}\right],\\
|A^{(s)}_{xy}| &= \left|\frac{\lambda}{\beta}D^{(s-1)}_{\vec x,\vec y}\delta_{x_0,y_0}(1-\delta_{\vec x,\vec y})\right| \leq \frac{\lambda D_{s-1}}{\beta|\vec x-\vec y|^p}(1-\delta_{\vec x,\vec y})e^{-|x_0-y_0|}.
\end{aligned}
$$

Hence, $\forall s=1,\ldots,(c+1)$,

$$
\begin{aligned}
|A^{(s)}_{xy}| &\leq Ke^{-|x_0-y_0|}\left[\frac{1}{|\vec x-\vec y|^p}(1-\delta_{\vec x,\vec y})+\frac{e}{2}\delta_{\vec x,\vec y}\right]\\
&\leq eKe^{-|x_0-y_0|}\left[\frac{(1-\delta_{\vec x,\vec y})}{|\vec x-\vec y|^p}+\delta_{\vec x,\vec y}\right]\leq eKF^{(1)}_{xy},\\
\sup_{x\in\Lambda}\sum_{y\in\Lambda}|A^{(s)}_{xy}| &\leq e\mathcal{O}(1)K,
\end{aligned}
$$

where, we define

$$K \equiv \frac{1}{\beta} \max\{\frac{J}{2}, \lambda D_1, \lambda D_2 \ldots, \lambda D_c\}, \tag{53}$$

and, for $w \in \mathbb{R}$, $w > 0$,

$$F^{(w)}_{xy} \equiv e^{-w|x_0-y_0|} \left[\frac{(1-\delta_{\vec{x},\vec{y}})}{|\vec{x}-\vec{y}|^p} + \delta_{\vec{x},\vec{y}} \right].$$

Then, fixing $\tau \in T_n$ and the sequence $\{s_{ij}\}$, we get

$$\begin{aligned}
\sum_{\substack{x_1,\ldots,x_n \in \Lambda: \\ x_1=z, x_2=z', x_i \neq x_j}} \prod_{\{i,j\}\in\tau} |A^{(s_{ij})}_{x_i x_j}| &= \sum_{\substack{x_1,\ldots,x_n \in \Lambda: \\ x_1=z, x_2=z', x_i \neq x_j}} \prod_{\{i,j\}\in\tau\setminus\bar{\tau}} |A^{(s_{ij})}_{x_i x_j}| \prod_{\{i,j\}\in\bar{\tau}} |A^{(s_{ij})}_{x_i x_j}| \\
&\leq [e\mathcal{O}(1)K]^{(n-k-2)} \sum_{\substack{x_{i_1},\ldots,x_{i_k} \in \Lambda: \\ x_{i_r} \neq x_{i_q} \forall r,q=1,\ldots,k}} \prod_{\{i,j\}\in\bar{\tau}} |A^{(s_{ij})}_{x_i x_j}| \\
&\leq [e\mathcal{O}(1)K]^{(n-k-2)} \sum_{\substack{x_{i_1},\ldots,x_{i_k} \in \Lambda: \\ x_{i_r} \neq x_{i_q} \forall r,q=1,\ldots,k}} eKF^{(1)}_{x_1 x_{i_1}} eKF^{(1)}_{x_{i_1} x_{i_2}} \ldots eKF^{(1)}_{x_{i_{k-1}} x_{i_k}} eKF^{(1)}_{x_{i_k} x_2}.
\end{aligned}$$

Applying iteratively the inequality (for $w_1 < w_2$)

$$\sum_{\substack{x_i \in \Lambda: \\ x_i \neq x,y}} F^{(w_1)}_{x x_i} F^{(w_2)}_{x_i y} \leq \mathcal{O}(1) F^{(w_1)}_{xy}, \tag{54}$$

which follows from

$$\sum_{\substack{\vec{x}_i \in \mathbb{Z}^d: \\ \vec{x}_i \neq \vec{x},\vec{y}}} \frac{1}{|\vec{x}-\vec{x}_i|^p} \frac{1}{|\vec{x}_i-\vec{y}|^p} \leq \frac{\mathcal{O}(1)}{|\vec{x}-\vec{y}|^p}, \qquad \sum_{\substack{z_0 \in \mathbb{R}: \\ z_0 \neq x_0, y_0}} e^{-w_1|x_0-z_0|} e^{-w_2|z_0-y_0|} \leq \mathcal{O}(1) e^{-w_1|x_0-y_0|},$$

(the formula is valid for any $w_1 < w_2$, in specific for $w_1 = 2/3$ and $w_2 = 1$, which we will take here), we get

$$\sum_{\substack{x_1,\ldots,x_n \in \Lambda: \\ x_1=z, x_2=z', x_i \neq x_j}} \prod_{\{i,j\}\in\tau} |A^{(s_{ij})}_{x_i x_j}| \quad \leq \quad [e\mathcal{O}(1)K]^{(n-1)} F^{(2/3)}_{zz'}. \tag{55}$$

Recall that

$$\sum_{\tau \in T_n} 1 = \sum_{\substack{d_1+\ldots+d_n=2n+2 \\ d_i \geq 1}} \sum_{\substack{\tau \in T_n: \\ \tau \approx (d1,\ldots,d_n)}} 1, \tag{56}$$

where the notation $\tau \approx (d_1, \ldots, d_n)$ means that the last sum above runs over the trees $\tau \in T_n$ that have fixed incidence indices $(d_1, \ldots, d_n)$. From the Cayley formula

$$\sum_{\substack{\tau \in T_n: \\ \approx (d1,\ldots,d_n)}} 1 = \frac{(n-2)!}{\prod_{i=1}^n (d_i - 1)!},$$

and, fixing $\tau \in T_n$, we have

$$\sum_{\{i,j\}\in\tau} \sum_{s_{ij}=1}^{c+1} 1 = (c+1)^{(n-1)}.$$

Hence, using (52), we get

$$\begin{aligned}
& \sum_{\substack{R\subset\Lambda:|R|\geq 2\\ z,z'\in R}} |\rho(R)|e^{|R|} \\
\leq\ & \sum_{n\geq N} \frac{e^n e^{n\beta C_2}\beta^{n-1}}{(n-2)!} \frac{(e^{C_4-C_6}C_3^{1/2m})^n}{C_5^{2(n-1)}} \left[\Gamma\left(\frac{1}{2m}\right)\right]^{-n} \\
& \times \sum_{\tau\in T_n} \sum_{\{i,j\}\in\tau} \sum_{s_{ij}=1}^{c+1} \prod_{k=1}^{n} \Gamma(d_k)[e\mathcal{O}(1)K]^{(n-1)} F_{zz'}^{(2/3)} \\
\leq\ & \sum_{n\geq N} \frac{e^n e^{n\beta C_2}\beta^{n-1}}{(n-2)!} \frac{(e^{C_4-C_6}C_3^{1/2m})^n}{C_5^{2(n-1)}} \left[\Gamma\left(\frac{1}{2m}\right)\right]^{-n} [e\mathcal{O}(1)K]^{(n-1)} \\
& \times F_{zz'}^{(2/3)}(c+1)^{(n-1)} \sum_{\substack{d_1+\ldots+d_n=2n+2\\ d_i\geq 1}} \prod_{k=1}^{n}(d_k-1)! \frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!} \\
\leq\ & \sum_{n\geq N} e^n e^{n\beta C_2}\beta^{n-1} \frac{(e^{C_4-C_6}C_3^{1/2m})^n}{C_5^{2(n-1)}} \left[\Gamma\left(\frac{1}{2m}\right)\right]^{-n} [e\mathcal{O}(1)K]^{(n-1)} F_{zz'}^{(2/3)}(c+1)^{(n-1)}4^n,
\end{aligned}$$

where we used the innequality

$$\sum_{\substack{d_1+\ldots+d_n=2n+2\\ d_i\geq 1}} 1 \leq 4^n.$$

Hence,

$$\begin{aligned}
\sum_{\substack{R\subset\Lambda:|R|\geq 2\\ z,z'\in R}} |\rho(R)|e^{|R|} \leq\ & e^N e^{N\beta C_2}\beta^N \frac{(e^{C_4-C_6}C_3^{1/2m})^N}{C_5^{2N}} \left[\Gamma\left(\frac{1}{2m}\right)\right]^{-N} [e\mathcal{O}(1)K]^N F_{zz'}^{(2/3)} \\
& \times (c+1)^N 4^N \sum_{n\geq 0} \left\{ e^2 e^{\beta C_2}\beta \frac{(e^{C_4-C_6}C_3^{1/2m})}{C_5^2} \left[\Gamma\left(\frac{1}{2m}\right)\right]^{-1} \mathcal{O}(1)4(c+1)K \right\}^n.
\end{aligned}$$

In short, we have proved the following result.

Lemma 3 *If β and $\beta K = \max\{\frac{J}{2}, \lambda D_1, \ldots, \lambda D_c\}$ are sufficiently small such that*

$$\varepsilon(\beta, K) \equiv e^2 e^{\beta C_2} 4(c+1)\beta K \frac{(e^{C_4-C_6}C_3^{1/2m})}{C_5^2} \left[\Gamma\left(\frac{1}{2m}\right)\right]^{-1} \mathcal{O}(1) < 1,$$

then, $\varepsilon(\beta, K)$ is a positive function and, for any $z \in \Lambda, z' \in \Lambda$ with $z \neq z'$

$$\sum_{\substack{R\subset\Lambda:|R|\geq 2\\ z,z'\in R}} |\rho(R)|e^{|R|} \leq [\varepsilon(\beta,K)]^N F^{(2/3)}_{zz'} = [\varepsilon(\beta,K)]^{|z_0-z'_0|+1} F^{(2/3)}_{zz'}. \tag{57}$$

¿From the lemma 3, we obtain

Corollary 4

$$\sup_{x\in\mathbb{Z}^{d+1}} \sum_{R:x\in R} |\rho(R)|e^{|R|} \leq \mathcal{O}(1)\varepsilon(\beta,K).$$

Proof. In fact, as $\rho(R) = 0$ if $|R| = 1$, we have

$$\begin{aligned}
\sup_{x\in\mathbb{Z}^{d+1}} \sum_{R:x\in R} |\rho(R)|e^{|R|} &= \sup_{x\in\mathbb{Z}^{d+1}} \sum_{\substack{R:x\in R\\ |R|\geq 2}} |\rho(R)|e^{|R|} \leq \sup_{x\in\mathbb{Z}^{d+1}} \sum_{\substack{z\in\mathbb{Z}^{d+1}:\\ z\neq x}} \sum_{\substack{R:|R|\geq 2\\ x,z\in R}} |\rho(R)|e^{|R|} \\
&\leq \sup_{x\in\mathbb{Z}^{d+1}} \sum_{z\in\mathbb{Z}^{d+1}:z\neq x} [\varepsilon(\beta,K)]^{|x_0-z_0|+1} F^{(2/3)}_{xz} \leq \mathcal{O}(1)\varepsilon(\beta,K),
\end{aligned}$$

since, for $w > 0$,

$$\sum_{z\in\mathbb{Z}^{d+1}:z\neq x} e^{-w|x_0-z_0|} \left[\frac{(1-\delta_{\vec{x},\vec{z}})}{|\vec{x}-\vec{z}|^p} + \delta_{\vec{x},\vec{z}}\right] = \mathcal{O}(1),$$

and so $\sum_{z\in\mathbb{Z}^{d+1}:z\neq x} F^{(2/3)}_{xz} \leq \mathcal{O}(1)$. ■

The results above establish the convergence of the cluster expansion for $\varepsilon(\beta, K)$ small enough such that $\mathcal{O}(1)\varepsilon(\beta, K) < 1$ (see condition (37)).

5 Decay of two-point truncated correlation

As it is well known, the convergence of the cluster expansion assures the decay of the correlation functions and lead to direct estimates. We present the main technical details related to the behavior of the truncated two-point function below.

Turning to the expression (43) which defines $\tilde{\rho}(R_i)$, we note that the index i of the term β^j_i is the same of the polymer R_i, and so $i \in \{1, 2, 3, \ldots, n\}$ as $j \in \{1, 2\}$.

Consider the expression (30) for $S_2(x_1; x_2)$, and recall that $i_1, i_2 \in \{1, 2, 3, \ldots, n\}$. Then, we have two distinct cases: $i_1 = i_2$ or $i_1 \neq i_2$. If $i_1 = i_2$, then $\{x_1, x_2\} \subset R_{i_1}$, $\{x_1, x_2\} \cap R_i = \emptyset\ \forall i \neq i_1$ and $\beta^1_{i_1} = \beta^1_{i_1} = \beta^2_{i_2} = \beta^2_{i_2} = 1$. If $i_1 \neq i_2$, then $x_1 \in R_{i_1}$, $x_1 \notin R_{i_2}$, $x_2 \in R_{i_2}$, $x_2 \notin R_{i_1}$, $\{x_1, x_2\} \cap R_i = \emptyset\ \forall i \notin \{i_1, i_2\}$, $\beta^1_{i_1} = \beta^2_{i_2} = 1$, and $\beta^2_{i_1} = \beta^1_{i_2} = 0$.

Hence, as $1 = (1 - \delta_{i_1,i_2}) + \delta_{i_1,i_2}$ we rewrite

$$S_2(x_1; x_2) = A_1(x_1, x_2) + A_2(x_1, x_2),$$

where

$$
\begin{aligned}
A_1(x_1,x_2) &\equiv \sum_{n\geq 1}\frac{1}{n!}\sum_{i_1,i_2=1}^{n}(1-\delta_{i_1,i_2})\sum_{\substack{R_1,\ldots,R_n\subset\Lambda,\ |R_j|\geq 2\\ R_{i_1}\ni x_1 R_{i_2}\ni x_2}}\phi^T(R_1,\ldots,R_n)\tilde\rho(R_1)\ldots\tilde\rho(R_n)\\
&= \sum_{n\geq 2}\frac{1}{(n-2)!}\sum_{\substack{R_1,\ldots,R_n\subset\Lambda,\ |R_j|\geq 2\\ R_1\ni x_1 R_2\ni x_2}}\phi^T(R_1,\ldots,R_n)\tilde\rho(R_1)\ldots\tilde\rho(R_n),\\
A_2(x_1,x_2) &\equiv \sum_{n\geq 1}\frac{1}{n!}\sum_{i_1,i_2=1}^{n}\delta_{i_1,i_2}\sum_{\substack{R_1,\ldots,R_n\subset\Lambda,\ |R_j|\geq 2\\ R_{i_1}\ni x_1 R_{i_2}\ni x_2}}\phi^T(R_1,\ldots,R_n)\tilde\rho(R_1)\ldots\tilde\rho(R_n)\\
&= \sum_{n\geq 1}\frac{1}{(n-1)!}\sum_{\substack{R_1,\ldots,R_n\subset\Lambda,\ |R_j|\geq 2\\ R_1\supset\{x_1,x_2\}}}\phi^T(R_1,\ldots,R_n)\tilde\rho(R_1)\ldots\tilde\rho(R_n),
\end{aligned}
$$

since, in $A_1(x_1,x_2)$ when $n=1$ we have $\sum_{i_1,i_2=1}^{1}(1-\delta_{i_1,i_2})=0$ and, for any $n>2$ the sum $\sum_{i_1,i_2=1}^{n}(1-\delta_{i_1,i_2})$ leads to $n(n-1)$ equal terms. And, in $A_2(x_1,x_2)$ the sum $\sum_{i_1,i_2=1}^{n}\delta_{i_1,i_2}$ gives n equal terms.

Thus,

$$|S_2(x_1;x_2)|\leq |A_1(x_1,x_2)|+|A_2(x_1,x_2)|.$$

Comparing (43) with (34), we note that if $R_i\cap\{x_1,x_2\}=\emptyset$ then $\tilde\rho(R_i)=\rho(R_i)$. If $R_i\cap\{x_1,x_2\}\neq\emptyset$, we can obtain the result (57) of the lemma 3 for $\tilde\rho(R_i)$ by changing $n_k(s)$ by $n_k(s)+1$. With such result, we change $\Gamma(d_k)$ by $\Gamma(d_k+1)$ in (51) and obtain an extra $\prod_{k=1}^{n}d_k$, which is bounded by $e^{2(n-1)}$. We use the lemma 2 to bound the factor $l=\int d\nu(\phi)\phi$ in (43). Hence, we can apply the lemma 3 and corollary 4 to estimate $\tilde\rho(R_i)$, (changing some multiplicative constants).

Now, let us find an upper bound for the term $|A_1(x_1,x_2)|$. We have

$$|A_1(x_1,x_2)|\leq\sum_{n\geq 2}\frac{1}{(n-2)!}B_n(x_1,x_2),$$

where

$$B_n(x_1,x_2)=\sum_{\substack{R_1,\ldots,R_n\subset\Lambda\\ |R_i|\geq 2,\, x_1\in R_1, x_2\in R_2}}\left|\phi^T(R_1,R_2,\ldots,R_n)|\tilde\rho(R_1)||\tilde\rho(R_2)||\tilde\rho(R_3)\ldots\tilde\rho(R_n)\right|.$$

Note that in (36), for $n\geq 2$, $\phi^T(R_1,\ldots,R_n)>0$ only if $g(R_1,\ldots,R_n)\in G_n$. Thus,

$$\sum_{\substack{R_1,\ldots,R_n\subset\Lambda\\ |R_i|\geq 2,\, x_1\in R_1, x_2\in R_2}}\left|\phi^T(R_1,R_2,\ldots,R_n)\right|[\cdot]=\sum_{g\in G_n}\left|\sum_{\substack{f\in G_n\\ \subset g}}(-1)^{|f|}\right|\sum_{\substack{R_1,\ldots,R_n\subset\Lambda:\,|R_i|\geq 2\\ g(R_1,\ldots,R_n)=g,\, x_1\in R_1, x_2\in R_2}}[\cdot].$$

By the Rota formula [19], we have

$$\left|\sum_{\substack{f\in G_n\\ f\subset g}}(-1)^{|f|}\right|\leq\sum_{\tau\in T_n:\tau\subset g}1\equiv N(g).$$

A proof of the Rota formula above can be found e.g. in [19] and [20].

We recall now that

$$\sum_{g\in G_n}[\cdot]=\sum_{\tau\in T_n}\sum_{g:\,\tau\subset g}\frac{1}{N(g)}[\cdot],$$

since in the double sum $\sum_\tau\sum_{g\supset t}$ each g will be repeated exactly $N(g)$ times.

Thus,

$$B_n(x_1,x_2)\le\sum_{\tau\in T_n}w_n(\tau,x_1,x_2),$$

where we have defined

$$w_n(\tau,x_1,x_2)\equiv\sum_{\substack{R_1,\dots,R_n\subset\Lambda:\,|R_i|\ge 2\\ g(R_1,R_2,\dots,R_n)\supset\tau,\,x_1\in R_1,x_2\in R_2}}|\tilde\rho(R_1)||\tilde\rho(R_2)||\tilde\rho(R_3)\dots\tilde\rho(R_n)|.$$

Using now the obvious bound

$$\sum_{R:\,R\cap R'\neq\emptyset}|\cdot|\le|R'|\sup_{x\in R'}\sum_{R:\,x\in R}|\cdot|,$$

and denoting again as $\bar\tau$ the subtree of τ which is the unique path joining vertex 1 to vertex 2, and denoting as $I_\tau=\{1,i_1,\dots,i_k,2\}$ the ordered set of the vertices of $\bar\tau$, one can easily check that

$$\begin{aligned}
w_n(\tau,x_1,x_2)\le&\prod_{i\notin I_\tau}^{n}\left[\sup_{x\in Z^d}\sum_{R_i:\,x\in R_i}|R_i|^{d_i-1}|\tilde\rho(R_i)|\right]\\
&\times\sum_{\substack{R_1,R_{i_1},\dots,R_{i_k},R_2:x_1\in R_1,x_2\in R_2\\ R_1\cap R_{i_1}\neq\emptyset,\dots R_{i_k}\cap R_2\neq\emptyset}}|R_1|^{d_1-1}|\tilde\rho(R_1)||R_2|^{d_2-1}|\tilde\rho(R_2)|\prod_{\substack{i\in I_\tau\\ i\neq1,2}}^{n}|R_i|^{d_i-2}|\tilde\rho(R_i)|\\
\le&\prod_{i\notin I_\tau}^{n}\left[\sup_{x\in Z^d}\sum_{R_i:\,x\in R_i}(d_i-1)!|\tilde\rho(R_i)|e^{|R_i|}\right](d_1-1)!(d_2-1)!\\
&\times\sum_{\substack{R_1,R_{i_1},\dots,R_{i_k},R_2:x_1\in R_1,x_2\in R_2\\ R_1\cap R_{i_1}\neq\emptyset,\dots R_{i_k}\cap R_2\neq\emptyset}}|\tilde\rho(R_1)|e^{|R_1|}|\tilde\rho(R_2)|e^{|R_2|}\prod_{\substack{i\in I_\tau\\ i\neq1,2}}(d_i-2)!|\tilde\rho(R_i)|e^{|R_i|},
\end{aligned}$$

since $|R|^n\le n!e^{|R|}$. Now, note that

$$\sum_{\substack{R_1,R_{i_1},\dots,R_{i_k},R_2:x_1\in R_1,x_2\in R_2\\ R_1\cap R_{i_1}\neq\emptyset,\dots R_{i_k}\cap R_2\neq\emptyset}}\le\sum_{x_{i_0}\in Z^d}\sum_{x_{i_1}\in Z^d}\dots\sum_{x_{i_k}\in Z^d}\sum_{\substack{R_1\\ x_1,x_{i_0}\in R_1}}\sum_{\substack{R_{i_1}\\ x_{i_0},x_{i_1}\in R_{i_1}}}\dots\sum_{\substack{R_{i_k}\\ x_{i_{k-1}},x_{i_k}\in R_{i_k}}}\sum_{\substack{R_2\\ x_{i_k},x_2\in R_2}}.$$

Hence, recalling (57) and applying iteratively the inequality (54) with $w_1=1/2$ and $w_2=2/3$,

$$\begin{aligned}
&\sum_{\substack{R_1,R_{i_1},\dots,R_{i_k},R_2:x_1\in R_1,x_2\in R_2\\ R_1\cap R_{i_1}\neq\emptyset,\dots R_{i_k}\cap R_2\neq\emptyset}}|\tilde\rho(R_1)|e^{|R_1|}|\tilde\rho(R_2)|e^{|R_2|}\prod_{i\in I_\tau}|\tilde\rho(R_i)|e^{|R_i|}\\
\le&\sum_{x_{i_0}\in Z^d}\sum_{x_{i_1}\in Z^d}\dots\sum_{x_{i_k}\in Z^d}[\varepsilon(\beta,K)]^{|(x_1)_0-(x_{i_0})_0|+1}F^{(2/3)}_{x_1x_{i_0}}\dots[\varepsilon(\beta,K)]^{|(x_{i_k})_0-(x_2)_0|+1}F^{(2/3)}_{x_{i_k}x_2}\\
\le&[\mathcal{O}(1)]^{k+1}[\varepsilon(\beta,K)]^{|(x_1)_0-(x_2)_0|+(k+2)}F^{(1/2)}_{x_1x_2},
\end{aligned}$$

since $\varepsilon(\beta,K)<1$ and $|(x_1)_0-(x_2)_0|\le|(x_1)_0-(x_{i_0})_0|+|(x_{i_0})_0-(x_{i_1})_0|+\ldots+|(x_{i_k})_0-(x_2)_0|$.

Thus, using the corollary 4 and noting that $|\{1,...,n\}\backslash I_\tau|=n-k-2$,

$$\begin{aligned}
w_n(\tau,x_1,x_2) &\le (d_1-1)!(d_2-1)!\left[\prod_{i\notin I_\tau}^{n}\sup_{x\in Z^d}\sum_{R_i:\,x\in R_i}(d_i-1)!|\rho(R_i)|e^{|R_i|}\right]\\
&\quad\times\left[\prod_{\substack{i\in I_\tau\\ i\neq 1,2}}^{n}(d_i-2)!\right][\varepsilon(\beta,K)]^{|(x_1)_0-(x_2)_0|+(k+2)}[\mathcal{O}(1)]^{k+1}F^{(1/2)}_{x_1x_2}\\
&\le [\mathcal{O}(1)]^n[\varepsilon(\beta,K)]^{|(x_1)_0-(x_2)_0|+n}F^{(1/2)}_{x_1x_2}\prod_{i=1}^{n}(d_i-1)!.
\end{aligned}$$

Finally, carrying out the sum over τ (and using, once again, the Cayley formula) we obtain

$$B_n(x_1,x_2)\le(n-2)![4\mathcal{O}(1)]^n[\varepsilon(\beta,K)]^{|(x_1)_0-(x_2)_0|+n}F^{(1/2)}_{x_1x_2}.$$

Taking β,K small enough to make $4\mathcal{O}(1)\varepsilon(\beta,K)<1$, for the contribution of A_1 to the correlations, we get the following bound:

$$\begin{aligned}
|A_1(x_1,x_2)| &\le \sum_{n\ge2}[4\mathcal{O}(1)]^n[\varepsilon(\beta,K)]^{|(x_1)_0-(x_2)_0|+n}F^{(1/2)}_{x_1x_2}\\
&\le \mathcal{O}(1)[\varepsilon(\beta,K)]^{|(x_1)_0-(x_2)_0|+2}F^{(1/2)}_{x_1x_2}.
\end{aligned}$$

In a similar and much easier way one can also prove a completely analogous bound for $|A_2(x_1,x_2)|$

$$|A_2(x_1,x_2)|\le\mathcal{O}(1)[\varepsilon(\beta,K)]^{|(x_1)_0-(x_2)_0|+1}F^{(1/2)}_{x_1x_2}.$$

Hence,

$$\begin{aligned}
|S_2(x;y)| &\le \mathcal{O}(1)[\varepsilon(\beta,K)]^{|x_0-y_0|}F^{(1/2)}_{xy}\\
&\le \mathcal{O}(1)[\varepsilon(\beta,K)]^{|x_0-y_0|}e^{-|x_0-y_0|/2}\left(\frac{1-\delta_{\vec{x},\vec{y}}}{|\vec{x}-\vec{y}|^p}+\delta_{\vec{x},\vec{y}}\right)\\
&\le \mathcal{O}(1)e^{-m'(\beta,K)|x_0-y_0|}\left(\frac{1-\delta_{\vec{x},\vec{y}}}{|\vec{x}-\vec{y}|^p}+\delta_{\vec{x},\vec{y}}\right),
\end{aligned}$$

where, since $\varepsilon(\beta,K)<1$, we write above

$$m'(\beta,K)\equiv-\log\{\varepsilon(\beta,K)\}+1/2>0.$$

These results prove part 2 of theorem 1 (the part 1 is a simpler case and may be proved in a similar way).

A lower bound can be obtained, as said, by analogous procedures (see e.g. [13]).

6 Four-point truncated correlation

Now we analyze the "time" decay of the partially truncated four-point function in order to determine the mass spectrum in the interval $(M,2M)$, where M is the one-particle mass. In this section, we restrict the interaction to even function of the spin variable ϕ.

We use the BS equation, and omit several technical details (related to those presented e.g. in [6]). The computation is carried out in a perturbative approach: we keep only the dominant term in a β expansion of the BS kernel.

Roughly, we show that, for the model considered here, in the intense noise regime $\beta \ll 1$, if the polynomial $U(\phi)$ is such that $\langle\phi^4\rangle > 3\langle\phi^2\rangle^2$, where $\langle\cdot\rangle$ is the expectation with respect to the single spin distribution $d\nu(\phi)$ (31), there is an isolated two-particle bound state with mass $M^* = 2M + \log(1-\zeta) + \mathcal{O}(\beta)$, where $\zeta = [\langle\phi^4\rangle - 3\langle\phi^2\rangle]/[\langle\phi^4\rangle - \langle\phi^2\rangle]$.

Sketch of the proof. We use the BS equation which writes the truncated four-point function $D(x_1, x_2; x_3, x_4) = S_4(x_1, x_2, x_3, x_4) - S_2(x_1, x_2)S_2(x_3, x_4)$ as

$$D(x_1, x_2; x_3, x_4) = D_0(x_1, x_2; x_3, x_4) + \sum_{y_1,y_2,y_3,y_4} D(x_1, x_2; y_1, y_2)K(y_1, y_2; y_3; y_4)D_0(y_3, y_4; x_3, x_4),$$

where $D_0(x_1, x_2; x_3, x_4) \equiv S_2(x_1, x_3)S_2(x_2, x_4) + S_2(x_1, x_4)S_2(x_2, x_3)$.

Due to translation invariance, D depends only on difference variable, i.e., $D = D(\xi, \eta, \tau)$ where here we take $\xi = x_2 - x_1$, $\eta = x_4 - x_3$ and $\tau = x_3 - x_2$. Using p, q and k, the conjugate variable (the Fourier transform) of ξ, η, τ, the BS equation becomes

$$\tilde{D}(p, q, k) = \tilde{D}_0(p, q, k) + \frac{1}{(2\pi)^{2(d+1)}} \int_{T^{d+1}} d^{d+1}p'\ d^{d+1}q'\ \tilde{D}(p, p', k)\tilde{K}(p', q', k)\tilde{D}_0(q', q, k),$$

or using the notation $(\tilde{D}(k)f)(p) \equiv \int_{T^{d+1}} d^{d+1}q\ \tilde{D}(p, q, k)f(q)$,

$$\tilde{D}(k) = \tilde{D}_0(k) + \frac{1}{(2\pi)^{2(d+1)}}\tilde{D}(k)\tilde{K}(k)\tilde{D}_0(k) = \tilde{D}_0(k)\left[1 - (2\pi)^{-2(d+1)}\tilde{K}(k)\tilde{D}_0(k)\right]^{-1}.$$

We restrict the analysis to the mass spectrum, i.e., we take $k = (k_0, \vec{k} = \vec{0})$, and consider $f(p)$ depending only on $\vec{p}$. From the definition of D_0, we get

$$\tilde{D}_0(p, q, k) = (2\pi)^{d+1}[\tilde{S}_2(p)\tilde{S}_2(q)\delta(k - p - q) + \tilde{S}_2(p)\tilde{S}_2(k - p)\delta(q - p)]. \tag{58}$$

Hence, $(\tilde{D}_0(k_0)f)(p) = (2\pi)^{d+1}\tilde{S}_2(p)\tilde{S}_2(k - p)[f(\vec{p}) + f(-\vec{p})]$. For $f(\vec{p}) = f(-\vec{p})$, we obtain

$$\begin{aligned}(\tilde{f}, \tilde{D}(k_0)\tilde{f}) = \quad & \int \tilde{f}(\vec{p}) \left\{2(2\pi)^{d+1} \int dp_0\ \tilde{S}_2(p)\tilde{S}_2(k_0 - p_0, -\vec{p})\right\} \\ & \times \left(\left[1 - \frac{1}{(2\pi)^{2(d+1)}}\tilde{K}(k_0)\tilde{D}_0(k_0)\right]^{-1} \tilde{f}\right)(\vec{p})d^dp. \end{aligned} \tag{59}$$

As $\{\ldots\}(k_0, \vec{p})$ above is analytic in $|Im\ k_0| < 2M$, the singularities in such region come from where the inverse of $1 - (2\pi)^{-2(d+1)}\tilde{K}(k_0)\tilde{D}_0(k_0)$ does not exist.

Now we follow replacing $\tilde{K}$ by the leading term in the perturbation expansion (in terms of β), which is named ladder approximation. We remark (once more time), as a comment on the reliability of such procedure, that in [6, 14, 15] a rigorous analysis (in similar "field theory") shows that the spectral properties calculated in the ladder approximation are maintained.

Using $K = D_0^{-1} - D^{-1}$ we obtain

$$\tilde{K}(p, q, k)|_{\mathcal{O}(\beta=0)} = \frac{\langle\phi^4\rangle - 3\langle\phi^2\rangle^2}{2\langle\phi^2\rangle^2\,[\langle\phi^4\rangle - \langle\phi^2\rangle^2]} = R \tag{60}$$

i.e., constant (not depending on β), local in space and time. Writing $[1-A]^{-1} = 1 + [1-A]^{-1}A$ we get

$$(\tilde{f}, \tilde{D}_0(k_0)\tilde{f}) = \int \overline{\tilde{f}}(\vec{p})\tilde{D}_0(p,q,k_0)\tilde{f}(\vec{q})\ dpdq + + \frac{R'}{1-R'I_D}\left(\int \overline{\tilde{f}}(\vec{p})\tilde{D}_0(p,q,k_0)\ dpdq\right)\left(\int \tilde{D}_0(p',q',k_0)\tilde{f}(\vec{q'})\ dp'dq\right) \quad (61)$$

where $dp \equiv d^{d+1}p$, etc, $R' = R/(2\pi)^{2(d+1)}$ and

$$I_D = \int dq'dq\ \tilde{D}_0(q',q,k_0) = 2(2\pi)^{d+1}\int dp\ \tilde{S}_2(p)\tilde{S}_2(k_0-p_0,-\vec{p}).$$

Thus, the singularity, and so the bound state, comes for $R'I_D = 1$. We follow the calculations separating in the expression for $\tilde{S}_2(p)$ the dominant one-particle contribution (as usual: we take the Lehmann spectral representation, see e.g. [12, 6])

$$\tilde{S}_2(p) = (2\pi)^d\tilde{c}_2(\vec{p})\frac{\sinh M(\vec{p})}{\cosh M(\vec{p}) - \cos p_0} + \int_{3M-\varepsilon}^{\infty}\frac{\sinh E}{\cosh E - \cos p_0}\ d\eta(E,\vec{p}),$$

where $\tilde{c}_2(\vec{p}) = \partial\tilde{\Gamma}(p_0 = i\xi,\vec{p})/\partial\xi|_{\xi=M(\vec{p})}$. We have, for the (first) dominant term, $\tilde{c}_2(\vec{p}) = \frac{\langle\phi^2\rangle}{(2\pi)^d} + \mathcal{O}(\beta)$. Thus, for the computation of (the leading part of) I_D we use $\tilde{S}_2(p) = \langle\phi^2\rangle\frac{\sinh M(\vec{p})}{\cosh M(\vec{p})-\cos p_0}$. Hence,

$$I_D = 2(2\pi)^{d+1}\int dp\langle\phi^2\rangle^2\frac{(e^{M(\vec{p})} - e^{-M(\vec{p})})^2}{(e^{M(\vec{p})} + e^{-M(\vec{p})} - e^{ip_0} - e^{-ip_0})(e^{M(\vec{p})} + e^{-M(\vec{p})} - e^{ik_0-ip_0} - e^{-ik_0+ip_0})}.$$

Writing $ik_0 = 2M - \varepsilon$, we get (taking the leading term $\beta = 0$) $I_D = 2(2\pi)^{2(d+1)}\langle\phi^2\rangle^2/(1-e^{-\varepsilon})$. And for the eigenvalue equation $R'I_D = 1$ we obtain

$$1 - e^{-\varepsilon} = \frac{\langle\phi^4\rangle - 3\langle\phi^2\rangle^2}{\langle\phi^4\rangle - \langle\phi^2\rangle^2} \equiv \zeta, \quad (62)$$

which leads, if $\zeta > 0$, to the bound state mass

$$M^* = 2M + \ln(1-\zeta). \quad (63)$$

■

We have the same bound state mass expression of [15], there calculated for a ferromagnetic system with continuous spin, nearest neighbor interaction and even single spin distribution.

As a final comment, let us say that we expect to apply the techniques presented here in the analysis of the correlation functions of other stochastic dynamical systems, such as these described by an anharmonic chain of oscillators, with conservative dynamics and different stochastic reservoirs [21] (in our case, for the intense noise, i.e., "high temperature" regime).

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