INDECOMPOSABLES IN DERIVED CATEGORIES OF GENTLE ALGEBRAS.

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In general, it is not easy to describe the indecomposable objects of the derived category of an algebra. In this paper, we show a nice way for doing this when the algebra is gentle. We have found that there is a connection between the derived category of a gentle algebra and a matrix problem presented by V. M. Bondarenko (*see* [B]). We show that the problem of finding the indecomposable objects of the derived category may be reduced to finding the indecomposable objects in that matrix problem.

In the first section we fix notations and describe our context and in the second we introduce the category of representations of a linearly ordered poset, Bondarenko's matrix problem. In Section 3, a functor is defined which will solve our problem. We define first the poset $\mathcal{Y}$ and then our desired functor. In the final section, the description of the indecomposables of $\mathbf{D}^b(A)$ is given.

1 Preliminaries and some basic facts about gentle algebras.

In this article, k is a field and A denotes an algebra of the form $k\mathbf{Q}/\mathbf{I} = k(\mathbf{Q},\mathbf{I})$ (for example, k is algebraically closed and A is indecomposable and basic). In general, unless otherwise mentionned, we assume that $\mathbf{Q}$ is connected and locally finite but, in the case of algebras, obviously $\mathbf{Q}$ is finite. As usual, $\mathbf{Q}_0$, resp. $\mathbf{Q}_1$, denotes the set of vertices, resp. arrows, of $\mathbf{Q}$. $\mathbf{I}$ is an admissible ideal of $k\mathbf{Q}$.

In most of our work, we assume that A is *gentle*. Gentle algebras form a class of biserial algebras that were introduced in [A-S] in 1987. They constitute an important and fairly large class of algebras. The characterization of gentle algebras by means of their bounden quiver was also stated by Assem and Skowroński in [A-S] and we recall it in the following Proposition.

PROPOSITION 1. *A is gentle if and only if the following conditions are satisfied.*

a. The number of arrows with a prescribed source (resp. target) is at most two.

b. For any arrow α of $\mathbf{Q}_1$ there is at most one arrow β with $s(\beta) = t(\alpha)$ (resp. γ with $s(\alpha) = t(\gamma)$) such that $\alpha\beta$ (resp. $\gamma\alpha$) is in $\mathbf{I}$.

c. *For any arrow α of $\mathbf{Q}_1$ there is at most one arrow β with $s(\beta) = t(\alpha)$ (resp. γ with $s(\alpha) = t(\gamma)$) such that $\alpha\beta$ (resp. $\gamma\alpha$) is not in $\mathbf{I}$.*

d. *$\mathbf{I}$ is generated by a set of paths of length two.*

Since we work most of the time with representations of or modules over algebras, and with *derived* categories, we will use freely the standard definitions and notations. For the benefit of the reader, let us establish that we will follow in general the notations and terminology of [Ri] and [H]. Unless otherwise specified, our modules are finitely generated, left modules.

Let us assume now that A is gentle (even though much of what follows is applicable in the general case too).

We denote by e_i the indecomposable idempotent corresponding to the vertex $i \in \mathbf{Q}_0$ and by $\mathbf{P}_1, \cdots, \mathbf{P}_i = A \cdot e_i, \cdots, \mathbf{P}_t$ the obvious representatives of all the indecomposable projectives of A.

We will use the notation $\mathbf{Pa}$ for the set of all paths of $k(\mathbf{Q}, \mathbf{I})$, that is of all paths of $\mathbf{Q}$ that are outside $\mathbf{I}$, while the notations $\mathbf{Pa}_{>l}$, $\mathbf{Pa}_{\geq l}$, respectively, denote the subsets of paths of length greater than l (resp. greater than or equal to l) (l any natural number).

It is clear that each element of A is represented univocously by a linear combination of paths of $\mathbf{Pa}$ so that we can assume that $\mathbf{Pa}$ is a basis of A.

With e_i we denote also the trivial path (of length 0) associated to the vertex i. If w is a path, $s(w)$ denotes its starting point, $t(w)$ its ending (tail) point and $l(w)$ its length.

A special role is played by the set $\mathbf{M}$, the set of maximal elements of $\mathbf{Pa}$. As a matter of fact a non trivial path w of $\mathbf{Q}$ lies in $\mathbf{Pa}$ if and only if it is a subpath of a maximal path $\tilde{w}$ not in $\mathbf{I}$ (that is, of an element of $\mathbf{M}$) . This maximal path has the form $\tilde{w} = \widehat{w} w \bar{w}$ with $\widehat{w}, \bar{w} \in \mathbf{Pa}$.
For gentle algebras, since - as we shall see - maximal paths intersect only in vertices, these paths are unique.

It is well known that the linear combinations of parallel paths are in a one-to-one correspondence with the non epic morphisms, $p = p(w)$ between two indecomposable projectives.

Notations and some useful remarks about well known facts related to derived categories.

Given A, we denote by $\mathbf{D}(A)$, resp. $\mathbf{D}^b(A)$, the derived category of A, resp. the derived category of bounded complexes of mod$A$; by $\mathbf{C}^b(proA)$, resp. $\mathbf{C}^{-,b}(proA)$ the category of bounded projective complexes, resp. of right bounded projective complexes with bounded cohomology (that is, complexes of projective modules with the property that the cohomology groups are non zero only at a finite number of places); and by $\mathbf{K}^b(proA)$, resp. $\mathbf{K}^{-,b}(proA)$ the corresponding homotopy categories.

We will also use the following notations. By $\mathfrak{p}(A)$ (resp., $\mathfrak{p}_{ht}(A)$) we denote the full subcategory of $\mathbf{C}^b(proA)$ (resp., $\mathbf{K}^b(proA)$) defined by the projective complexes such that the image of every differential map is contained in the radical of the corresponding projective module. Since any projective complex is the sum of one complex with this property and two complex where, alternativelly, all differential maps are 0's or isomorphisms (which is, hence, isomorphic to the zero object in the derived category) we can always assume that we reduce ourselves to consider projective complexes of this form.

It is well known that $\mathbf{D}^b(A)$ is equivalent to $\mathbf{K}^{-,b}(proA)$ (see, for example, [KZ], Prop. 6.3.1 and [HAR]). We state this result, and give a sketch of Zimmermann's proof for the benefit of the reader as it is written in the above mentionned book.

THEOREM 1. $\mathbf{D}^-(A)$ *is equivalent to* $\mathbf{K}^-(proA)$. *The image of* $\mathbf{D}^b(A)$ *under this equivalence is* $\mathbf{K}^{-,b}(proA)$.

Sketch of Proof. First, one constructs by induction, starting from a given complex $M^\bullet$ bounded on the right and using projective covers, a projective complex, $P^\bullet$, isomorphic to $M^\bullet$ in the derived category. As a matter of fact, one sees that there is a complexes-morphism form $P^\bullet$ to $M^\bullet$ which induces an isomorphism in homologies. The result follows. ▪

Given $M^\bullet \in \mathbf{D}(A)$, we denote with $P^\bullet_{M^\bullet}$ the projective resolution of $M^\bullet$ (see [KZ] or [G]), and with $H^i(M^\bullet)$ the i-th cohomology module.

If $\mathcal{A}$ is a Krull-Schmidt category, we will denote by $\mathrm{ind}_0\mathcal{A}$ the set of objects of the spectroid $\mathrm{ind}\,\mathcal{A}$ [GR].

In order to simplify our exposition, let us introduce two easy constructions, as follows.

For $P^\bullet \in \mathbf{C}^{-,b}(proA) \setminus \mathbf{C}^b(proA)$, let s be the maximal number such that $P^s \neq 0$ and $H^i(P^\bullet) = 0$ for $i \leq s$. Then, $\alpha(P^\bullet)^\bullet$ denotes the complex given by

$$\alpha(P^\bullet)^i = \begin{cases} P^i & \text{, if } i \geq s, \\ 0 & \text{, otherwise;} \end{cases}$$

$$d^i_{\alpha(P^\bullet)^\bullet} = \begin{cases} d^i_{P^\bullet} & \text{, if } i \geq s, \\ 0 & \text{, otherwise.} \end{cases}$$

For $P^\bullet \in \mathbf{C}^b(proA)$ ($P^\bullet \neq 0^\bullet$), let t be the maximal number such that $P^i = 0$ for $i < t$. Then, $\beta(P^\bullet)^\bullet$ denotes the complex given by

$$\beta(P^\bullet)^i = \begin{cases} P^i & \text{, if } i \geq t, \\ \ker d^t_{P^\bullet} & \text{, if } i = t-1, \\ 0 & \text{, otherwise;} \end{cases}$$

$$d^i_{\beta(P^\bullet)^\bullet} = \begin{cases} d^i_{P^\bullet} & \text{, if } i \geq t, \\ i_{\ker d^t_{P^\bullet}} & \text{, if } i = t-1, \\ 0 & \text{, otherwise,} \end{cases}$$

where $i_{\ker d^t_{P^\bullet}}$ is the cannonical inclusion.

LEMMA 1. *Let $M^\bullet \in \mathbf{K}^{-,b}(proA) \setminus \mathbf{K}^b(proA)$ be an indecomposable. Then $\beta(\alpha(M^\bullet)^\bullet)^\bullet$ is also indecomposable in $\mathbf{D}^b(A)$ and*

$$M^\bullet \cong P^\bullet_{\beta(\alpha(M^\bullet)^\bullet)^\bullet}.$$

PROOF. Obvious. ▪

LEMMA 2. *There exist spectroids* $\mathrm{ind}\mathfrak{p}_{ht}(A)$*,* $\mathrm{ind}\mathfrak{p}(A)$ *and* $\mathrm{ind}\,\mathbf{K}^b(proA)$ *of* $\mathfrak{p}_{ht}(A)$*,* $\mathfrak{p}(A)$ *and* $\mathbf{K}^b(proA)$*, respectively, such that* $\mathrm{ind}_0\mathfrak{p}_{ht}(A) = \mathrm{ind}_0\mathfrak{p}(A) = \mathrm{ind}_0\mathbf{K}^b(proA)$*.*

Proof. Obvious. ▪

Let $\overline{\mathcal{X}(A)} = \{\, M^\bullet \in \mathrm{ind}_0\,\mathfrak{p}(A) \mid P^\bullet_{\beta(M^\bullet)^\bullet} \notin \mathbf{K}^b(proA)\,\}$. Let $\cong_{\mathcal{X}}$ be the equivalence relation on the set $\overline{\mathcal{X}(A)}$ defined by $M^\bullet \cong N^\bullet$ iff $P^\bullet_{\beta(M^\bullet)^\bullet} \cong P^\bullet_{\beta(N^\bullet)^\bullet}$ in $\mathbf{K}^{-,b}(proA)$. We use $\mathcal{X}(A)$ for a fixed set of representatives of the quotient set $\overline{\mathcal{X}(A)}$ over the equivalence relation $\cong_{\mathcal{X}}$.

From the lemmas 1 and 2 we obtain the following

COROLLARY 1. *There exist spectroids* $\mathrm{ind}\,\mathbf{D}^b(A)$ *and* $\mathrm{ind}\,\mathfrak{p}(A)$ *of* $\mathbf{D}^b(A)$ *and* $\mathfrak{p}(A)$*, respectively, such that* $\mathrm{ind}_0\,\mathbf{D}^b(A) = \mathrm{ind}_0\,\mathfrak{p}(A) \cup \{\beta(M^\bullet)^\bullet \mid M^\bullet \in \mathcal{X}(A)\}$*.*

Remark. 1. *If A has finite global dimension, we have $\mathcal{X}(A) = \emptyset$ and $\mathrm{ind}_0\,\mathbf{D}^b(A) = \mathrm{ind}_0\,\mathfrak{p}(A)$.*

Let T be the translation functor $\mathbf{D}(A) \to \mathbf{D}(A)$. In analogy with [D] we will use the following definitions.

DEFINITION 1. *Let k be an algebraically closed field. Then*

- *A is called* derived tame *(*see [GK]*) if, for each cohomology dimension vector $(d_i)_{i\in\mathbb{Z}}$, there are a finite number of bounded complexes of $A - k[x]$-bimodules $C_1^\bullet, \cdots, C_n^\bullet$ such that each C_j^i is free and of finite rank as right $k[x]$-module and such that every indecomposable $X^\bullet \in \mathbf{D}^b(A)$ with $\dim H^i(X^\bullet) = d_i$ is isomorphic to $C_j^\bullet \otimes_{k[x]} S$ for some j and some simple $k[x]$-module S.*

- *A is called* derived discrete *(see [V]) if for every cohomology dimension vector $(d_i)_{i\in\mathbb{Z}}$, we have up to isomorphism a finite number of indecomposables $X^\bullet \in \mathbf{D}^b(A)$ with $\dim H^i(X^\bullet) = d_i$.*

- *A is called* derived finite *if there exists a finite number of indecomposables*

$$X_1^\bullet, \cdots, X_n^\bullet \in \mathbf{D}^b(A)$$

such that every indecomposable object $X^\bullet \in \mathbf{D}^b(A)$ is isomorphic to $T^i(X_j^\bullet)$ for some $i \in \mathbb{Z}$ and some j.

2 Bondarenko's category of representations of posets.

In this section k is a field (as usual) and $\mathcal{Y}$ is a linearly ordered set (may be infinite) provided with a (fixed) involution σ (*see* [B]).

We begin by defining the category $\mathcal{S}(\mathcal{Y}, k)$.

The objects of $\mathcal{S}(\mathcal{Y}, k)$ are (finite) square block matrices, $B = B_i^j$ $(i, j \in \mathcal{Y})$, called *representations* or *$\mathcal{Y}$-matrices* with all the entries of all the blocks sitting in k, and verifying the following properties. (Notice that we represent the row x of blocks of B by B_x, and the column x, by B^x. Notice also that some blocks may be empty.)

a. The horizontal and vertical partitions of B are compatible. (If B, C are two block matrices (not necessarily square blocks), we say that the horizontal partition of B is compatible with the vertical partition of C if the number of rows in each B_x is equal to the number of columns of each C^x - so that we can multiply CB *by blocks* -, and similarly we define what it means that the vertical partition of B is compatible with the horizontal partition of C.)

b. If $x, y \in \mathcal{Y}$ are such that $\sigma(x) = y$, then all matrices in B_x have the same number of rows than all matrices in B_y (and, consequently, all matrices in B^x have the same number of columns as the matrices in B^y).

c. $B^2 = 0$.

A morphism of $\mathcal{S}(\mathcal{Y}, k)$ from B to C is a block matrix T_i^j $(i, j \in \mathcal{Y})$ with entries in k such that the following are satisfied.

a. the horizontal (resp. vertical) partition of T is compatible with the vertical (resp. horizontal) partition of B (resp. C).

b. $TC = BT$,

c. If $j < i$, then $T_i^j = 0$ (*i.e. all blocks below the main diagonal are 0*).

d. If $\sigma(x) = y$, then $T_x^x = T_y^y$.

Since the matrices T are triangular, in order that a T be invertible it is necessary and sufficient that the diagonal blocks, T_i^i, are all invertible.

It is clear that $\mathcal{S}(\mathcal{Y},k)$ is an additive k-category. It was shown in [BD] that finding the indecomposables of $\mathcal{S}(\mathcal{Y},k)$ can be reduced to finding the indecomposables of a matrix problem introduced and solved in [NR]. Presently, we show that, when A is gentle, finding the indecomposables of $\mathfrak{p}(A)$ is equivalent to finding the indecomposables of certain subcategory of $\mathcal{S}(\mathcal{Y},k)$.

3 The functor.

We will start with a finite projective complex, $P^\bullet$, of length m, with the property that the images of all differential maps are contained in the radical of the corresponding modulo (in other words, $P^\bullet \in \mathfrak{p}(A)$).

$$P^\bullet : \qquad P^n \overset{\partial^n}{\to} \cdots \overset{\partial^{n+m-2}}{\to} P^{n+m-1} \overset{\partial^{n+m-1}}{\to} P^{n+m} \qquad , n, m \in \mathbb{Z}.$$

Let us say that in each P^j of the complex $P^\bullet$, the indecomposable $\mathbf{P}_i$ appears, $d_{i,j}$ times or, simplifying our notations, that $\mathbf{P}_i^{d_{i,j}}$ is the component of P^j envolving the indecomposable $\mathbf{P}_i$. Thus, we can rewrite our complex as

$$\oplus_{i=1}^t \mathbf{P}_i^{d_{i,n}} \overset{\partial^n}{\to} \cdots \overset{\partial^{n+m-2}}{\to} \oplus_{i=1}^t \mathbf{P}_i^{d_{i,n+m-1}} \overset{\partial^{n+m-1}}{\to} \oplus_{i=1}^t \mathbf{P}_i^{d_{i,n+m}}.$$

As it is well known, each morphism between projectives (these being finite direct sums of indecomposables) is given by a block matrix, each block giving the morphism component that corresponds to each pair of indecomposables.

In other words, each block matrix corresponds to a morphism $\mathbf{P}_r^{d_{r,j}} \to \mathbf{P}_s^{d_{s,j+1}}$. And, as we know, the paths $w \in \mathbf{Pa}, s(w) = r, t(w) = s$ form a basis of the morphisms space, Hom $(\mathbf{P}_r, \mathbf{P}_s)$, but in our particular case of the category $\mathfrak{p}(A)$ we can assume that only paths $w \in \mathbf{Pa}_{\geq 1}$ are involved.

(If w is as indicated, it defines the morphism $p(w)$ from $\mathbf{P}_r$ to $\mathbf{P}_s$ consisting in multiplication times w *on the right*: $u \mapsto v = uw$. Any homomorphism from $\mathbf{P}_r$ to $\mathbf{P}_s$ is associated then to a linear combination of paths like w.)

Actually, as we have said already, it is convenient for us to refer our non zero paths to maximal paths, that is to elements of $\mathbf{M}$. Let us recall that, for instance, the non trivial path w determines a unique maximal path $\tilde{w} = \widehat{w}w\bar{w}$. Thus, $\widehat{w} = u$ is a special element in the basis of $\mathbf{P}_r$ having $s(u) = s(\tilde{w})$, $t(u) = s(w)$ and $v = uw \neq 0$ will be also a subpath of the same $\tilde{u} = \tilde{w}$ and a special element in the basis of $\mathbf{P}_s$. It is important to notice that the pair of non zero paths with the same origin (u, v), with $\tilde{u} = \tilde{v}$ and with $l(v) > l(u) \geq 1$ *determines* $w \in \mathbf{Pa}_{\geq 1}$ such that $v = uw$.

Hence, in order to represent our complex, we need then to give a matrix, say, $\mathbf{A}= (\mathbf{A}_{r,j}^{s,j+1})$ determining the sequence of morphisms $\partial^j, (j = n, ..., n+m-1)$ which in turn determine our complex. In particular, we have to represent the family of morphisms $p(w)$ which appear in $\partial^j : P^j \to P^{j+1}$. To facilitate to remember, it will be convenient that we use a formal sum

$$\partial^j : \quad \sum_{w \in \mathbf{Pa}_{\geq 1}} p(w)\mathbf{A}_{w,j},$$

where $\mathbf{A}_{w,j}$ denotes the matrix block that expresses the "multiplicities" of the morphism $p(w)$ in the component corresponding to $\mathbf{P}_s^{d_{s,j+1}}$ of the restriction of ∂^j to $\mathbf{P}_r^{d_{r,j}}$. Let us explain this in a detailed way:

Fixed the place j the component of ∂^j going from $\mathbf{P}_r^{d_{r,j}}$ to $\mathbf{P}_s^{d_{s,j+1}}$ is represented by a matrix $\mathbf{A}_{r,j}^{s,j+1} \in \mathrm{Mat}(d_r \times d_s; k(\langle p(w_1), \ldots, p(w_t) \rangle))$, where w_i's are the parallel non trivial paths from r to s and $k(\langle p(w_1), \ldots, p(w_t) \rangle)$ is the k-vector space with basis $\{p(w_1), \ldots, p(w_t)\}$. It is clear that $\mathbf{A}_{r,j}^{s,j+1}$ can be writing uniquely as

$$\mathbf{A}_{r,j}^{s,j+1} = \sum_{i=1}^{t} p(w_i)\mathbf{A}_{w_i,j},$$

where $\mathbf{A}_{w_i,j} \in \mathrm{Mat}(d_r \times d_s; k)$.

(It should be kept in mind that our convention is that the indecomposable projectives appearing in the domain of our ∂^j, say, correspond to rows, whereas the indecomposable appearing in the co-domain (target) correspond to columns.)

The preceding ideas lead us to the definition of our poset $\mathcal{Y}$. It has to be a product of two posets: the first corresponding to the paths and the second to the places j. As a matter of fact, we introduce a poset $\mathcal{Y}_w$ for each maximal path $w \in \mathbf{M}$. It is the set of the subpaths u of w such that $s(u) = s(w)$, ordered by the end points $t(u)$. Then our definitions are the following.

$$\mathcal{Y} = (\dot{\cup}_{w \in \mathbf{M}} \mathcal{Y}_w) \times \mathbb{Z},$$

(where the first component is an ordered disjoint union - as for this, we assume to have given some linear ordering, fixed, to $\mathbf{M}$ - and the second one is, respectively, the set of integers and where we order the two products antilexicographically, this means that $[u, i] < [v, j]$ if and only if $i < j$ or ($i = j$ and $\tilde{u} < \tilde{v}$) or ($i = j, \tilde{u} = \tilde{v}$ and $l(u) < l(v)$). It should be observed that it is possible that a (trivial) path u belongs to two different maximal paths. If it is so, the two occurrencies of u must be regarded, obviously, as *different*.

$$\oplus_{i=1}^{t} \mathbf{P}_i^{d_{i,n}} \xrightarrow{\partial^n} \cdots \xrightarrow{\partial^{n+m-2}} \oplus_{i=1}^{t} \mathbf{P}_i^{d_{i,n+m-1}} \xrightarrow{\partial^{n+m-1}} \oplus_{i=1}^{t} \mathbf{P}_i^{d_{i,n+m}}.$$

Next, we indicate how to define the involution σ. We state that $[u,i] \cong_\sigma [v,j]$ if and only if $i = j$ and $t(u) = t(v)$. This means that, for each point j, $[u,j] \cong_\sigma [v,j]$ if and only if the paths $u, v \in \dot{\cup}_{w\in\mathbf{M}}\mathcal{Y}_w$ correspond to *the same* indecomposable projective. Then σ be the involution corresponding to $\cong_\sigma$.

In order to justify this definition we prove that there are not more than two paths u, v such that $t(u) = t(v)$. (Then, in case there is only one, u, we write that $\sigma(u) = u$ and, when there are two, u, v, that σ interchanges them.)

LEMMA 3. *Let $A= k\mathbf{Q}/\mathbf{I}$ be a gentle algebra. Then two maximal paths in $\mathbf{M}$ cannot have a common arrow.*

Proof. Let us assume that w, w' are two maximal paths and α is an arrow belonging to both of them.

Then, if α is followed by one arrow for w and for w', we get a contradiction with the definition of *gentle* algebras; if α if the last arrow of only one of w, w', we get a contradiction to the maximality of this path; if α is the last arrow of both, let us consider instead the first arrow such that all the following are common arrows and perform similar arguments. ▪

COROLLARY 2. *In the conditions of this lemma, $\mathbf{Q}$ cannot have a vertex which is the end point of three, different, maximal paths of $\mathbf{M}$.*

COROLLARY 3. *Let $A= k\mathbf{Q}/\mathbf{I}$ be a gentle algebra and let u, v be two non trivial paths from $\dot{\cup}_{m\in\mathbf{M}}\mathcal{Y}_m$ such that $t(u) = t(v)$. Then, if $u \neq v$, u, v cannot have a common arrow.*

At this moment, we are going to use the notation u, v, w for three non trivial paths from $\dot{\cup}_{m\in\mathbf{M}}\mathcal{Y}_m$, which are, respectively, subpaths of the maximal paths $\tilde{u}, \tilde{v}, \tilde{w}$ of $\mathbf{M}$, and which are such that $t(u) = t(v) = t(w)$.

PROPOSITION 2. *Let $A= k\mathbf{Q}/\mathbf{I}$ be a gentle algebra and let us keep our general notations. Then, at least two of the paths u, v, w must be equal.*

Proof. Clear. ▪

In order to fullfil all our promises, it will be enough to define a functor, $\mathbf{F}$, from the category $\mathfrak{p}(A)$ to the category $\mathcal{S}(\mathcal{Y}, k)$ and show that it respects and preserves indecomposable objects.

Given the complex $P^\bullet \in \mathfrak{p}(A)$, we must begin by defining $\mathbf{F}(P^\bullet) \in \mathcal{S}(\mathcal{Y}, k)$.

This will be a matrix representing the sequence of differential maps given by $P^\bullet$. We first perform, as explained above, the decomposition of each projective (and, consequently, of each differential map) as a direct sum of indecomposables. Then, $\mathbf{F}(P^\bullet)$ is a block matrix with entries in k. Its non zero blocks appear only at places $([u,j],[v,j+1])$ corresponding to non trivial paths w that determine u, v by the relations $\tilde{w} = uw\bar{w}$, $v = uw$

$$\mathbf{F}(P^\bullet)_{[u,j]}^{[v,j+1]} = \mathbf{A}_{w,j}.$$

(Let us recall that the requirement that the image of each differential map is contained in the radical of the target projective is translated by the condition that all paths w involved at each place have length at least one).

It is clear then that, for instance, the block row u of one of our matrices at the position $[v,j] = [uw,j]$ represents $\partial^j(u)$ via the "multiplicities" of $p(w)$. Notice also that all blocks outside the diagonal of places $(j, j+1)$ (*i.e.* the diagonal above the main diagonal) are 0 blocks.

We see that the requirement that all products $\partial^j \partial^{j+1}$ are equal to zero, is translated as the requirement that all products of consecutive blocks are equal to zero. Or, equivalently, that the big matrix satisfies the condition that its square is equal to 0.

After these observations, we hope our definition of $\mathbf{F}(P^\bullet)$ becomes absolutely clear.

Now, let us indicate how to define $\mathbf{F}$ of a morphism $\varphi^\bullet : P^\bullet \to P'^\bullet$ between two complexes in $\mathfrak{p}(A)$. At each place, the morphism $\varphi^\bullet$ is a homomorphism from the projective, P^j to the projective P'^j; that is, a block matrix between the direct sums of indecomposable projectives. By representing the blocks of ϕ_j similarly as how we did with the differentials maps, by $\phi_{w,j}$, and the blocks of the differential maps of $\mathbf{F}(P'^\bullet)$ by $A'_{w,j}$, we must have that

$$\sum_{w=w_1w_2, w_1\in \mathbf{Pa}, w_2\in \mathbf{Pa}_{\geq 1}} \phi_{w_1,j} A'_{w_2,j} = \sum_{w=w_3w_4, w_3\in \mathbf{Pa}_{\geq 1}, w_4\in \mathbf{Pa}} A_{w_3,j}\phi_{w_4,j+1}.$$

In this way, $\mathbf{F}$ becomes defined also at morphisms as this matrix associated to $\varphi^\bullet$.

Let us state the definitions for $\mathbf{F}$.

In objects,

$$\mathbf{F}(P^\bullet)_{[u,i]}^{[v,j]} = \begin{cases} \mathbf{A}_{w,j} & \text{, if } j = i+1, v = uw, w \in \mathbf{Pa}_{\geq 1}; \\ 0 & \text{, otherwise,} \end{cases}$$

where block $F(P^\bullet)_{[u,i]}$ (resp. $F(P^\bullet)^{[u,i]}$) has $d_{t(u),i}$ rows (resp. columns).

In morphisms,

$$\mathbf{F}(\varphi^\bullet)_{[u,i]}^{[v,j]} = \begin{cases} \phi_{w,j} & \text{, if } i = j \text{ and } v = uw, w \in \mathbf{Pa}; \\ 0 & \text{, otherwise.} \end{cases}$$

This gives a morphism of $\mathcal{S}(\mathcal{Y}, k)$.

Example 1. *Let us consider the example given by the quiver*

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[loop above, "x"] \arrow[r, "a"] & \underset{2}{\bullet} \arrow[loop above, "y"] \end{tikzcd}$$

with the relations $x^2 = y^2 = 0$, which defines a gentle algebra over any field k.

First we look for the set of maximal paths. We see that there is only one maximal path, so that $\mathbf{M} = \{xay\}$, and $\mathbf{Pa}_{\geq 1} = \{xay, x, a, y, xa, ay\}$. Hence, the poset $\mathcal{Y}$ will be

$$\{e_1 < x < xa < xay\} \times \mathbb{Z},$$

and the involution will be given by $\sigma([e_1, j]) = [x, j]$ and $\sigma([xa, j]) = [xay, j]$.

We see that the differential maps correspond to the formal sums

$$\partial^j = A_{x,j}p(x) + A_{a,j}p(a) + A_{y,j}p(y) + A_{xa,j}p(xa) + A_{ay,j}p(ay) + A_{xay,j}p(xay).$$

Now, let us consider the following projective complex $P^\bullet$:

$$\cdots \to 0 \to P^1 = \mathbf{P}_1 \xrightarrow{\delta^1} P^2 = \mathbf{P}_1^2 \oplus \mathbf{P}_2 \to 0 \to \cdots,$$

where

$$\delta^1 = \begin{pmatrix} 2p(x) & 0 & p(a) + 3p(ay) + 2p(xay) \end{pmatrix}.$$

Then we have $A_{x,1} = (2\ 0)$, $A_{a,1} = (1)$, $A_{xa,1} = (0)$, $A_{ay,1} = (3)$ and $A_{xay,1} = (2)$.

Example 2. *Let us consider now the example given by the quiver*

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \underset{2}{\bullet} \end{tikzcd}$$

with two maximal paths: $\mathbf{M} = \{a, b\}$, which also defines a gentle algebra over any field k.

Here, the ordered set $\mathcal{Y}$ is $\{e_{s(a)} < a < e_{s(b)} < b\} \times \mathbb{Z}$ and the involution is given by $\sigma([e_{s(a)}, j] = [e_{s(b)}, j]$ (the second occurrence of e_1 must be though as different from the first one) and $\sigma([a, j]) = [b, j]$.

We see that the differential maps correspond to the formal sums

$$\partial^j = A_{a,j}p(a) + A_{b,j}p(b).$$

Now, let us consider the following projective complex $P^\bullet$:

$$\cdots \to 0 \to P^1 = \mathbf{P}_1^2 \oplus \mathbf{P}_2 \xrightarrow{\delta^1} P^2 = \mathbf{P}_2 \to 0 \to \cdots ,$$

where

$$\delta^1 = \begin{pmatrix} p(a) + 2p(b) & p(b) & 0 \end{pmatrix}^T .$$

Then we have $A_{a,1} = (1\ 0)^T$, $A_{b,1} = (2\ 1)^T$.

Let $\mathcal{U}$ be the full subcategory of $\mathcal{S}(\mathcal{Y}, k)$ defined by the objects of $\operatorname{Im} \mathbf{F}$. We can prove the following lemma which has, clearly, the following corollary, where, as usual, we use the symbol ind_0 to denote the set of indecomposables of the Krull-Schmidt category.

LEMMA 4. *1.* $\ker \mathbf{F} = 0$.

2. $X \cong Y$ *in* $\mathcal{U}$ *if and only if* $X \cong Y$ *in* $\operatorname{Im} \mathbf{F}$.

COROLLARY 4. ind_0 $\operatorname{Im} \mathbf{F} = (\operatorname{ind}_0 \mathcal{S}(\mathcal{Y}, \mathrm{k})) \cap \operatorname{Im} \mathbf{F} = \operatorname{ind}_0 \mathcal{U}$.

Proof of the lemma.

1. This is obvious by the definition of $\mathbf{F}$.
2. The implication from right to left is now obvious. To see the reverse implication, we construct a functor $\mathbf{H}$ from $\mathcal{U}$ to $\operatorname{Im} \mathbf{F}$ defining it as the identity on objects and, for a morphism φ, stipulating that

$$\mathbf{H}(\varphi)_{[u,i]}^{[v,j]} = \begin{cases} \varphi_{[u,i]}^{[v,j]} & \text{, if } j = i, \tilde{u} = \tilde{v}; \\ 0 & \text{, otherwise.} \end{cases}$$

■

4 Description of the indecomposables.

In this section we solve our problem, *i.e.* we give a description of the indecomposable objects of $\mathbf{D}^b(A)$.

We need some notations. Let us remember that k is a field and A is a gentle finite dimensional k-algebra of the form $k\mathbf{Q}/\mathbf{I}$. As it is usually done, for each arrow $a \in \mathbf{Q}_1$ we introduce its formal inverse a^{-1} and stipulate that $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$, and that $(a^{-1})^{-1} = a$. Lets us recall also that if p is a path, $p = a_1 \cdots a_n$, it is defined naturally that $p^{-1} = a_n^{-1} \cdots a_1^{-1}$. Of course, $s(p^{-1}) = t(p)$ and $t(p^{-1}) = s(p)$.

By a *walk* w (resp. a *generalized walk*) of length $n > 0$ we mean a sequence $w_1 \cdots w_n$ where each w_i is either of the form p or p^{-1}, p being an arrow (resp. a path of length > 0) and where $s(w_{i+1}) = t(w_i)$ for $1 \leq i < n$. Again, $s(w) = s(w_1)$ and $t(w) = t(w_n)$. The notion of *inverse* of a walk (resp. generalized walk) is also known, and it is clear that passage to inverses is an involutory transformation.

If we have a closed walk (resp. generalized walk), *i.e.* it happens that $s(w) = t(w)$ we consider also its , *rotations*, $w[j]$, which are the walks (generalized wals) $w_{j+1} \cdots w_n w_1 \cdots w_j$ $(j = 1, \cdots, n-1)$.

The product (= concatenation) of two walks (resp. generalized walks) $w = w_1 \cdots w_n$ and $w' = w'_1 \cdots w'_{n'}$ is defined as the walk $ww' = w_1 \cdots\ w_n w'_1 \cdots\ w'_{n'}$ provided that $t(w_n) = s(w'_1)$.

We will consider two equivalence relations on the set of generalized walks, which will be denoted by $\cong_s$ and by $\cong_r$. By definition, $\cong_s$ is the equivalence relation on the set of all generalized walks, generated by stablishing that $u \cong_s w \Leftrightarrow u = w^{-1}$; and $\cong_r$ is the equivalence relation on the set of all closed generalized walks wich identifies each generalized walk with its rotations and their inverses.

Strings and Generalized strings.

By definition, a *string* is a walk $w = w_1 \cdots w_n$ such that $w_{i+1} \neq w_i^{-1}$, for $1 \leq i < n$ and such that no subword of w or of w^{-1} is in $\mathbf{I}$. The set of all strings will be denoted by St.

With $\overline{GSt}$ let us denote the set of all generalized walks $w = w_1 \cdots w_n$ satisfying

$$\begin{array}{l} w_i w_{i+1} \in \mathbf{I} \text{ if } w_i, w_{i+1} \in \mathbf{Pa}_{\geq 1}, \\ w_{i+1}^{-1} w_i^{-1} \in \mathbf{I} \text{ if } w_i^{-1}, w_{i+1}^{-1} \in \mathbf{Pa}_{\geq 1}, \\ w_i w_{i+1} \in St \text{ otherwise.} \end{array}$$

GSt denotes a fixed set of representatives of the quotient of $\overline{GSt}$ over the equivalence relation $\cong_s$ plus all trivial paths (paths of length 0) and its elements will be called *generalized strings*.

Similarly, we define the *generalized bands* in the following way.

Firstly, let us introduce a function, μ defined on the set of generalized walks. Given the generalized walk $w = w_1 \cdots w_n$, we put, for $1 < i \leq n$

$$\mu_w(0) = 0, \mu_w(i) = \mu_w(i-1) + 1 \text{ (if } w_i \in \mathbf{Pa}_{\geq 1}) \text{ or } \mu_w(i) = \mu_w(i-1) - 1 \text{ (otherwise).}$$

After this, let us consider the set $\overline{GBa}$ of all closed generalized walks $w = w_1 \cdots w_n$ (*i.e.* $e(w_n) = s(w_1)$) such that $w^2 \in \overline{GSt}$, such that $\mu_w(n) = \mu_w(0)$ and such that they are not themselves powers. We use GBa for a fixed set of representatives of the quotient set of $\overline{Ba}$ over the equivalence relation $\cong_r$ and we call its elements *generalized bands*.

Remark. 2. *In practice, we assume in general that $\mu_w(0) \leq \mu_w(n)$. One is allowed to do this (inverting w if necessary) because if $\mu_w(0) \geq \mu_w(n)$, then $\mu_{w^{-1}}(0) \leq \mu_{w^{-1}}(n)$.*

Next, we associate to generalized strings and bands certain finite projective complexes which, as we shall see, give all the indecomposables in the category $\mathfrak{p}(A)$.

DEFINITION 2. *Let $w = w_1 \cdots w_n$ be a generalized string. Then $P_w^\bullet$ is the projective complex $\cdots P_w^i \xrightarrow{d_w^i} P_w^{i+1} \cdots$ defined as follows. The modules are*

$$P_w^i = \oplus_{j=0}^n \delta(\mu(j), i)\mathbf{P}_{c(j)},$$

where μ, $\mathbf{P}_{c(j)}$ were already defined and where δ is the Kronecker-delta, $c(j)$ is a short for $t(w_j)$ and $c(0) = s(w)$. The differential maps are $d_w^i = (d_{j,k}^i)_{0\leq j,k \leq n}$ where, for each $i \in \mathbb{Z}$,

$$d_{jk}^i = \begin{cases} P(w_{j+1}) & \text{, if } w_{j+1} \in \mathbf{Pa}_{>0}, \mu(j) = i \text{ and } k = j+1; \\ P(w_j^{-1}) & \text{, if } w_j^{-1} \in \mathbf{Pa}_{>0}, \mu(j) = i \text{ and } k = j-1; \\ 0 & \text{, otherwise.} \end{cases}$$

The analogous definition is a little more elaborated in the case of generalized bands. Let us call Ind $k[x]$ the set of all indecomposable polynomials, except $\{x^d \mid d \geq 1\}$, with coefficients in our field k. If $f(x) \in \mathrm{Ind}k[x]$ let us denote with $F_{f(x)}$ the corresponoding Frobenius matrix.

DEFINITION 3. *Let $w = w_1 \cdots w_n$ be a generalized band and let $f(x) \in \mathrm{Ind}k[x]$. Then $P_{w,f}^\bullet$ is the projective complex $\cdots P_{w,f}^i \xrightarrow{d_w^i} P_{w,f}^{i+1} \cdots$ defined as follows. The modules are*

$$P_{w,f}^i = \oplus_{j=0}^{n-1} \delta(\mu(j), i)\mathbf{P}_{c(j)}^{\deg f}.$$

The differential maps are $d_{w,f}^i = (d_{j,k}^i)_{0\leq j,k<n}$ where, for each $i \in \mathbb{Z}$,

$$d_{jk}^i = \begin{cases} P(w_{j+1})\mathbf{Id}_{deg f(x)} & \text{, if } w_{j+1} \in \mathbf{Pa}_{>0}, \mu(j) = i \text{ and } k = j+1; \\ P(w_j^{-1})\mathbf{Id}_{deg f(x)} & \text{, if } w_j^{-1} \in \mathbf{Pa}_{>0}, \mu(j) = i \text{ and } k = j-1; \\ P(w_n)F_{f(x)} & \text{, if } w_n \in \mathbf{Pa}_{>0}, \mu(j) = i,\ j = n-1 \text{ and } k = 0; \\ P(w_n^{-1})F_{f(x)} & \text{, if } w_n^{-1} \in \mathbf{Pa}_{>0}, \mu(j) = i,\ j = 0 \text{ and } k = n-1; \\ 0 & \text{, otherwise.} \end{cases}$$

Indecomposasble matrix representations.

Now we are going to describe (conveniently for us) the indecomposable representations of a Bondarenko's poset.

Let $\mathcal{Y} = (\mathcal{Y}, \sigma)$ be a linearly ordered poset with involution and let $\mathbf{Q}(\mathcal{Y})$ be the quiver defined as follows.

$$\mathbf{Q}(\mathcal{Y})_0 = \mathcal{Y}/\sigma,$$

$$\mathbf{Q}(\mathcal{Y})_1 = \mathcal{Y} \times \mathcal{Y} = \mathcal{Y}^2.$$

Next, let us introduce the following additional definitions.
For $\alpha \in \mathcal{Y}$ $\overline{\alpha}$ denotes the class of α modulo $\cong_\sigma$.
If $\alpha, \beta \in \mathcal{Y}$,

$$s((\alpha, \beta)) = \overline{\alpha}$$

and

$$t((\alpha, \beta)) = \overline{\beta}.$$

p_i $(i = 1, 2)$ is the natural projection of $\mathcal{Y}^2$ onto $\mathcal{Y}$, i.e., for instance, $p_1((\alpha, \beta)) = \alpha = p_2((\alpha, \beta)^{-1})$.

Let $\overline{St(\mathcal{Y})}$ be the set of words $w = w_1 \dots w_n$ such that $p_2(w_i) \neq p_1(w_{i+1})$ for $1 \leq i < n$. We will denote by $St(\mathcal{Y})$ a set of representatives of such words under the relation $\cong_s$ and of paths of lenght 0. We call the elements of $St(\mathcal{Y})$ $\mathcal{Y}$-*strings*.

Consider the set of words $w = w_1 \dots w_n$ such that $s(w) = t(w)$, $w^2 \in \overline{St(\mathcal{Y})}$ and which are not themselves powers. Let $Ba(\mathcal{Y})$ be the set of representatives of such words under $\cong_r$. We call the elements of $Ba(\mathcal{Y})$ $\mathcal{Y}$-*bands*. For each $\mathcal{Y} - string$ $w = w_1 \cdots w_n$ we define B_w, an $\mathcal{Y}$-matrix and at the same time a representation of $\mathbf{Q}(\mathcal{Y})$ in the following way.

Let us consider a k-vector space with basis some set of $n+1$ vectors $v_0, \cdots v_n$ and, given $y \in \mathcal{Y}$, the subspace $M_w(\overline{y})$ generated by the set of v_i's such that $c(i) = \overline{y}$ (see definition 2).

Now, for each $(s, t) \in \mathbf{Q}_1(\mathcal{Y})$, let us define the linear map $M_w(s, t)$ by the following rules.

$$M_w(s,t)(v_i) = \begin{cases} v_{i+1} & \text{, if } w_{i+1} = (s,t)\ ; \\ v_{i-1} & \text{, if } w_i^{-1} = (s,t); \\ 0 & \text{, otherwise.} \end{cases}$$

Next, let $(B_w)_s^t$ be the matrix representing of $M_w((s, t))$ in the fixed basis.

Analogously, for each $\mathcal{Y}$-band $w = w_1 \cdots w_n$ and each indecomposable polynomial $f(x) = \alpha_1 + \cdots + \alpha_d x^{d-1} + x^d$ (not equal to x^d) let us define $B_{w,f}$ in the following way.

Let us consider the k-vector space with basis some set of $n \times d$ vectors v_{ij}, $(i = 0, \cdots, n-1; j = 1, \cdots, d)$ and, given $y \in \mathcal{Y}$, the subspace $M_{w,f}(\overline{y})$ generated by the set of v_{ij}'s such that $c(i) = \overline{y}$.

Now, for each $(s,t) \in \mathbf{Q}_1(\mathcal{Y})$, we define the linear map $M_{w,f}(s,t)$ by the following rules.

$$M_w((s,t))(v_{ij}) = \begin{cases} v_{i+1j} & \text{, if } i \neq n-1 \text{ and } w_{i+1} = (s,t); \\ v_{i-1j} & \text{, if } i \neq 0 \text{ and } w_i^{-1} = (s,t); \\ v_{0j+1} & \text{, if } i = n-1,\ j \neq d \text{ and } w_n = (s,t); \\ v_{n-1j+1} & \text{, if } i = 0, j \neq d \text{ and } w_n^{-1} = (s,t); \\ -\sum_r \alpha_r v_{0r} & \text{, if } i = n-1, j = d \text{ and } w_n = (s,t); \\ -\sum_r \alpha_r v_{n-1r} & \text{, if } i = 0,\ j = d \text{ and } w_n^{-1} = (s,t); \\ 0 & \text{, otherwise.} \end{cases}$$

Next, let $(B_{w,f})_s^t$ be the matrix representing of $M_{w,f}((s,t))$ in the fixed basis.

From Proposition 1 in [B] (se also [BD]) we obtain the following theorem.

THEOREM 2. *Let $\mathcal{Y} = (\mathcal{Y}, \sigma)$ be a linearly ordered set with involution. Then*

$$\mathrm{ind}_o \mathcal{S}(\mathcal{Y}, k) = \{B_u \,|\, u \in St(\mathcal{Y})\} \dot\cup \{B_{v,f} \,|\, v \in Ba(\mathcal{Y}), f \in \mathrm{Ind}k[x]\}.$$

▪

The main theorem.

Now we return to our context of $A = k\mathbf{Q}/\mathbf{I}$. In order to simplify our exposition, let us introduce now some additional notations.

For a walk $w = w_1 \cdots w_n$ we write $\mu(w)$ for the minimum of the $\mu_w(i)$, $i = 0, \cdots, n$.

$\mathbf{Q}_c = \{a \in \mathbf{Q}_1 \,|\, \exists\ a_1, \ldots, a_m \in \mathbf{Q}_1$ such that $s(a_{i+1}) = t(a_i), s(a_1) = t(a_m), a_1 = a$ and $a_i a_{i+1} = a_m a_1 = 0\}$.

$\overline{GSt}_c = \{w = w_1 \ldots w_n \in GSt |\ l(w) > 0$ and $\exists a \in \mathbf{Q}_c$ such that $aw \in GSt$, and $\mu(w) = 0\}$.

$\overline{GSt}^c = \{w = w_1 \ldots w_n \in GSt |\ l(w) > 0$ and $\exists a \in \mathbf{Q}_c$ such that $wa^{-1} \in GSt$, and $\mu(w) = \mu_w(n)\}$.

$GSt_c = \{w = w_1 \ldots w_n \in \overline{GSt}_c |$ if $w_1 \in \mathbf{Q}_c$ then $w_2 \ldots w_n \notin \overline{GSt}_c\}$.

$GSt^c = \{w = w_1 \ldots w_n \in \overline{GSt}^c |$ if $w_n^{-1} \in \mathbf{Q}_c$ then $w_1 \ldots w_{n-1} \notin \overline{GSt}^c\}$.

$GSt_c^c = GSt_c \cap GSt^c$.

For every $w \in \mathbf{Pa}_{>0}$

$$\check{w} = \begin{cases} a, \text{ if } \exists\ a \in \mathbf{Q}_1 \text{ such that } aw = 0; \\ 0, \quad \text{otherwise.} \end{cases}$$

LEMMA 5. *Let A be a gentle algebra and let $w \in \mathbf{Pa}_{>0}$. Then we have*

1. *ker* $p(w) = A\breve{w}$;

2. $\beta(P^\bullet_w)^\bullet = P^\bullet_w$ *for any* $w = w_1 w_2$ *such that* w_1^{-1} *and* w_2 *are belong to* $\mathbf{Pa}_{>0}$.

PROOF. 1 is obvious and, for 2, let us observe that, since A is gentle, $\ker p(w_1^{-1}) \cap \ker p(w_2) = 0$.∎

As consequence we obtain the following

LEMMA 6. *Let* A *be a gentle algebra. Then we have*

1. *For every* $w \in GBa$ *and* $f \in \mathrm{Ind}k[x]$, *we have* $\beta(P^\bullet_{w,f})^\bullet = P^\bullet_{w,f}$;

2. *In order to have that* $P^\bullet_{\beta(P^\bullet_w)^\bullet} \notin \mathbf{K}^b(proA)$ *it is necessary and sufficiernt that either* $w \in GSt_c$ *or* $w \in GSt^c$.

∎

Now we are in a good position to state and proof our main theorem which gives all the indecomposable objects of the derived category.

THEOREM 3. *Let* $A = k\mathbf{Q}/\mathbf{I}$ *be a gentle algebra and let us keep our foregoing notations. We have that (*T *being the translation functor)*

$$\mathrm{ind}_0\mathbf{D}^b(A) = \{T^i(P^\bullet_w) \,|\, w \in GSt, i \in \mathbb{Z}\} \cup \{T^i(P^\bullet_{w,f}) \,|\, w \in GBa, f \in \mathrm{Ind}k[x], i \in \mathbb{Z}\}$$

$$\cup \{T^i(\beta(P^\bullet_w)^\bullet \,|\, w \in GSt_c, i \in \mathbb{Z}\} \cup \{T^i(\beta(P^\bullet_w)^\bullet \,|\, w \in GSt^c \setminus GSt^c_c\ , i \in \mathbb{Z}\}.$$

PROOF.

Step 1. Let us define

$$\gamma : \overline{GSt(A)} \times \mathbb{Z} \to \overline{St(\mathcal{Y}(A))}$$

by

$$\gamma(w, m) = \gamma(w_1, m) \cdots \gamma(w_n, m),$$

$$\gamma(w_i, m) = ([\widehat{w}_i, \mu(i-1) + m], [\widehat{w}_i w_i, \mu(i) + m]) \text{ in case } w_i \in \mathbf{Pa}_{>0},$$

$$\gamma(w_i, m) = ([\widehat{w_i^{-1}}, \mu(i) + m], [\widehat{w_i^{-1}} w_i^{-1}, \mu(i-1) + m])^{-1} \text{ in case } w_i^{-1} \in \mathbf{Pa}_{>0},$$

where we assume that $w = w_1 \cdots w_n$ and that $m \in \mathbb{Z}$.

We are going to show that $\gamma(w, m) \in \overline{St(\mathcal{Y}(A))}$, that is that $t(\gamma(w_i, m)) = s(\gamma(w_{i+1}, m))$ and that $p_2(\gamma(w_i, m)) \neq p_1(\gamma(w_{i+1}, m))$ for ewach $1 \le i < n$. We distinguish four cases (a), (b), (c), (d).

(a) $w_i, w_{i+1} \in \mathbf{Pa}_{>0}$.

Since $t(\widehat{w}_i w_i) = t(w_i) = s(w_{i+1}) = t(\widehat{w_{i+1}})$, we have $t(\gamma(w_i, m)) = \overline{[\widehat{w}_i w_i, \mu(i) + m]} = \overline{[\widehat{w_{i+1}}, \mu(i) + m]} = s(\gamma(w_{i+1}, m))$.

On the other hand, since $\widehat{w_{i+1}}w_{i+1} \neq 0$ but $w_iw_{i+1} = 0$, we have $\widehat{w}_iw_i \neq \widehat{w_{i+1}}$ and hence $p_2(\gamma(w_i, m)) = [\widehat{w}_iw_i, \mu(i) + m] \neq [\widehat{w_{i+1}}, \mu(i) + m] = p_1(\gamma(w_{i+1}, m))$.

(b) $w_i, w_{i+1}^{-1} \in \mathbf{Pa}_{>0}$.

Since $t(\widehat{w}_iw_i) = t(w_i) = s(w_{i+1}) = t(w_{i+1}^{-1})$, we have $t(\gamma(w_i, m)) = \overline{[\widehat{w}_iw_i, \mu(i) + m]} = \overline{[\widehat{w_{i+1}^{-1}}w_{i+1}^{-1}, \mu(i) + m]} = s(\gamma(w_{i+1}, m))$.

On the other hand, since $w_iw_{i+1} \in St$, we have $\widehat{w}_iw_i \neq \widehat{w_{i+1}^{-1}}w_{i+1}^{-1}$, so that $p_2(\gamma(w_i, m)) = [\widehat{w}_iw_i, \mu(i) + m] \neq [\widehat{w_{i+1}^{-1}}w_{i+1}^{-1}, \mu(i) + m] = p_1(\gamma(w_{i+1}, m))$.

(c) $w_i^{-1}, w_{i+1} \in \mathbf{Pa}_{>0}$.

This, in a sense, is the dual of case (b).

(d) $w_i^{-1}, w_{i+1}^{-1} \in \mathbf{Pa}_{>0}$.

This, in a sense, is the dual of case (a).

Since, for any $u, v \in \mathbf{Pa}_{>0}$, $\widehat{u} = \widehat{v}$ and $\widehat{u}u = \widehat{v}v$ if and only if $u = v$, γ is mono. We show now that

$$\operatorname{Im}\gamma = \{w \in \overline{St(\mathcal{Y}(A))} | B_w \in \operatorname{Im}\mathbf{F}\}.$$

Let $w = w_1 \cdots w_n \in St(\mathcal{Y}(A))$ and $B_w \in \operatorname{Im}\mathbf{F}$. Then, for some $x_i \in \mathbf{Pa}_{>0}$, $w_i = ([\widehat{x}_i, \mu_w(i-1)], [\widehat{x}_ix_i, \mu_w(i)])$ if $w_i \in \mathbf{Q}(\mathcal{Y})_1$ and $w_i = ([\widehat{x}_i, \mu_w(i)], [\widehat{x}_ix_i, \mu_w(i-1)])^{-1}$ if $w_i^{-1} \in \mathbf{Q}(\mathcal{Y}(A))_1$.

For convenience, let us put

$$y_i = \begin{cases} x_i & \text{, if } w_i \in \mathbf{Q}(\mathcal{Y})_1; \\ x_i^{-1} & \text{, if } w_i^{-1} \in \mathbf{Q}(\mathcal{Y})_1. \end{cases}$$

We will show that $y = y_1 \cdots y_n \in \overline{GSt(A)}$, that is that $y_iy_{i+1} \in \overline{GSt(A)}$ for $1 \leq i < n$. As for this, we again distinguish four cases.

(a) $w_i, w_{i+1} \in \mathbf{Q}(\mathcal{Y})_1$;

In this case $t(x_i) = t(\widehat{x_{i+1}}) = s(x_{i+1})$. On the other hand, since $\widehat{x}_ix_i \neq \widehat{x_{i+1}}$ and since $\widehat{x_{i+1}}x_{i+1} \neq 0$, we have $x_ix_{i+1} = 0$, so that $y_iy_{i+1} \in \overline{GSt(A)}$.

(b) $w_i, w_{i+1}^{-1} \in \mathbf{Q}(\mathcal{Y})_1$; Here $t(x_{i+1}) = t(x_i)$. Since A is gentle and $\widehat{x}_ix_i \neq \widehat{x_{i+1}}x_{i+1}$, we have that $x_ix_{i+1}^{-1} \in St$, so that, once more, $y_iy_{i+1} \in \overline{GSt(A)}$.

(c) $w_i^{-1}, w_{i+1} \in \mathbf{Q}(\mathcal{Y})_1$;

This, in a sense, is the dual of case (b).

(d) $w_i^{-1}, w_{i+1}^{-1} \in \mathbf{Q}(\mathcal{Y})_1$;

This, in a sense, is the dual of case (a).

Let γ_b be the restriction of γ on GBa. Then it is clear that

$$\operatorname{Im}\gamma_b = \{w \in \overline{Ba(\mathcal{Y}(A))} | B_{w,1} \in \operatorname{Im}\mathbf{F}\}.$$

Step 2. Let u be in GSt, v in GBa and $f \in Ind\, k[x]$. Then it is easy to see that $F(P_u^\bullet) = B_{\gamma(u)}$ and $F(P_{v,f}^\bullet) = B_{\gamma_b(v),f}$.

Step 3. It follows from Steps 1 and 2, and Theorem 2 that
$ind_0\mathbf{K}^b(\text{pro}A) = \{T^i(P_u^\bullet), T^i(P_{v,f}^\bullet) \mid u \in GSt, v \in GBa, f \in Ind\, k[x] \text{ and } i \in \mathbb{Z}\}$.

Step 4. We end up our proof with the following observation. It follows from Lemma 6 that $\{\beta(M^\bullet)^\bullet | M^\bullet \in \text{ind}\,_0\mathfrak{p}(A)$ and $P^\bullet_{\beta(M^\bullet)^\bullet} \notin \mathbf{K}^b(\text{pro}A)\} = \{T^i(\beta(P_w^\bullet)^\bullet), T^i(\beta(P_r^\bullet)^\bullet) \mid w \in GSt_c, r \in GSt^c \setminus GSt_c^c \text{ and } i \in \mathbb{Z}\}$. ▪

Joining together the preceeding results and our definitions above, we obtain the following theorem.

THEOREM 4. *Let k be an algebraically closed field and $A = k\mathbf{Q}/\mathbf{I}$ be a gentle algebra. Then*

a. A is derived tame;

b. A is derived discrete if and only if $GBa = \emptyset$;

c. A is derived finite if and only if $\mid GSt \mid < \infty$.

▪

Remark. 3. *If A has finite global dimension, the statement a. of the theorem follows from [Ri1].*

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