Derived categories of Schur algebras

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Abstract

In this paper we classify the derived tame Schur and infinitesimal Schur algebras and describe indecomposable objects in their derived categories.

1 Introduction

Let A be a finite-dimensional algebra of the form kQ/I over an algebraically closed field k, $A-\text{mod}$ be the category of left A-modules and let $\mathbf{D}^b(A)$ be the bounded derived category of the category $A-\text{mod}$. The category $\mathbf{D}^b(A)$ is well-understood for some classes of algebras A. For example, the description of indecomposable objects of $\mathbf{D}^b(A)$ is well-known for hereditary algebras of finite and tame type [H] and for the tubular algebras [HR].

We say that A is derived tame (see [GK]) if for each sequence $\mathbf{n} = (n_i)_{i\in\mathbb{Z}}$ of positive integers the indecomposable complexes in $\mathbf{D}^b(A)$ of cohomology dimension $\mathbf{n}$ can be parametrized using only one continuous parameter.

The goal of this paper is to prove the following theorem.

Theorem 1. *Let p be the characteristic of k.*

(i) Let $S = S(n,d)$ be a Schur algebra. Then S is derived tame if and only if one of the following holds:

a) $p > d$;

b) $p = 2$, $n = 2$, $d = 3$;

c) $p = 2$, $n = 2$, $d = 5$ or $d = 7$;

d) $n = 2$, $p \leq d < 2p$ and $d \neq 3$ if $p = 2$;

e) $p = 2$, $n \geq 3$, $d = 2$ or $d = 3$;

f) $p = 3$, $n = 3$, $d = 4$ or $d = 5$.

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(ii) Let $S = S(n,d)_r$ be an infinitesimal Schur algebra. Then S is derived tame if and only if one of the following holds:

a) $p > 2$, $n \geq 2$, $d < p$;

b) $p = 2$, $n = 2$, $d = 3$;

c) $p = 2$, $n = 2$, d odd, $r = 1$;

d) $p = 2$, $n = 2$, $r = 2$, $d = 5$ or $d = 7$;

e) $p = 2$, $n = 2$, $r = 1$, $d = 2$;

f) $n = 2$, $r = 1$, $2 < p \leq d < 2p$.

The structure of the paper is the following. In Section 2 the definition of Schur algebras and some preliminary results from [E] and [DEMN] are given. In Section 3 we show that the problem of classification of all indecomposable objects in $\mathbf{D}^b(A)$ can be reduced to the problem of classification of indecomposable objects in the category $\mathfrak{p}(A)$, some subcategory of the category of bounded projective complexes $\mathbf{C}^b(A-\text{pro})$ and we prove Theorem 1. In Section 4 the description of all indecomposable objects in $\mathbf{D}^b(A)$ for derived tame Schur and infinitesimal Schur algebras is given.

2 Preliminaries

Let V be an n-dimensional vector space over an algebraically closed field of characteristic p. Then the symmetric group S_d has a natural action on $V^{\otimes d}$ which makes it a module for the group algebra of S_d. The endomorphism ring of the module $V^{\otimes d}$ is the Schur algebra $S(n,d)$. There is an equivalence between the category of modules for $S(n,d)$ and the category of polynomial representations of $GL(n)$ which are homogeneous of degree d [G]. Schur algebras are quasi-hereditary [P] like the algebras corresponding to the blocks of category $\mathcal{O}$ [BGG]. All Schur algebras of finite representation type, i.e., those algebras that have only finitely many non-isomorphic indecomposable modules, were classified in [E].

The Schur algebra is of finite representation type if one of the following holds:

a) $n = 2$ and $d < p^2$ or $n \geq 3$ and $d < 2p$;
b) $p = n = 2$ and $d = 5$ or $d = 7$.

Moreover, the corresponding blocks are Morita equivalent to one of the algebras A_m below:

$$\underset{m\geq 1}{A_m}: \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \overset{3}{\bullet} \arrow[l, shift left, "\beta_2"] & \cdots & \overset{m-1}{\bullet} \arrow[r, shift left, "\alpha_{m-1}"] & \overset{m}{\bullet} \arrow[l, shift left, "\beta_{m-1}"] \end{tikzcd} \qquad \begin{array}{l} \alpha_i\alpha_{i+1} = 0, \beta_{i+1}\beta_i = 0, \\ \beta_i\alpha_i = \alpha_{i+1}\beta_{i+1}, \alpha_1\beta_1 = 0. \end{array}$$

According to [D], infinite representation type Schur algebras either have tame or wild representation type. The algebra is of tame representation type if it has at most finitely many one-parameter families of indecomposable modules in each dimension. Otherwise, the algebra is wild. The Schur algebras of tame representation type were classified in [DEMN] and are covered by the following cases: a) $p = n = 3$ and $d = 7$ or $d = 8$;

b) $p = 3$, $n = 2$, $d = 9$ or $d = 10$ or $d = 11$;
c) $p = n = 2$, $d = 4$ or $d = 9$.

The corresponding non-semisimple blocks are Morita equivalent to the algebras given by the following quivers with relations.

1. For algebras $S(2,4)$, $S(2,9)$, $p = 2$

$$D_3 : \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\beta_2"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha_2"] \end{tikzcd} \qquad \begin{array}{l} \alpha_1\beta_1 = \alpha_2\beta_2 = 0, \\ \alpha_1\beta_2\alpha_2 = \beta_2\alpha_2\beta_1 = 0. \end{array}$$

2. For algebras $S(2,9)$, $S(2,10)$, $S(2,11)$, $p = 3$

$$D_4 : \begin{tikzcd} & \overset{2}{\bullet} \arrow[d, shift left, "\alpha_2"] & \\ \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \underset{0}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[u, shift left, "\beta_2"] \arrow[r, shift left, "\beta_3"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha_3"] \end{tikzcd} \qquad \begin{array}{l} \alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_1 = \alpha_3\beta_2 = 0, \\ \alpha_1\beta_3 = \alpha_2\beta_3 = 0, \beta_2\alpha_2 = \beta_3\alpha_3, \\ \alpha_1\beta_2\alpha_2 = \beta_2\alpha_2\beta_1 = 0. \end{array}$$

3. For the algebra $S(3,7)$, $p = 3$

$$R_4 : \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \overset{3}{\bullet} \arrow[l, shift left, "\beta_2"] \arrow[r, shift left, "\alpha_3"] & \overset{4}{\bullet} \arrow[l, shift left, "\beta_3"] \end{tikzcd} \qquad \begin{array}{l} \alpha_1\beta_1 = \alpha_1\alpha_2 = \beta_2\beta_1 = 0, \\ \beta_1\alpha_1 = \alpha_2\beta_2, \beta_2\alpha_2 = \alpha_3\beta_3. \end{array}$$

4. For the algebra $S(3,8)$, $p = 3$

$$H_4 : \begin{tikzcd} & \overset{2}{\bullet} \arrow[d, shift left, "\alpha_2"] & \\ \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \underset{0}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[u, shift left, "\beta_2"] \arrow[r, shift left, "\beta_3"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha_3"] \end{tikzcd} \qquad \begin{array}{l} \alpha_1\beta_1 = \alpha_1\beta_2 = \alpha_1\beta_3 = 0, \\ \alpha_3\beta_1 = \alpha_3\beta_3 = \alpha_2\beta_1 = 0, \\ \beta_2\alpha_2 = \beta_1\alpha_1 + \beta_3\alpha_3. \end{array}$$

In [DNP1] the authors introduced certain subalgebras $S(n,d)_r$ which they called *infinitesimal Schur algebras*. The representation theory of these algebras is connected to the theory of polynomial representations of the group scheme G_rT. Here $G = GL(n)$ over an algebraically closed field of characteristic $p > 0$, G_rT is the inverse image of the diagonal torus $T \subset G$ under the rth iteration of the Frobenius morphism. The infinitesimal Schur algebras $S(n,d)_r$ of finite representation type were classified in [DNP2]. They belong to the following cases:

a) $p \geq 2$, $n \geq 3$, $d < 2p$, $r \geq 2$;
b) $p \geq 2$, $n \geq 3$, $d < p$, $r = 1$;
c) $p = 3$, $n = 3$, $r = 1$, $d = 4$ or $d = 5$;
d) $p = 2$, $n = 3$, $r = 1$, $d = 2$ or $d = 3$;
e) $p \geq 2$, $n = 2$, $d < p^2$, $r \geq 2$;
f) $p = 2$, $n = 2$, $r \geq 3$, $d = 5$ or $d = 7$;
g) $p = 2$, $n = 2$, $r = 2$, d odd;
h) $p \geq 2$, $n = 2$, $r = 1$.

The infinitesimal Schur algebras in cases a), b), e), f) coincide with the corresponding Schur algebras. In all other cases the corresponding non-semisimple blocks are Morita equivalent to the algebras given by the following quivers with relations.

1. For algebras $S(3,4)_1$, $S(3,5)_1$, $p = 3$ and $S(3,2)_1$, $S(3,3)_1$, $p = 2$

$$G: \quad \begin{tikzcd} & \overset{2}{\bullet} \arrow[d, shift left, "\alpha_2"] & \\ \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \underset{0}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[u, shift left, "\beta_2"] \arrow[r, shift left, "\beta_3"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha_3"] \end{tikzcd} \qquad \begin{array}{l} \beta_1\alpha_1 = \beta_2\alpha_2 = \beta_3\alpha_3, \\ \text{all other products} = 0. \end{array}$$

2. For algebras $S(2,d)_2$, d odd, $p = 2$ and $S(2,d)_1$, $p \geq 2$

$$\underset{\substack{m=2n+1\\ n>0}}{F_m}: \quad \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \overset{3}{\bullet} \arrow[l, shift left, "\beta_2"] & \cdots & \overset{m-1}{\bullet} \arrow[r, shift left, "\alpha_{m-1}"] & \overset{m}{\bullet} \arrow[l, shift left, "\beta_{m-1}"] \end{tikzcd} \qquad \begin{array}{l} \alpha_i\alpha_{i+1} = 0, \beta_{i+1}\beta_i = 0, \alpha_1\beta_1 = 0, \\ \beta_i\alpha_i = \alpha_{i+1}\beta_{i+1}, \beta_{m-1}\alpha_{m-1} = 0. \end{array}$$

The infinitesimal Schur algebras of tame representation type were classified in [DEMN]. They belong to the following classes:

a) $p \geq 5$, $n = 3$, $p \leq d \leq 2p - 1$, $r = 1$;
b) $p = 3$, $n = 3$, $d = 3$, $r = 1$;
c) $p = 3$, $n = 4$, $r = 1$, $d = 3$ or $d = 4$ or $d = 5$;
d) $p = 3$, $n = 3$, $r \geq 2$, $d = 7$ or $d = 8$;
e) $p = 3$, $n = 2$, $r \geq 3$, $d = 9$ or $d = 10$ or $d = 11$;

f) $p = 2$, $n = 4$, $r = 1$, $d = 2$ or $d = 3$;
g) $p = 2$, $n = 2$, $d = 4$, $r \geq 2$;
h) $p = 2$, $n = 2$, $d = 9$, $r \geq 3$.

The algebras in e) have non-semisimple blocks of type D_4. The algebra $S(3,7)_r$, $r \geq 2$, $p = 3$ has a block of type R_4. The algebra $S(3,8)_r$, $r \geq 2$, $p = 3$ has a block of type H_4. All algebras in g) with $r > 2$ and all algebras in h) with $r > 3$ have blocks of type D_3. Other algebras have the blocks which are Morita equivalent to the following quivers with relations:

1. For algebras $S(4,3)_1$, $S(4,4)_1$, $S(4,5)_1$, $p = 3$ and $S(4,2)_1$, $S(4,3)_1$, $p = 2$

$$B: \quad \begin{tikzcd} & \overset{2}{\bullet} \arrow[d, shift left, "\alpha_2"] & \\ \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \overset{0}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[u, shift left, "\beta_2"] \arrow[r, shift left, "\beta_3"] \arrow[d, shift left, "\beta_4"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha_3"] \\ & \underset{4}{\bullet} \arrow[u, shift left, "\alpha_4"] & \end{tikzcd} \qquad \begin{array}{l} \beta_1\alpha_1 = \beta_2\alpha_2 = \beta_3\alpha_3 = \beta_4\alpha_4, \\ \text{all other products} = 0. \end{array}$$

2. For algebras $S(3,d)_1$, $p \geq 5$, $p \leq d \leq 2p-1$ and $S(3,3)_1$, $p = 3$

$$B_1: \quad \begin{tikzcd} & \overset{2}{\bullet} \arrow[d, shift left, "\alpha_2"] & \\ \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \overset{0}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[u, shift left, "\beta_2"] \arrow[r, shift left, "\beta_3"] \arrow[d, shift left, "\beta_4"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha_3"] \\ & \underset{4}{\bullet} \arrow[u, shift left, "\alpha_4"] & \end{tikzcd} \qquad \begin{array}{l} \beta_1\alpha_1 = \beta_2\alpha_2 = \beta_3\alpha_3 = \beta_4\alpha_4, \\ \alpha_i\beta_4 = \beta_i\alpha_i = 0 \text{ for } i = 1,2,3. \end{array}$$

3. For algebras $S(2,4)_2$, $S(2,9)_3$, $p = 2$

$$D: \quad \begin{tikzcd} & \overset{2}{\bullet} \arrow[d, shift left, "\alpha_2"] & \\ \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \underset{0}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[u, shift left, "\beta_2"] \arrow[r, shift left, "\beta_3"] & \overset{3}{\bullet} \arrow[l, shift left, "\alpha_3"] \end{tikzcd} \qquad \begin{array}{l} \beta_1\alpha_1 = \beta_2\alpha_2, \alpha_1\beta_1 = \alpha_1\beta_2 = 0, \\ \alpha_2\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 = 0, \\ \alpha_1\beta_3\alpha_3 = \alpha_2\beta_3\alpha_3 = \beta_3\alpha_3\beta_2 = \beta_3\alpha_3\beta_1 = 0. \end{array}$$

3 Derived representation type

Let A be a finite-dimensional algebra of the form kQ/I over an algebraically closed field k, $A-\mathrm{mod}$ be the category of left A-modules. We will follow in general the notation and terminology of [Ri] and [H].

Given A, we denote by $\mathbf{D}(A)$ (resp., $\mathbf{D}^-(A)$ or $\mathbf{D}^b(A)$) the derived category of $A-\mathrm{mod}$ (resp., the derived category of right bounded complexes of $A-\mathrm{mod}$ or the derived category of bounded complexes of $A-\mathrm{mod}$); by $\mathbf{C}^b(A-\mathrm{pro})$ (resp., $\mathbf{C}^-(A-\mathrm{pro})$ or $\mathbf{C}^{-,b}(A-\mathrm{pro})$) the category of bounded projective complexes (resp., of right bounded projective complexes or of right bounded projective complexes with bounded cohomology (that is, complexes of projective modules with the property that the cohomology groups are non zero only at a finite number of places)); and by $\mathbf{K}^b(A-\mathrm{pro})$ (resp., $\mathbf{K}^-(A-\mathrm{pro})$ or $\mathbf{K}^{-,b}(A-\mathrm{pro})$) the corresponding homotopy categories.

We identify the homotopy category $\mathbf{K}^b(A-\mathrm{pro})$ with the full subcategory of perfect complexes in $\mathbf{D}^b(A)$. Recall that a complex is perfect if it is isomorphic to a bounded complex of finitely generated projective A-modules.

We will also use the following notations. By $\mathfrak{p}(A)$ we denote the full subcategory of $\mathbf{C}^b(A-\mathrm{pro})$ defined by the projective complexes such that the image of every differential map is contained in the radical of the corresponding projective module. Since any projective complex is the sum of one complex with this property and two complex where, alternativelly, all differential maps are 0's or isomorphisms (which is, hence, isomorphic to the zero object in the derived category) we can always assume that we reduce ourselves to consider projective complexes of this form.

It is well known that $\mathbf{D}^b(A)$ is equivalent to $\mathbf{K}^{-,b}(A-\mathrm{pro})$ (see, for example, [KZ], Prop. 6.3.1 and [Har]).

Proposition 1. *[Har]* $\mathbf{D}^-(A)$ *is equivalent to* $\mathbf{K}^-(A-\mathrm{pro})$*. The image of* $\mathbf{D}^b(A)$ *under this equivalence is* $\mathbf{K}^{-,b}(A-\mathrm{pro})$.

Given $M^\bullet \in \mathbf{D}^b(A)$ we denote by $P^\bullet_{M^\bullet}$ the projective resolution of $M^\bullet$ (see [KZ]) and by $H^i(M^\bullet)$ the i-th cohomology module.

We call a category $\mathcal{C}$ *basic* if it satisfies the following conditions:

- all its objects are pairwise non-isomorphic;
- for each object x there are no non-trivial idempotents in $\mathcal{C}(x,x)$.

A full subcategory $\mathcal{S} \subset \mathcal{C}$ is called a *skeleton* of $\mathcal{C}$ if it is basic and each object $x \in \mathcal{C}$ is isomorphic to a direct summand of a (finite) direct sum of some objects of $\mathcal{S}$. It is evident that if $\mathcal{C}$ is a category with unique direct decomposition property, then it has a skeleton and the last one is unique up to isomorphism. We will denote it by $\mathbf{Sk}\,\mathcal{C}$ and the set of its objects by $\mathbf{Ver}\,\mathcal{C}$.

In order to simplify our exposition, let us introduce two easy constructions, as follows.

For $P^\bullet \in \mathbf{C}^{-,b}(A-\mathrm{pro}) \setminus \mathbf{C}^b(A-\mathrm{pro})$, let s be the maximal number such that $P^s \neq 0$ and $H^i(P^\bullet) = 0$ for $i \leq s$. Then, $\alpha(P^\bullet)^\bullet$ denotes the *brutal truncation* of $P^\bullet$ below s (see [**?**]), i.e. the complex given by

$$\alpha(P^\bullet)^i = \begin{cases} P^i & \text{, if } i \geq s; \\ 0 & \text{, otherwise,} \end{cases}$$

$$\partial^i_{\alpha(P^\bullet)^\bullet} = \begin{cases} \partial^i_{P^\bullet} & \text{, if } i \geq s; \\ 0 & \text{, otherwise.} \end{cases}$$

For $P^\bullet \neq 0^\bullet \in \mathbf{C}^b(A-\mathrm{pro})$, let t be the maximal number such that $P^i = 0$ for $i < t$. Then, $\beta(P^\bullet)^\bullet$ denotes the *(good) truncation* of $P^\bullet$ below t (see [W]), i.e. the complex given by

$$\beta(P^\bullet)^i = \begin{cases} P^i & \text{, if } i \geq t; \\ \operatorname{Ker} \partial^t_{P^\bullet} & \text{, if } i = t-1; \\ 0 & \text{, otherwise,} \end{cases}$$

$$\partial^i_{\beta(P^\bullet)^\bullet} = \begin{cases} \partial^i_{P^\bullet} & \text{, if } i \geq t; \\ i_{\operatorname{Ker} \partial^t_{P^\bullet}} & \text{, if } i = t-1; \\ 0 & \text{, otherwise,} \end{cases}$$

where $i_{\operatorname{Ker} \partial^t_{P^\bullet}}$ is the obvious inclusion.

Lemma 1. *Let $M^\bullet \in \mathbf{K}^{-,b}(A-\mathrm{pro}) \setminus \mathbf{K}^b(A-\mathrm{pro})$ be an indecomposable. Then $N^\bullet = \beta(\alpha(M^\bullet)^\bullet)^\bullet$ is also indecomposable in $\mathbf{D}^b(A)$ and*

$$M^\bullet \cong P^\bullet_{N^\bullet}.$$

Proof. Obvious. □

Lemma 2. *There exist skeletons $\mathbf{Sk}\,\mathfrak{p}(A)$ and $\mathbf{Sk}\,\mathbf{K}^b(A-\mathrm{pro})$ of $\mathfrak{p}(A)$ and $\mathbf{K}^b(A-\mathrm{pro})$, respectively, such that $\mathbf{Ver}\,\mathfrak{p}(A) = \mathbf{Ver}\,\mathbf{K}^b(A-\mathrm{pro})$.*

Proof. Obvious. □

Let $\overline{\mathcal{X}(A)} = \{\, M^\bullet \in \mathbf{Ver}\,\mathfrak{p}(A) \mid P^\bullet_{\beta(M^\bullet)^\bullet} \notin \mathbf{K}^b(A-\mathrm{pro}) \,\}$. Let $\cong_{\mathcal{X}}$ be the equivalence relation on the set $\overline{\mathcal{X}(A)}$ defined by $M^\bullet \cong_{\mathcal{X}} N^\bullet$ if and only if $P^\bullet_{\beta(M^\bullet)^\bullet} \cong P^\bullet_{\beta(N^\bullet)^\bullet}$ in $\mathbf{K}^{-,b}(A-\mathrm{pro})$. We use the notation $\mathcal{X}(A)$ for a fixed set of representatives of the quotient set $\overline{\mathcal{X}(A)}$ over the equivalence relation $\cong_{\mathcal{X}}$.

From Lemmas 1 and 2 we obtain the following

Corollary 1. *There exist skeletons $\mathbf{Sk}\,\mathbf{D}^b(A)$ and $\mathbf{Sk}\,\mathfrak{p}(A)$ of $\mathbf{D}^b(A)$ and $\mathfrak{p}(A)$, respectively, such that $\mathbf{Ver}\,\mathbf{D}^b(A) = \mathbf{Ver}\,\mathfrak{p}(A) \cup \{\beta(M^\bullet)^\bullet \mid M^\bullet \in \mathcal{X}(A)\}$.*

Remark 1. *If A has a finite global dimension then $\mathcal{X}(A) = \emptyset$ and $\mathbf{Ver}\,\mathbf{D}^b(A) = \mathbf{Ver}\,\mathfrak{p}(A)$.*

Let T be the translation functor $\mathbf{D}(A) \to \mathbf{D}(A)$. By analogy to [D] we will use the following definitions.

Definition 1. *Let k be an algebraically closed field and A be a finite-dimensional k-algebra. Then*

- *A is called* derived wild *if there exists a complex of $A - k\langle x,y\rangle$- bimodules $M^\bullet$ such that each M^i is free and of finite rank as right $k\langle x,y\rangle$-module and such that the functor $M \otimes_{k\langle x,y\rangle} -$ preserves indecomposability and isomorphism classes.*

- *A is called* derived tame *(*see [GK]*) if, for each cohomology dimension vector $(d_i)_{i\in\mathbb{Z}}$, there exist a localization $R = k[x]_f$ with respect to some $f \in k[x]$ and a finite number of bounded complexes of $A - R$-bimodules $C_1^\bullet, \cdots, C_n^\bullet$ such that each C_j^i is free and of finite rank as right R-module and such that every indecomposable $X^\bullet \in \mathbf{D}^b(A)$ with $\dim H^i(X^\bullet) = d_i$ is isomorphic to $C_j^\bullet \otimes_R S$ for some j and some simple R-module S.*

- *A is called* derived discrete (see [V]) *if for every cohomology dimension vector $(d_i)_{i\in\mathbb{Z}}$, we have up to isomorphism a finite number of indecomposables $X^\bullet \in \mathbf{D}^b(A)$ with $\dim H^i(X^\bullet) = d_i$.*

- *A is called* derived finite *if we have a finite number of indecomposables*

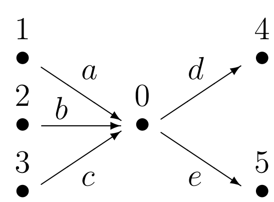
$$X_1^\bullet, \cdots, X_n^\bullet \in \mathbf{D}^b(A)$$

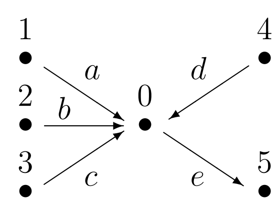
such that every indecomposable object $X^\bullet \in \mathbf{D}^b(A)$ is isomorphic to $T^i(X_j^\bullet)$ for some $i \in \mathbb{Z}$ and some j.

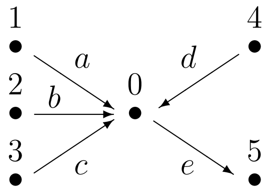
We denote by $\mathbf{P}_i$ the indecomposable projective corresponding to the vertex $i \in Q_0$ and by $p(w)$ the morphism between two indecomposable projectives corresponding to the path w of Q.

3.1

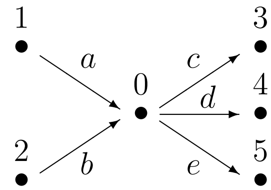
We list below the wild algebras which are used in the proof of the Theorem 1.

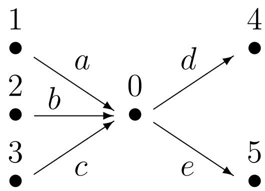
W_1 :


W_2 :


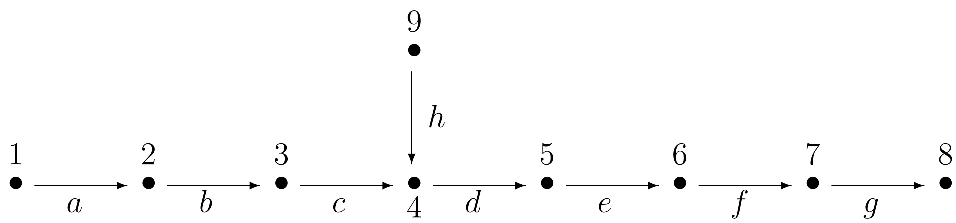
W_3 : 

$de = 0$

W_4 : 

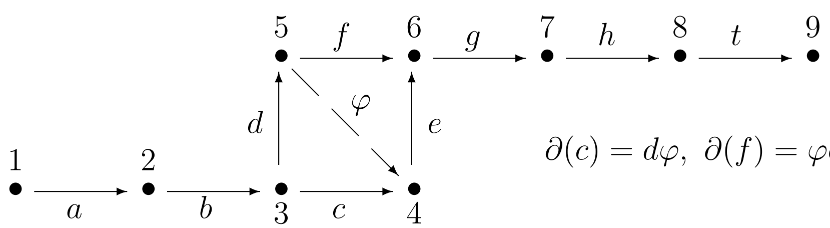
W_5 : 

$cd = 0$

W_6 : 

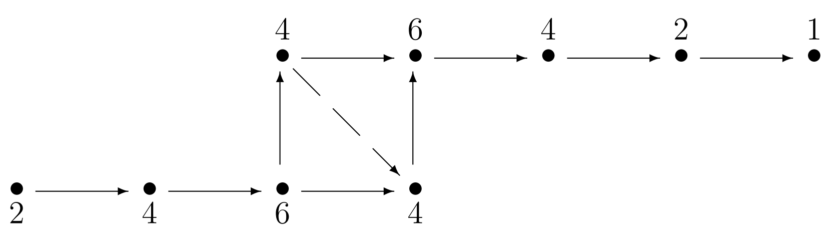
The wildness of the algebras $W_1 - W_6$ follows from [U].

Let us consider the following box (see [D] or [Ro] for definition) whose wildness will be used in the proof of the Theorem 1.

W : 

$\partial(c) = d\varphi,\ \partial(f) = \varphi e$

Consider the following dimension vector $\vec{d}$:



Since $f_W(\vec{d}) = -1$, the wildness of the box W follows from [D].

3.2

Theorem 2. *(i) Let $S = S(n, d)$ be a Schur algebra. Then S is derived tame if and only if every block of S is Morita equivalent to A_1 or A_2;*

(ii) Let $S = S(n, d)_r$ be an infinitesimal Schur algebra. Then S is derived tame if and only if every block of S is Morita equivalent to A_1 or F_3.

Proof. It follows from [DEMN] that if $S(n, d)$ or $S(n, d)_r$ is not wild then any of its non-semisimple blocks is Morita equivalent to one of the algebras in Section 2. We show that all algebras in this list with the exception of A_1, A_2 and F_3 are derived wild. Let A be one of the algebras from Section 2 and let B be one of the algebras $W_1 - W_6$. Since B is wild, there exists $B - k\langle x, y\rangle$-bimodule $M = M(B)$ such that the functor $M \otimes_{k\langle x,y\rangle} -$ preserves indecomposability and isomorphism classes. We set $d_i = \dim M(i)$ and denote by $[M(x)]$ the matrix corresponding to the map $M(x) : M(s(x)) \to M(e(x))$ with respect to some fixed basis.

(1) Let $A = G$, $M = M(W_1)$ and let $N^\bullet$ be the following complex of $A - k\langle x, y\rangle$-bimodules:

$$\cdots \to 0 \to N^1 = \oplus_{i=1}^3 \mathbf{P}_i^{d_i} \xrightarrow{\partial^1} \mathbf{P}_0^{d_0} \xrightarrow{\partial^2} \mathbf{P}_1^{d_4} \oplus \mathbf{P}_2^{d_5} \to 0 \to \cdots,$$

where

$$\partial^1 = \begin{pmatrix} p(\alpha_1)[M(a)] & p(\alpha_2)[M(b)] & p(\alpha_3)[M(c)] \end{pmatrix}^T, \partial^2 = \begin{pmatrix} p(\beta_1)[M(d)] & p(\beta_2)[M(e)] \end{pmatrix}.$$

(2) Let $A = B$, $M = M(W_2)$ and let $N^\bullet$ be the following complex of $A - k\langle x, y\rangle$-bimodules:

$$\cdots \to 0 \to N^1 = \oplus_{i=1}^4 \mathbf{P}_i^{d_i} \xrightarrow{\partial^1} \mathbf{P}_0^{d_0} \xrightarrow{\partial^2} \mathbf{P}_1^{d_5} \to 0 \to \cdots,$$

where

$$\partial^1 = \begin{pmatrix} p(\alpha_1)[M(a)] & p(\alpha_2)[M(b)] & p(\alpha_3)[M(c)] & p(\alpha_4)[M(d)] \end{pmatrix}^T, \partial^2 = p(\beta_1)[M(e)].$$

(3) Let $A = B_1$, $M = M(W_3)$ and let $N^\bullet$ be the following complex of $A - k\langle x, y\rangle$-bimodules:

$$\cdots \to 0 \to N^1 = \oplus_{i=1}^4 \mathbf{P}_i^{d_i} \xrightarrow{\partial^1} \mathbf{P}_0^{d_0} \xrightarrow{\partial^2} \mathbf{P}_1^{d_5} \to 0 \to \cdots,$$

where

$$\partial^1 = \begin{pmatrix} p(\alpha_1)[M(a)] & p(\alpha_2)[M(b)] & p(\alpha_3)[M(c)] & p(\alpha_4)[M(d)] \end{pmatrix}^T, \partial^2 = p(\beta_4)[M(e)].$$

(4) Let $A = D$, $M = M(W_4)$ and let $N^\bullet$ be the following complex of $A - k\langle x, y\rangle$-bimodules:

$$\cdots \to 0 \to N^1 = \mathbf{P}_1^{d_1} \oplus \mathbf{P}_2^{d_2} \xrightarrow{\partial^1} \mathbf{P}_0^{d_0} \xrightarrow{\partial^2} \mathbf{P}_1^{d_3} \oplus \mathbf{P}_2^{d_4} \oplus \mathbf{P}_0^{d_5} \to 0 \to \cdots,$$

where

$$\partial^1 = \begin{pmatrix} p(\alpha_1)[M(a)] & p(\alpha_2)[M(b)] \end{pmatrix}^T, \partial^2 = \begin{pmatrix} p(\beta_1)[M(c)] & p(\beta_2)[M(d)] & p(\beta_3\alpha_3)[M(e)] \end{pmatrix}.$$

(5) Let $A = D_3$, $M = M(W_6)$ and let $N^\bullet$ be the following complex of $A - k\langle x,y\rangle$-bimodules:

Set $N^i = \mathbf{P}_2^{d_i}$ for $i \in \{1,2,4,5,6,7,8\}$, $N^3 = \mathbf{P}_2^{d_3} \oplus \mathbf{P}_1^{d_9}$, $\partial^1 = p(\beta_2\alpha_2)[M(a)]$, $\partial^2 = \begin{pmatrix} p(\beta_2\alpha_2)[M(b)] & 0 \end{pmatrix}$, $\partial^3 = \begin{pmatrix} p(\beta_2\alpha_2)[M(c)] & p(\alpha_1)[M(h)] \end{pmatrix}^T$, $\partial^4 = p(\beta_2\alpha_2)[M(d)]$, $\partial^5 = p(\beta_2\alpha_2)[M(e)]$, $\partial^6 = p(\beta_2\alpha_2)[M(f)]$ and $\partial^7 = p(\beta_2\alpha_2)[M(g)]$.

(6) Let $A = D_4$, $M = M(W_5)$ and let $N^\bullet$ be the following complex of $A - k\langle x,y\rangle$-bimodules:

$$\cdots \to 0 \to N^1 = \oplus_{i=1}^3 \mathbf{P}_i^{d_i} \xrightarrow{\partial^1} \mathbf{P}_0^{d_0} \xrightarrow{\partial^2} \mathbf{P}_1^{d_4} \oplus \mathbf{P}_2^{d_5} \to 0 \to \cdots,$$

where

$$\partial^1 = \begin{pmatrix} p(\alpha_1)[M(a)] & p(\alpha_3)[M(b)] & p(\alpha_2)[M(c)] \end{pmatrix}^T, \partial^2 = \begin{pmatrix} p(\beta_1)[M(d)] & p(\beta_2\alpha_2)[M(e)] \end{pmatrix}.$$

(7) Let $A = H_4$, $M = M(W_5)$ and let $N^\bullet$ be the following complex of $A - k\langle x,y\rangle$-bimodules:

$$\cdots \to 0 \to N^1 = \mathbf{P}_1^{d_1} \oplus \mathbf{P}_3^{d_2} \oplus \mathbf{P}_2^{d_3} \xrightarrow{\partial^1} \mathbf{P}_0^{d_0} \xrightarrow{\partial^2} \mathbf{P}_1^{d_4} \oplus \mathbf{P}_2^{d_5} \to 0 \to \cdots,$$

where

$$\partial^1 = \begin{pmatrix} p(\alpha_1)[M(a)] & p(\alpha_3)[M(b)] & p(\alpha_2)[M(c)] \end{pmatrix}^T, \partial^2 = \begin{pmatrix} p(\beta_3)[M(d)] & p(\beta_1)[M(e)] \end{pmatrix}.$$

(8) Let $A \in \{A_m, m > 2, F_r, r > 3, R_4\}$.

Since box W is wild, there exists $W - k\langle x,y\rangle$-bimodule M such that the functor $M \otimes_{k\langle x,y\rangle} -$ preserves indecomposability and isomorphism classes. Denote by $N^\bullet$ the following complex of $A - k\langle x,y\rangle$-bimodules.

Set $N^i = \mathbf{P}_3^{d_i}$ for $1 \le i \le 3$, $N^j = \mathbf{P}_2^{d_{j+1}}$ for $5 \le j \le 8$, $N^4 = \mathbf{P}_2^{d_5} \oplus \mathbf{P}_3^{d_4}$,

$$\partial^1 = p(\beta_2\alpha_2)[M(a)], \partial^2 = p(\beta_2\alpha_2)[M(b)], \partial^3 = \begin{pmatrix} p(\beta_2)[M(d)] & p(\beta_2\alpha_2)[M(c)] \end{pmatrix},$$

$$\partial^4 = \begin{pmatrix} p(\alpha_2\beta_2)[M(f)] & p(\beta_2)[M(e)] \end{pmatrix}^T, \partial^5 = p(\alpha_2\beta_2)[M(g)],$$

$$\partial^6 = p(\alpha_2\beta_2)[M(h)], \partial^7 = p(\alpha_2\beta_2)[M(t)].$$

It is not difficult to verify that the functor $N^\bullet \otimes_{k\langle x,y\rangle} -$, which acts from the category of finite-dimensional $k\langle x,y\rangle$-modules to the category $\mathfrak{p}(A)$, where A is one of the algebras from (1)-(8), preserves indecomposability and isomorphism classes. So, A is derived wild.

It follows from [H] that algebra A_1 is derived tame. It follows from [BM] that the algebra A_2 is derived tame. The derived tameness of the algebra F_3 follows from Theorem 4 (see Section 4).

□

3.3 Proof of Theorem 1.

It follows from Theorem 2 that a Schur (resp., infinitesimal Schur) algebra $S(n,d)$ (resp., $S(n,d)_r$) is derived tame if all its blocks are of type A_1 or A_2 (resp., A_1 or F_3). Hence, in order to classify derived tame Schur (resp., infinitesimal Schur) algebras it is enough to choose among representation tame and representation finite algebras those that have all blocks of type A_1, A_2 or F_3.

Let $S(n,d)$ (resp., $S(n,d)_r$) be a Schur (resp., infinitesimal Schur) algebra. It follows from [DN] that it is semisimple if and only if it satisfies conditions a), b) in (i) (resp., a), b), c) in (ii)). If $S(n,d)$ is of type c) in (i) then any of its non-semisimple blocks is Morita equivalent to A_2 by 5.4, 5.5 and 1.3 in [E]. Suppose now that $S(n,d)$ is a Schur algebra of finite representation type with $n=2$ and $d<p^2$ [E]. Such algebra has all non-semisimple blocks of type A_2 if and only if it satisfies d) in (i) by Proposition 5.1 in [E]. Let $S(n,d)$ be a Schur algebra of finite representation type with $n \geq 3$ and $d < 2p$. The basic algebra of $S(n,d)$ is a direct sum of all blocks of the group algebra kS_d of the symmetric group S_d (see 1.4 in [E]). The irreducible representations of S_d are parametrized by the partitions of d. Moreover, two such representations belong to the same block if the corresponding partitions have the same p-core [JK]. By 4.1 in [E], if a block of kS_d has s partitions with $\leq n$ parts then it is equivalent to the algebra A_s. Applying this we conclude that $S(n,d)$ with $n \geq 3$ and $d < 2p$ has all non-semisimple blocks Morita equivalent to A_2 if and only if it satisfies the conditions e) and f) in (i).

Now let $S(n,d)_r$ be an infinitesimal Schur algebra satisfying d) in (ii). Then its non-semisimple blocks are Morita equivalent to F_3 by [DNP2], Section 6.3. Finally, the remaining infinitesimal Schur algebras of finite representation type have non-semisimple blocks Morita equivalent to the algebra F_3 if and only if they satisfy e) and f) in (ii) by Propositions 2.4 and 3.2 in [DNP3]. This completes the proof of Theorem 1.

4 Indecomposables in derived categories of algebras A_1, A_2 and F_3

4.1 $A = A_1$.

It follows from [H] that the projective complexes

$$P_i^\bullet : \quad \cdots \to 0 \to P^i = \mathbf{P}_1 \to 0 \to \cdots ,$$

where $i \in \mathbb{Z}$, constitute an exhaustive list of pairwise non-isomorphic indecomposable objects of $\mathbf{D}^b(A_1)$.

Corollary 2. *An algebra of type A_1 is derived finite.*

4.2 $A = A_2$.

Let $\alpha = \alpha_1, \beta = \beta_1$ and let e_1, e_2 be the idempotents corresponding to the vertices. Consider the following projective complexes of $\mathbf{D}^b(A_2)$.

- $P^\bullet_{e_1}: \quad \cdots \to 0 \to P^1 = \mathbf{P}_1 \to 0 \to \cdots$
- $P^\bullet_{e_2}: \quad \cdots \to 0 \to P^1 = \mathbf{P}_2 \to 0 \to \cdots$
- $P^\bullet_{\alpha}: \quad \cdots \to 0 \to P^1 = \mathbf{P}_1 \stackrel{p(\alpha)}{\to} \mathbf{P}_2 \to 0 \to \cdots$
- $P^\bullet_{\beta}: \quad \cdots \to 0 \to P^1 = \mathbf{P}_2 \stackrel{p(\beta)}{\to} \mathbf{P}_1 \to 0 \to \cdots$
- $P^\bullet_{(\beta\alpha)^s}: \quad \cdots \to 0 \to P^1 = \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \cdots \stackrel{p(\beta\alpha)}{\to} P^{s+1} = \mathbf{P}_2 \to 0 \to \cdots$
- $P^\bullet_{\alpha(\beta\alpha)^s}: \quad \cdots \to 0 \to P^1 = \mathbf{P}_1 \stackrel{p(\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \cdots \stackrel{p(\beta\alpha)}{\to} P^{s+2} = \mathbf{P}_2 \to 0 \to \cdots$
- $P^\bullet_{(\beta\alpha)^s\beta}: \quad \cdots \to 0 \to P^1 = \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \cdots \stackrel{p(\beta\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta)}{\to} P^{s+2} = \mathbf{P}_1 \to 0 \to \cdots$
- $P^\bullet_{\alpha(\beta\alpha)^s\beta}: \quad \cdots \to 0 \to P^1 = \mathbf{P}_1 \stackrel{p(\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \cdots \stackrel{p(\beta\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta\alpha)}{\to} \mathbf{P}_2 \stackrel{p(\beta)}{\to}$ $P^{s+3} = \mathbf{P}_1 \to 0 \to \cdots$

Theorem 3. *The projective complexes $T^i(P^\bullet_{e_1})$, $T^i(P^\bullet_{e_2})$, $T^i(P^\bullet_{\alpha})$, $T^i(P^\bullet_{\beta})$, $T^i(P^\bullet_{(\beta\alpha)^s})$, $T^i(P^\bullet_{\alpha(\beta\alpha)^s})$, $T^i(P^\bullet_{(\beta\alpha)^s\beta})$ and $T^i(P^\bullet_{\alpha(\beta\alpha)^s\beta})$, where $i \in \mathbb{Z}$, $s \in \mathbb{N}$, constitute an exhaustive list of pairwise non-isomorphic indecomposable objects of $\mathbf{D}^b(A_2)$.*

Proof. The proof follows from [BM]. □

Corollary 3. *An algebra of type A_2 is derived discrete.*

4.3 $A = F_3$.

We will construct a certain box (see [D] or [Ro] for definition) corresponding to the algebra F_3. Consider the path algebra $B = kQ/I$, where $Q_0 = \{1[i], 2[i], 3[i] \mid i \in \mathbb{Z}\}$, $Q_1 = \{a[i], b[i], c[i], d[i], f[i] \mid i \in \mathbb{Z}\}$, $s(a[i]) = s(c[i]) = s(f[i]) = e(b[i-1]) = e(d[i-1]) = e(f[i-1]) = 2[i]$, $s(b[i]) = e(a[i-1]) = 1[i]$, $s(d[i]) = e(c[i-1]) = 3[i]$ and $I =< a[i]b[i+1] + c[i]d[i+1] \mid i \in \mathbb{Z} >$.

Consider a normal box $\mathcal{B} = (B, V)$ with a kernel $\overline{V}$ freely generated by the set $\{\alpha[i], \beta[i], \gamma[i], \delta[i], \varphi[i] \mid i \in \mathbb{Z}\}$, where $s(\alpha[i]) = s(\gamma[i]) = s(\varphi[i]) = e(\beta[i]) = e(\delta[i]) = e(\varphi[i]) = 2[i]$, $s(\beta[i]) = e(\alpha[i]) = 1[i]$, $s(\delta[i]) = e(\gamma[i]) = 3[i]$ and with the differential ∂ given by the formulas:

$$\partial(f[i]) = a[i]\beta[i+1] + c[i]\delta[i+1] + \alpha[i]b[i] + \gamma[i]d[i],$$

$$\partial(\varphi[i]) = \alpha[i]\beta[i] + \gamma[i]\delta[i], \partial(x) = 0 \text{ for } x \notin \{f[i], \varphi[i] \mid i \in \mathbb{Z}\}.$$

The category of the representations of the box $\mathcal{B}$ will be denoted by $\operatorname{rep}(\mathcal{B})$.

Proposition 2. *The category* $\mathfrak{p}(F_3)$ *is equivalent to* $\operatorname{rep}(\mathcal{B})$.

Proof. Consider the functor $G : \operatorname{rep}(\mathcal{B}) \to \mathfrak{p}(F_3)$ defined as follows. The modules are

$$G(M)^j = \oplus_{i\in\{1,2,3\}} \mathbf{P}_i^{\dim_k M(i[j])}$$

and the differential maps are

$$\partial_M^j = \begin{pmatrix} 0 & p(\alpha_1)[M(b[j])] & 0 \\ p(\beta_1)[M(a[j])] & p(\beta_1\alpha_1)[M(f[j])] & p(\alpha_2)[M(c[j])] \\ 0 & p(\beta_2)[M(d[j])] & 0 \end{pmatrix},$$

where $[M(x[j])]$ is the matrix corresponding to the map $M(x[j])$ in some fixed basis. It is easy to see that G is a representation equivalence. □

We recall some definitions and results related to the bunches of semi-chains considered by Bondarenko in [B] and Deng in [De] in a form convenient for our purposes (see also [CB] for an alternative approach). We will use the classification of indecomposables representations of a bunch of semi-chains given in [B]. We will use some notation from [DG].

Definition 2. *A bunch of semi-chains* $\mathbf{C} = \{\mathbf{I}, E_i, F_i, \sim\}$ *is defined by following data:*

1. *A set* $\mathbf{I}$ *of indices;*
2. *Two semi-chains (i.e., partially ordered sets with the condition that each element is incomparable with at most one other)* E_i *and* F_i *given for each* $i \in \mathbf{I}$*;*

   Put $\mathbf{E} := \cup_{i\in\mathbf{I}} E_i$*,* $\mathbf{F} := \cup_{i\in\mathbf{I}} F_i$ *and* $|\mathbf{C}| := \mathbf{E} \cup \mathbf{F}$*.*
3. *An equivalence relation* $\sim$ *on* $|\mathbf{C}|$ *such that each equivalence class consists of at most 2 elements and if* $a \sim b \neq a$ *for some* $a \in E_i$ *(resp.,* $a \in F_i$*), then* a *is comparable with all elements of* E_i *(resp.,* F_i*).*

We consider the ordering on $|\mathbf{C}|$, which is just the union of all orderings on E_i and F_i (i.e., $a < b$ means that a and b belong to the same semi-chain E_i or F_i and $a < b$ in this semi-chain).

We construct a certain box associated with a bunch of semi-chains $\mathbf{C}$. Given $u \in |\mathbf{C}|$ we denote by $\bar{u}$ the corresponding element of $|\mathbf{C}|/\sim$. Consider the path algebra $A(\mathbf{C}) = kQ = kQ(\mathbf{C})$, where $Q_0 = |\mathbf{C}|/\sim$, $Q_1 = \{a_u^v \mid u \in E_i, v \in F_i, i \in \mathbb{Z}\}$, (we also will assume that $a_u^v = 0$ for all other cases), $s(a_u^v) = \bar{u}$ and $e(a_u^v) = \bar{v}$.

Consider a normal box $\mathcal{C} = (A(\mathbf{C}), V = V(\mathbf{C}))$ with a kernel $\overline{V}$ freely generated by the set $\{\varphi_u^v, \psi_m^n \mid \ u, v \in E_i, m, n \in F_i, u < v, n < m, i \in \mathbb{Z}\}$ (we also will suppose that $\varphi_u^v = \psi_m^n = 0$ for all other cases), where $s(\varphi_u^v) = \bar{u}$, $s(\psi_m^n) = \bar{m}$, $e(\varphi_u^v) = \bar{v}$, $e(\psi_m^n) = \bar{n}$ and with the differential ∂ given by the formulas:

$$\partial(a_u^v) = \sum_{n>u} \varphi_u^n a_n^v + \sum_{m>v} a_u^m \psi_m^v,$$

$$\partial(\varphi_u^v) = \pm \sum_{u<n<v} \varphi_u^n \varphi_n^v,$$

$$\partial(\psi_u^v) = \pm \sum_{v<n<u} \psi_u^n \psi_n^v.$$

The choice of signs in the last formulas guarantees the condition $\partial^2 = 0$.

The category of representations of the bunch of semi-chains $\mathbf{C}$ is then defined as the category $\operatorname{rep} \mathcal{C}$. One can easily verify that this definition gives just the same representations as the definition in [B].

Consider an equivalence relation $\sim_c$ on $|\mathbf{C}|$ given by the following rule: $a \sim_c b$ if and only if either $a = b$ or a and b belong to the same semi-chain E_i or F_i and a is incomparable with b. In case the equivalence class $x \in |\mathbf{C}|/\sim_c$ consists of a unique element $a \in |\mathbf{C}|$ we will identify x with a and write $x \in |\mathbf{C}|$. Consider a relation $\sim$ on $|\mathbf{C}|/\sim_c$ given by the following rule: $a \sim b$ if and only if either $a = b$ and a consists of two elements or $a, b \in |\mathbf{C}|$ and $a \sim b \neq a$.

Definition 3. *Let $\mathbf{C} = \{\mathbf{I}, E_i, F_i, \sim\}$ be a bunch of semi-chains.*

- *A* $\mathbf{C}$-word *is a sequence $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$, where $w_k \in |\mathbf{C}|/\sim_c$ and each r_k is either $\sim$ or $-$, such that for all possible values of k:*

  (a) $r_k = \sim$ if and only if $w_{k-1} \sim w_k$;

  (b) $r_k = -$ if and only if $w_{k-1} \subset E_i$, $w_k \subset F_i$ for some $i \in \mathbb{Z}$ or vice versa;

  (c) $r_{k+1} \neq r_k$.

  Possibly $m = 0$, i.e., $w \in |\mathbf{C}|/\sim_c$.

- *Call a $\mathbf{C}$-word $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$ a* $\mathbf{C}$-cycle *if $w_m = w_0$, $r_1 = \sim$ and $r_m = -$.*

- *Call a $\mathbf{C}$-word $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$* full *if, whenever $w_0 \in |\mathbf{C}|$ and w_0 is not a unique element in its equivalence class $\overline{w_0}$, then $r_1 = \sim$, and whenever $w_m \in |\mathbf{C}|$ and w_m is not a unique element in its equivalence class $\overline{w_m}$, then $r_m = \sim$.*

- *Call a $\mathbf{C}$-cycle $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$* aperiodic *if the sequence $w_0 r_1 w_1 \cdots r_m$ can not be written as a multiple self-concatenation $v \cdots v$ of a shorter sequence v.*

  Given a $\mathbf{C}$-word $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$ set $w^* := w_m r_m \cdots r_2 w_1 r_1 w_0$.

- *Call a* $\mathbf{C}$*-word* $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$ simple *if, from* $w = u \sim u^* \sim u \cdots$ *for some* $\mathbf{C}$*-word* u*, it follows that* $w = u$*.*

Denote by $Ind\, k[x]$ the set of indecomposable polynomials with highest coefficient 1 except $\{x^d | d \geq 1\}$.

Definition 4. *Let* $\mathbf{C} = \{\mathbf{I}, E_i, F_i, \sim\}$ *be a bunch of semi-chains.*

- *Given a full* $\mathbf{C}$*-word* $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$*, we set* $d_l(w) = 1$ *if* $w_0 \sim w_0$ *and either* $r_1 = -$ *or* $m = 0$*, and we set* $d_l(w) = 0$ *otherwise.*

- *Given a full* $\mathbf{C}$*-word* $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$*, we set* $d_r(w) = 1$ *if* $w_m \sim w_m$*,* $m > 0$ *and* $r_m = -$*, and we set* $d_r(w) = 0$ *otherwise.*

- *By a* usual string *we mean a simple full* $\mathbf{C}$*-word* $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$ *such that* $d_l(w) + d_r(w) = 0$*.*

- *By a* special string *we mean a pair* (w, k)*, where* $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$ *is a simple full* $\mathbf{C}$*-word such that* $0 < d_l(w) + d_r(w) < 2$ *and* $k \in \{0, 1\}$*.*

- *By a* bispecial string *we mean a quadruple* (w, k, l, n)*, where* $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$ *is a simple full* $\mathbf{C}$*-word such that* $d_l(w) + d_r(w) = 2$ *and* $k, l \in \{0, 1\}$*,* $n \in \mathbb{N}$*.*

- *By a* string *we mean usual or special or bispecial string.*

- *By a* band *we mean a pair* (w, f)*, where* $f \in Ind\, k[x]$ *and* w *is an aperiodic* $\mathbf{C}$*-cycle.*

For each string and each band, Bondarenko constructed in [B] some indecomposable representation of $\mathbf{C}$ and proved that up to some isomorphisms between of them (see [B] for details) they constitute an exhaustive list of pairwise non-isomorphic indecomposable representations.

From now on we will consider the bunch of semi-chains $\mathbf{C} = \mathbf{C}(F_3) := \{\mathbb{Z}, E_i, F_i, \sim\}$, where $E_i = \{y[i]^- < x[i] < z[i], y[i]^+ < x[i]\}$, $F_i = \{r[i] < p[i]^- < q[i] > p[i]^+ > r[i]\}$ and the equivalence relation $\sim$ is given by $r[i] \sim x[i+2]$, $q[i] \sim z[i+1]$. We denote by $p[i]$ (resp., $y[i]$) the class $\{p[i]^-, p[i]^+\} \in |\mathbf{C}(F_3)|/\sim_c$ (resp., $\{y[i]^-, y[i]^+\} \in |\mathbf{C}(F_3)|/\sim_c$). We denote by $\mathcal{C}(F_3)$ the corresponding box.

We associate to indecomposable representations of $\mathbf{C}(F_3)$ certain finite projective complexes which, as we shall see, give all indecomposables in the category $\mathfrak{p}(F_3)$.

Definition 5. *Let* M *be an indecomposable representation of the bunch of semi-chains* $\mathbf{C}(F_3)$*. Then* $P(M)^\bullet$ *is the projective complex* $\cdots \to P(M)^i \xrightarrow{\partial^i_{P(M)^\bullet}} P(M)^{i+1} \to \cdots$ *defined as follows. The modules are*

$$P(M)^i = \mathbf{P}_1^{d_{1,i}} \oplus \mathbf{P}_2^{d_{2,i}} \oplus \mathbf{P}_3^{d_{3,i}}$$

and the differential maps are

$$\partial^i_{P(M)^\bullet} = \begin{pmatrix} 0 & p(\alpha_1)A_i & 0 \\ p(\beta_1)B_i & p(\beta_1\alpha_1)C_i & p(\alpha_2)D_i \\ 0 & p(\beta_2)R_i & 0 \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} M^{r[i]}_{x[i]} & M^{p[i]^-}_{x[i]} & M^{p[i]^+}_{x[i]} & M^{q[i]}_{x[i]} \\ 0 & 0 & 0 & 0 \\ M^{r[i]}_{y[i]^+} & M^{p[i]^-}_{y[i]^+} & M^{p[i]^+}_{y[i]^+} & M^{q[i]}_{y[i]^+} \end{pmatrix}, R_i = \begin{pmatrix} -M^{r[i]}_{x[i]} & -M^{p[i]^-}_{x[i]} & -M^{p[i]^+}_{x[i]} & -M^{q[i]}_{x[i]} \\ 0 & 0 & 0 & 0 \\ M^{r[i]}_{y[i]^-} & M^{p[i]^-}_{y[i]^-} & M^{p[i]^+}_{y[i]^-} & M^{q[i]}_{y[i]^-} \end{pmatrix},$$

$$C_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M^{r[i]}_{z[i]} & M^{p[i]^-}_{z[i]} & M^{p[i]^+}_{z[i]} & M^{q[i]}_{z[i]} \end{pmatrix}, B_i = \begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D_i = \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$M^{v[i]}_{u[i]}$ *is the matrix corresponding to the map* $M(a^{v[i]}_{u[i]})$ *in some fixed basis and where* $d_{i,j}$ *and dimensions of all blocks are uniquely defined by dimensions of the matrix* $M^{v[i]}_{u[i]}$.

Proposition 3. *The projective complexes* $P(M)^\bullet$*, where* $M \in \mathbf{Ver}\,\mathrm{rep}\,(\mathcal{C}(F_3))$*, constitute an exhaustive list of pairwise non-isomorphic indecomposable objects of* $\mathfrak{p}(F_3)$*.*

Proof. Let $\mathcal{A} = (A, V)$ be the box corresponding to F_3 (see above). Let us consider the sub-box $\mathcal{A}' = (A', V')$ of $\mathcal{A}$, where $A'_0 = A_0$, $A'_1 = \{a[i], c[i] \mid i \in \mathbb{Z}\}$ and $\overline{V'} = 0$. It is easy to see that $\mathcal{A}' \cong \amalg_{\mathbb{Z}} \mathcal{A}''$, where $\mathcal{A}''$ is the principal box corresponding to the hereditary algebra $A'' : \bullet \leftarrow \bullet \rightarrow \bullet$ (i.e., $\mathcal{A}'' = (A'', A'')$).

Consider the trivial category D' with the set of vertices $|\mathbf{C}(F_3)|/_\sim$ and the functor $F' : A' \to D'$ which maps:

$$1[i] \to \overline{x[i+1]} \oplus \overline{p[i-1]^+} \oplus \overline{y[i+1]^+},$$

$$2[i] \to \overline{x[i+1]} \oplus \overline{p[i-1]^+} \oplus \overline{p[i-1]^-} \oplus \overline{z[i]},$$

$$3[i] \to \overline{x[i+1]} \oplus \overline{p[i-1]^-} \oplus \overline{y[i+1]^-},$$

$$a[i] \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c[i] \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Construct the amalgamation $B = A \amalg^{A'} B'$, or, the same, the couniversal square:

$$\begin{array}{ccc} A' & \xrightarrow{F'} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{F} & B \end{array}$$

Consider now the box $\mathcal{A}^F = (B, V^F)$, where $V^F = \mathcal{B} \otimes_{\mathcal{A}} V \otimes_{\mathcal{A}} \mathcal{B}$.

We say that the box $\mathcal{A}^F$ is obtained from $\mathcal{A}$ *by reducing the sub-box* $\mathcal{A}'$. By [D], F induces the representation equivalence $F^* : \operatorname{rep}(\mathcal{A}^F) \to \operatorname{rep}(\mathcal{A})$.

Straightforward calculation, which we omit, shows that $B = A(\mathbf{C}(F_3))$ and there exists the sub-bimodule U of the bimodule $\overline{V^F}$ such that $V^F = V(\mathbf{C}(F_3)) \oplus U$, $\partial(t) \in V(\mathbf{C}(F_3))$ for each $t \in B_1$ and $\partial(u) \in V(\mathbf{C}(F_3))$ for each $u \in V(\mathbf{C}(F_3))$. Hence it follows from the definition of the representations of a box that $\mathbf{Ver}\operatorname{rep}(\mathcal{C}(F_3)) = \mathbf{Ver}\operatorname{rep}(\mathcal{A}^F)$. Then it is ease to see that $(GF^*)(M) \cong P(M)^\bullet$ for each indecomposable representation M of $\mathbf{C}(F_3)$, where G is as in Proposition 2. □

Definition 6. *Consider the bunch of semi-chains* $\mathbf{C}(F_3)$.

- *Denote by S the set of all usual strings w and special strings (w, k), where $w = w_0 r_1 w_1 r_2 w_2 \cdots r_m w_m$, such that one of the following conditions hold:*

  (a) $w_0 = q[t-1]$, $r_1 = \sim$, $w_1 = z[t]$ for some $t \in \mathbb{Z}$ and if $w_k \in \{x[i+1], y[i], z[i], r[i-1], p[i], q[i-1]\}$ then $i \geq t$;

  (b) $w_m = q[t-1]$, $r_m = \sim$, $w_{m-1} = z[t]$ for some $t \in \mathbb{Z}$ and if $w_k \in \{x[i+1], y[i], z[i], r[i-1], p[i], q[i-1]\}$ then $i \geq t$;

  (c) $w_0 = r[t-1]$, $r_1 = \sim$, $w_1 = x[t+1]$, $w_2 \neq r[t+1]$ for some $t \in \mathbb{Z}$ and if $w_k \in \{x[i+1], y[i], z[i], r[i-1], p[i], q[i-1]\}$ then $i \geq t$;

  (d) $w_m = r[t-1]$, $r_m = \sim$, $w_{m-1} = x[t+1]$, $w_{m-2} \neq r[t+1]$ for some $t \in \mathbb{Z}$ and if $w_k \in \{x[i+1], y[i], z[i], r[i-1], p[i], q[i-1]\}$ then $i \geq t$.

- *Denote by Ψ the set of all $M \in \mathbf{Ver}\operatorname{rep}(\mathbf{C}(F_3))$ which correspond to the elements from S.*

Theorem 4. *The projective complexes $P(M)^\bullet, M \in \mathbf{Ver}\operatorname{rep}(\mathbf{C}(F_3))$ and $\beta(P(M)^\bullet)^\bullet$, $M \in \Psi$, constitute an exhaustive list of pairwise non-isomorphic indecomposable objects of $\mathbf{D}^b(F_3)$.*

Proof. It is ease to see that $\operatorname{Ker} p(\alpha_1) = \operatorname{Ker} p(\beta_2) = 0$, $\operatorname{Ker} p(\alpha_2) = \operatorname{Ker} p(\beta_1) = \operatorname{Ker} p(\beta_1\alpha_1)$, and $\operatorname{Ker} p(\beta_1)$ has the following minimal projective resolution:

$$\cdots \to \mathbf{P}_1 \oplus \mathbf{P}_3 \xrightarrow{\varphi} \mathbf{P}_2 \xrightarrow{\psi} \mathbf{P}_1 \oplus \mathbf{P}_3 \xrightarrow{\varphi} \mathbf{P}_2 \xrightarrow{\psi} \mathbf{P}_1 \oplus \mathbf{P}_3 \xrightarrow{\varphi} \mathbf{P}_2 \xrightarrow{\psi} \mathbf{P}_1 \oplus \mathbf{P}_3 \to \operatorname{Ker} p(\beta_1) \to 0,$$

where

$$\varphi = \begin{pmatrix} p(\alpha_1) & p(\beta_2) \end{pmatrix}^T, \psi = \begin{pmatrix} p(\beta_1) & p(\alpha_2) \end{pmatrix}.$$

Straightforward calculation, which we omit, shows that

$$\mathcal{X}(F_3) = \{P(M)^\bullet \mid M \in \Psi\}.$$

Hence the theorem follows from Corollary 1 and Proposition 3. □

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