DENSITY OF RECURRENT POINTS ON INVARIANT MANIFOLDS OF SYMPLECTIC AND VOLUME PRESERVING DIFFEOMORPHISMS.

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ABSTRACT. In this paper we consider symplectic and volume preserving diffeomorphisms of compact manifolds, and prove that, C^r generically, $1 \leq r \leq \infty$, the invariant manifolds of hyperbolic periodic points contain a dense subset of recurrent points. As a consequence, each of these invariant manifolds is contained in its omega limit set, and its closure is a chain transitive set.

1. INTRODUCTION AND STATEMENT OF RESULTS.

The behavior of invariant manifolds is an important point in the study of conservative systems. Poincaré was probably the first one to draw the trellis of invariant manifolds that arises in the presence of a tranverse homoclinic orbit, and became astonished with the complexity of the phenomenon. Most of the interesting facts about the dynamics on invariant manifolds are known only in dimension two, and carrying these results to higher dimensions is a challenging task.

Recently, the interest on the subject was revived by the work of Franks and Le Calvez [FrCa] on the dynamics of area preserving maps of the two dimensional sphere. They define a more general notion of regions of instability, and extend to these context many of the classical results, with no need for a twist condition. They show that the union of invariant manifolds of hyperbolic periodic orbits form a dense subset of these regions, and that their boundaries have connecting orbits.

In this work, we show that, generically, the invariant manifolds of hyperbolic periodic orbits of symplectic and volume preserving diffeomorphisms contain a dense subset of recurrent points. This results in two other aspects of recurrent behavior. Each of these invariant manifolds accumulates all over itself, a condition best formulated in terms of an ω-limit set, and the closure of each one is a chain transitive set. It is not known whether the restriction of a diffeomorphism to the closure of an invariant manifold is topologically transitive, even in dimension two, a question that could lead to the understanding of other properties like density of homoclinic and periodic orbits.

It is worth mentioning some related results in dimension two. A result of Mather [Ma] shows that, generically, the invariant manifolds (in fact the four branches) of hyperbolic periodic points have the same closure. Another, is that the existence of recurrent points in the invariant manifolds is sufficient for the existence of homoclinic points, when the surface is the sphere or the torus [Ol1], and that the

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same phenomenon holds in higher genus, for diffeomorphisms in most homotopy classes [Ol2]. A third one, is a result of Robinson [Ro2] and Pixton [Pi], that also holds for dissipative diffeomorphisms. They show that, if the invariant manifolds intersect only at the periodic orbit, and if the unstable manifold accumulates on a point z of the stable manifold, then, by a small perturbation supported in a small neighborhood of z, it is possible to turn z into a homoclinic point.

We start by recalling some basic facts. Good references for most of what we say are the books of Abraham and Marsden [AbMa], Palis and de Melo [PaMe], and Roig and Domingues [RoDo].

We consider a C^∞ compact manifold M with a volume or a symplectic structure. A volume structure is given by a nowhere zero n-form, where n is the dimension of M, and a symplectic structure is given by a symplectic form, that is, a nondegenerate closed 2-form. In both cases we denote the form by η, and assume it to be C^∞.

Let $D^r_\eta(M)$ be the space of all C^r diffeomorphisms f of M, such that $f^*\eta = \eta$, equipped with the C^r topology, $1 \leq r \leq \infty$. $D^r_\eta(M)$ is a topological group with respect to composition. It is also a Baire space, and we say that a property is C^r generic if it holds for diffeomorphisms in a residual subset of $D^r_\eta(M)$.

Let p be a periodic point of f. We will always denote its period by ξ. We say that p is elementary if 1 is not an eigenvalue of $df^\xi(p)$, and that p is hyperbolic if no eigenvalue of $df^\xi(p)$ has absolute value 1. If f is volume preserving or symplectic, then hyperbolic periodic points are of saddle type, and the stable and unstable invariant manifolds of p are

$$W^s(p) = \left\{x \in M\,;\ f^{\xi n}x \to p,\ as\ n \to \infty\right\}$$

and

$$W^u(p) = \left\{x \in M\,;\ f^{\xi n}x \to p,\ as\ n \to -\infty\right\},$$

respectively. These are injectively immersed submanifolds that intersect transversely at p. When we wish to stress their dependence on f, we will write $W^s(p,f)$ and $W^u(p,f)$.

The main result is the following.

Theorem 1. *$D^r_\eta(M)$ contains a residual subset R, such that, if p is a hyperbolic periodic point of $f \in R$, then $W^u(p)$ contains a dense subset of f-recurrent points, and $W^s(p)$ contains a dense subset of f^{-1}-recurrent points.*

Fix a Riemannian metric on M, and denote distances by d and open balls of radius ε centered at p by $B_\varepsilon(p)$.

Let $W^u_\varepsilon(p)$ be the connected component of $W^u(p) \cap B_\varepsilon(p)$ which contains p. For every ε sufficiently small, $W^u_\varepsilon(p)$ is an embedded disk whose dimension is that of $W^u(p)$. $W^u_\varepsilon(p)$ is called the unstable manifold of size ε at p. The set $D^u_\varepsilon(p) = \overline{W^u_\varepsilon(p) - f^{-\xi}W^u_\varepsilon(p)}$ is called a fundamental domain of $W^u(p)$. $D^u_\varepsilon(p)$ is diffeomorphic to the product of a compact interval and a sphere (when $W^u(p)$ is a curve, $D^u_\varepsilon(p)$ consists of two compact intervals). We have that $W^u(p) - \{p\} = \cup_{n\in\mathbb{Z}} f^{\xi n} D^u_\varepsilon(p)$, and if $x \in W^u(p) - \{p\}$ then the orbit of x under f^ξ has at most two points in $D^u_\varepsilon(p)$. Similarly, for every ε sufficiently small, we have the stable manifold of size ε at p, $W^s_\varepsilon(p)$, and fundamental domains $D^s_\varepsilon(p)$, with analogous properties.

For a subset E of M and $-\infty \leq j < k \leq +\infty$, we use $O_j^k(E,f)$ to denote the set $\bigcup_{i=j}^k f^i(E)$, and $O(E,f)$ to denote $O_{-\infty}^{+\infty}(E,f)$. The ω-limit set of E is the set

$$\omega(E,f) = \left\{ y \in M \, ; \; y = \lim_{i\to\infty} f^{n_i} x_i \, , \; x_i \in E \, , \; n_i \to \infty \right\} .$$

If E is non-empty, then $\omega(E,f)$ is non-empty , compact, f-invariant and

$$\overline{O_k^\infty(E,f)} = O_k^\infty(E,f) \cup \omega(E,f) \;, \; \text{ for any } k.$$

If p is a hyperbolic periodic point of f with period ξ and $D_\varepsilon^u(p)$ is a fundamental domain of $W^u(p)$, we define the ω-limit set of $W^u(p)$ as $\omega(D_\varepsilon^u(p), f^\xi)$ and denote it by $\omega(W^u(p))$. The definition is independent of ε, and $\overline{W^u(p)} = W^u(p) \cup \omega(W^u(p))$. Similarly, for the stable manifold, we use f^{-1} to define its α-limit set $\alpha(W^s(p))$ as $\omega(D_\varepsilon^s(p), f^{-\xi})$.

Corollary 1. *$D_\eta^r(M)$ contains a residual subset R, such that, if p is a hyperbolic periodic point of $f \in R$, then $W^u(p) \subset \omega(W^u(p))$ and $W^s(p) \subset \alpha(W^s(p))$.*

Corollary 1 follows from the following proposition.

Proposition 1. *Let h be a homeomorphism of a topological space X, and $x \in X$ an h-recurrent point. Then x is h^ξ-recurrent for any $\xi \geq 1$.*

Proof. If g is any homeomorphism of X and $y \in X$, let $\omega(y,g)$ denote the ω-limit set of y with respect to g. The following properties are immediate consequences of the definition of $\omega(y,g)$.

(a) $g^i\omega(y,g^j) = \omega(g^iy, g^j)$, for any $i,j \in \mathbb{Z}$ and $y \in X$.

(b) $\omega(y,g)$ is g-invariant for any $y \in X$.

(c) Let $a,b,c \in X$. If $a \in \omega(b,g)$ and $b \in \omega(c,g)$, then $a \in \omega(c,g)$.

Firstly, we show that there is $m \in \{0,...,\xi-1\}$ such that $h^m x \in \omega(x,h^\xi)$. Let n_k be a sequence such that $\lim_{k\to\infty} n_k = +\infty$ and $\lim_{k\to\infty} h^{n_k}x = x$. For each k, let $r_k \in \{0,...,\xi-1\}$ be the number such that ξ divides $n_k + r_k$. The limit set of the sequence $(h^{n_k+r_k}x)_{k\in\mathbb{N}}$ is contained in $\{x,...,h^{\xi-1}x\}$, implying that $h^m x$ is a limit point of $(h^{n_k+r_k}x)_{k\in\mathbb{N}}$ for some $m \in \{0,...,\xi-1\}$. Therefore, $h^m x \in \omega(x,h^\xi)$.

Now, we show that $h^{im}x \in \omega(x,h^\xi)$ for every integer $i > 1$. Assuming by induction that $h^{im}x \in \omega(x,h^\xi)$, we have that $h^{(i+1)m}x = h^m h^{im} x \in h^m\omega(x,h^\xi)$, and by (a) it follows that $h^{(i+1)m}x \in \omega(h^m x, h^\xi)$. Since $h^m x \in \omega(x,h^\xi)$, by (c) we have that $h^{(i+1)m}x \in \omega(x,h^\xi)$.

Finally, we have $h^{\xi m}x \in \omega(x,h^\xi)$, and by (b) $x \in \omega(x,h^\xi)$. □

Let now $h: X \to X$ be a homeomorphism of a compact metric space X. An invariant set A is chain transitive, if given any two points x and y of A, there is an ε-chain in A from x to y, that is, there is a sequence $x = x_0, x_1, ..., x_n = y$, such that $x_i \in A$ and $d(h(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i \leq n-1$.

The following result was observed by John Franks, during a talk he gave at the Courant Institute in May of 2000.

Corollary 2. *$D_\eta^r(M)$ contains a residual subset R, such that, if p is a hyperbolic periodic point of $f \in R$, then $\overline{W^u(p)}$ and $\overline{W^s(p)}$ are chain transitive sets under f^ξ, where ξ is the period of p.*

Corollary 2 follows from the existence of complete Lyapounov functions, a result which we now briefly review. We refer the reader to [Fr] for details.

A point $x \in X$ is called chain recurrent if for every $\varepsilon > 0$ there is an ε-chain from x to itself. The set of chain recurrent points is called the chain recurrent set of h, and we denote it by $\mathcal{R}(h)$. This set is always compact and invariant under h. It is easy to see that if X contains a dense subset of recurrent points, then $\mathcal{R}(h) = X$. If we define a relation $\sim$ on $\mathcal{R}(h)$ by $x \sim y$ if for every $\varepsilon > 0$ there is an ε-chain from x to y and another from y to x, then $\sim$ is an equivalence relation on $\mathcal{R}(h)$, and its equivalence classes are called the chain transitive components of $\mathcal{R}(h)$. We have that $\mathcal{R}(h) = \mathcal{R}(h^{-1})$, that the chain transitive components of $\mathcal{R}(h)$ and $\mathcal{R}(h^{-1})$ are the same, and that an invariant set C is chain transitive under h if and only if it is under h^{-1}. Chain transitive sets are interesting because the restriction of a homeomorphism to such a set has no attractors or repellers.

A complete Lyapounov function for h is a continuous function $g : X \to \mathbb{R}$ with the following properties:

(L1) If $x \notin \mathcal{R}(h)$, then $g(h(x)) < g(x)$.

(L2) If $x, y \in \mathcal{R}(h)$, then $g(x) = g(y)$ if and only if $x \sim y$.

(L3) $g(\mathcal{R}(h))$ is a compact nowhere dense subset of $\mathbb{R}$.

Every homeomorphism of a compact metric space has a complete Lyapounov function.

From (L3), it follows that if C is a connected subset of $\mathcal{R}(h)$, then $g(C)$ consists of a single point, and by (L2), we have that C is contained in a chain transitive component of $\mathcal{R}(h)$. In particular, if $\mathcal{R}(h)$ is connected, then $\mathcal{R}(h)$ is chain transitive, and if X is connected and $\mathcal{R}(h) = X$, then X is chain transitive.

$\overline{W^u(p)}$ and $\overline{W^s(p)}$ are invariant under f^ξ. Corollary 2 then follows from considering the restrictions of f^ξ to $\overline{W^u(p)}$ and of $f^{-\xi}$ to $\overline{W^s(p)}$. In fact, since f^ξ-recurrent points are dense in $\overline{W^u(p)}$, we have that $\mathcal{R}(f^\xi \big/_{\overline{W^u(p)}}) = \overline{W^u(p)}$, and since $\overline{W^u(p)}$ is connected, we have that $\overline{W^u(p)}$ is chain transitive under $f^\xi \big/_{\overline{W^u(p)}}$. Therefore, any two points of $\overline{W^u(p)}$ can be connected by an ε-chain for f^ξ contained in $\overline{W^u(p)}$. The reasoning for $\overline{W^s(p)}$ is the same.

2. Proof of Theorem 1.

In this section we prove Theorem 1. For the well known properties of conservative systems listed below, we refer the reader to the work of Robinson [Ro1].

Firstly, we note that it is enough to show that $D^r_\eta(M)$ contains a residual subset R^+, such that, if p is a hyperbolic periodic point of $f \in R^+$, then $W^u(p, f)$ contains a dense subset of f-recurrent points. In fact, the inversion $\iota(f) = f^{-1}$ is a homeomorphism of $D^r_\eta(M)$, and therefore $R^- = \iota(R^+)$ is also a residual subset of $D^r_\eta(M)$. Let $R = R^+ \cap R^-$. Consider $f \in R$ and p a hyperbolic periodic point of f. Since $f \in R^+$, we have that $W^u(p, f)$ contains a dense subset of f-recurrent points, and since $f \in R^-$, we have $f^{-1} \in R^+$, implying that $W^s(p, f) = W^u(p, f^{-1})$ contains a dense subset of f^{-1}-recurrent points.

If p is a hyperbolic periodic point of period ξ, we are going to use o to denote its orbit $\{f^i p \ ; \ 0 \leq i \leq \xi - 1\}$. The unstable invariant manifold of o is $W^u(o) = \bigcup_{i=0}^{\xi-1} W^u(f^i p) = \{x \in M \ ; \ d(f^n x, o) \to 0, \ as \ n \to -\infty\}$, and we say that D is a fundamental domain of $W^u(o)$, if D is a fundamental domain of $W^u(f^i p)$ for some $i \in \{0, ..., \xi - 1\}$. It follows that $W^u(o) - o = O(D, f)$. Stable manifolds of hyperbolic periodic orbits are defined in the same way, and have similar properties.

Now we describe the sets involved in the construction of the residual set R^+.

We consider a collection of sets $A_{n,k,m}$ indexed by $n,k,m \in \mathbb{N}$ defined in the following way.

$f \in A_{n,k,m}$ when it satisfies the following two conditions:

(1) *In the symplectic case, every periodic point p of f with period less than or equal to n is elementary and all the eigenvalues of $df^{\xi}(p)$ have multiplicity one. In the volume preserving case with dimension of M greater than or equal to 3, every periodic point p of f with period less than or equal to n is hyperbolic.*

(2) *If o is a hyperbolic periodic orbit of f with period less than or equal to n, then $W^u(o)$ has a fundamental domain D, depending only on f and o, and not on k nor m, which satisfies the following condition:*

(A) *D contains a finite subset $Z(k)$, depending only on f, o, D and k, and not on m, which satisfies the following two conditions:*

(A1) *$Z(k)$ is $\frac{1}{k}$-dense in D,*
(A2) *If $z \in Z(k)$, then $O_1^{\infty}(z,f) \cap B_{\frac{1}{m}}(z) \neq \emptyset$.*

Let us assume for the moment that each $A_{n,k,m}$ is open and dense in $D^r_{\eta}(M)$, and let us show that $R^+ = \bigcap_{n,k,m} A_{n,k,m}$ is the desired residual set.

Consider $f \in R^+$ and a hyperbolic periodic orbit o of f. Fix n such that the period of o is less than or equal to n. We have that $f \in A_{n,k,m}$ for every k and m. Let D be the fundamental domain of o which satisfies (A) for every k and m. Defining $Z = \bigcup_k Z(k)$, we have that Z is a dense subset of D. Let $z \in Z$, and fix k such that $z \in Z(k)$. Since $f \in A_{n,k,m}$ for every m, we have that $O_1^{\infty}(z,f) \cap B_{\frac{1}{m}}(z) \neq \emptyset$ for every m, implying that every point of Z is f-recurrent. Therefore, $O(Z,f)$ is a dense subset of $W^u(o)$ made of f-recurrent points.

Now, let us show that each $A_{n,k,m}$ is open. The set of diffeomorphisms which satisfy (1) is open and dense in $D^r_{\eta}(M)$. This is Theorem 1A and 1B of [Ro1] (note that his notion of elementary periodic points is different, and that if p is a 1-elementary periodic point (according to him), then the eigenvalues of $df^{\xi}(p)$ have multiplicity 1). In order to see that the set of diffeomorphisms which satisfy (2) is open, firstly note that if f satisfies (1), then the set of hyperbolic periodic orbits of f with period $\leq n$ is finite, has constant cardinality and depend continuously on diffeomorphisms in a neighborhood of f. Furthermore, their local invariant manifolds also depend continuously on diffeomorphisms on a neighborhood of f. Since (A), (A1) and (A2) are open conditions defined in terms of finitely many hyperbolic periodic orbits and finite subsets of fundamental domains of their local invariant manifolds, we have that the set of diffeomorphisms which satisfy (2) is open.

It remains to show that each $A_{n,k,m}$ is a dense subset of $D^r_{\eta}(M)$. For this, we need two simple lemmas, the first one being the following.

Lemma 1. *Let $f \in D^r_{\eta}(M)$, V be a neighborhood of f, and p be a hyperbolic periodic point of f. Given $z \in W^u(p)$ and $\delta > 0$, there is $g \in V$ such that:*
(B1) *$g(x) = f(x)$ if $x \notin B_{\delta}(z)$, and*
(B2) *$O_1^{\infty}(z,g) \cap B_{\delta}(z) \neq \emptyset$.*

Since this result is more or less well known and its proof is different in flavor from the induction that will follow, we postpone it to section 3.

Let p be a hyperbolic periodic point of period ξ and $D = \overline{W^u_\varepsilon(p) - f^{-\xi}W^u_\varepsilon(p)}$ a fundamental domain of $W^u(p)$. The set $D^\circ := W^u_\varepsilon(p) - \overline{f^{-\xi}W^u_\varepsilon(p)}$ is obtained from D by deleting its boundary as a submanifold of M. For the next lemma, it is important to notice that for a point $z \in D^\circ$ and every δ sufficiently small, we have $f\left(\overline{B_\delta(z)}\right) \cap O^0_{-\infty}(D, f) = \emptyset$.

Lemma 2. *Let $f \in D^r_\eta(M)$, $o_1, ..., o_l$ be distinct hyperbolic periodic orbits of f, and D_i be a fundamental domain of $W^u(o_i)$ for $1 \le i \le l$. Given $\delta > 0$, a neighborhood V of f, and a finite subset Z of $\bigcup_{i=1}^{l} D_i^\circ$ there is $g \in V$ such that:*

(C1) $O^0_{-\infty}(D_i, g) = O^0_{-\infty}(D_i, f)$, *for* $1 \le i \le l$ (D_i is a fundamental domain of both $W^u(o_i, g)$ and $W^u(o_i, f)$, and the local unstable invariant manifolds of o_i with respect to f and g coincide).

(C2) $O^\infty_1(z, g) \cap B_\delta(z) \neq \emptyset$, $\forall\ z \in Z$.

Proof. Since $Z \subset \bigcup_{i=1}^{l} D_i^\circ$ and the sets $D_1, ..., D_l$ are fundamental domains of distinct periodic orbits, by decreasing δ if necessary, we may assume that

(D) $f\left(\overline{B_\delta(z)}\right) \cap O^0_{-\infty}(D_i, f) = \emptyset$, for every $z \in Z$ and $1 \le i \le l$.

The proof is by induction on the cardinality of Z.

When Z has one element, we write $Z = \{z\}$. By Lemma 1 applied to f, V and z, there is $g \in V$ which satisfies (C2) and $g(x) = f(x)$ if $x \notin B_\delta(z)$. This last condition implies that $g^{-1}(y) = f^{-1}(y)$ if $y \notin f(B_\delta(z)) = g(B_\delta(z))$, and by (D), we have that $O^0_{-\infty}(D_i, g) = O^0_{-\infty}(D_i, f)$ for $1 \le i \le l$, and g also satisfies (C1).

Suppose now that the lemma holds for sets of cardinality s, and write $Z = \{z_1, ..., z_s, z_{s+1}\}$. By induction, there is $g_1 \in V$ which satisfies the following two conditions:

(C1') $O^0_{-\infty}(D_i, g_1) = O^0_{-\infty}(D_i, f)$, for $1 \le i \le l$, and

(C2') $O^\infty_1(z_j, g_1) \cap B_\delta(z_j) \neq \emptyset$, for $1 \le j \le s$.

Since (D) is an open condition, we may choose g_1 to satisfy

(D') $g_1\left(\overline{B_\delta(z)}\right) \cap O^0_{-\infty}(D_i, g_1) = \emptyset$, for every $z \in Z$ and $1 \le i \le l$.

Let $V_1 \subset V$ be a neighborhood of g_1 such that any $g \in V_1$ satisfies

(C2") $O^\infty_1(z_j, g) \cap B_\delta(z_j) \neq \emptyset$, for $1 \le j \le s$.

By Lemma 1 applied to g_1, V_1 and z_{s+1}, there is $g_2 \in V_1$ which satisfies the following two conditions:

(B1') $g_2(x) = g_1(x)$ *if* $x \notin B_\delta(z_{s+1})$, *and*

(B2') $O^\infty_1(z_{s+1}, g_2) \cap B_\delta(z_{s+1}) \neq \emptyset$.

From (C2") and (B2'), we have that g_2 satisfies (C2). From (B1'), we have that $g_2^{-1}(y) = g_1^{-1}(y)$ if $y \notin g_1(B_\delta(z_{s+1})) = g_2(B_\delta(z_{s+1}))$, and by (D') it follows that $O^0_{-\infty}(D_i, g_2) = O^0_{-\infty}(D_i, g_1)$, for $1 \le i \le l$. From (C1') we conclude that g_2 also satisfies (C1). □

Finally, we are ready to show that the sets $A_{n,k,m}$ are dense in $D^r_\eta(M)$.

Let U be an open subset of $D^r_\eta(M)$. As remarked before, diffeomorphisms that satisfy (1) form an open and dense subset of $D^r_\eta(M)$. Let $f \in U$ satisfy (1), and denote by $o_1, ..., o_l$ the hyperbolic periodic orbits of f with period $\le n$. Let D_i be a fundamental domain of $W^u(o_i, f)$, $1 \le i \le l$.

Let $V \subset U$ be a neighborhood of f such that any $g \in V$ satisfies (1) and has exactly l hyperbolic periodic orbits of period $\le n$. For each $1 \le i \le l$, choose a finite subset Z_i $\frac{1}{k}$-dense in D_i, and let $Z = \bigcup_{i=1}^{l} Z_i$. Lemma 2 applied to f, V, o_i, D_i, Z and $\delta = \frac{1}{m}$, gives $g \in V$ which satisfies (C1) and (C2).

From (C1), we have that $o_1, ..., o_l$ are exactly the hyperbolic periodic orbits of g with period $\leq n$, and that D_i is also a fundamental domain of $W^u(o_i, g)$ for $1 \leq i \leq l$. Each of the fundamental domains D_i satisfies condition (A), and therefore g satisfies (2). Notice that each D_i depends only on g and o_i, and since in Lemma 2, Z and δ are independent, each Z_i depends only on g, o_i and D_i.

3. Proof of Lemma 1.

We need the following local perturbation result:

Lemma 3. *Let U be a neighborhood of the identity in $D^r_\eta(M)$, $z \in M$, and $\delta > 0$. There is $\rho > 0$ such that for any $y \in B_\rho(z)$, there is $\phi \in U$ satisfying the following:*
(E1) $\phi(x) = x$ if $x \notin B_\delta(z)$, *and*
(E2) $\phi(z) = y$.

Firstly, we show how Lemma 3 implies Lemma 1.

Let f, V, z and δ be as in Lemma 1. Let U be a neighborhood of id, the identity map of M, such that if $\phi \in U$ then $f \circ \phi \in V$. Let ρ be the number provided by Lemma 3, and $y \in B_\rho(z)$ an f-recurrent point. There is $\phi \in U$ that satisfies (E1) and (E2), and such that $g := f \circ \phi \in V$. If $x \notin B_\delta(z)$, then $g(x) = f(\phi(x)) = f(x)$, implying that g satisfies (B1). If $k = \inf\{n \geq 1\,;\, f^n(y) \in B_\delta(z)\}$, then it is easy to see that $g^i(z) = f^i(y)$ for $1 \leq i \leq k$, and g satisfies (B2).

In order to prove Lemma 3, we consider the symplectic and volume preserving cases separately.

3.1. The symplectic case.

In this case, the dimension of M is even, say $2n$, and we can take symplectic coordinates $(u, v) = (u_1, ..., u_n, v_1, ..., v_n)$ around z, namely, a chart $\psi : B \to M$, where B is an open ball centered at $(0,0) \in \mathbb{R}^{2n}$, $\psi(0,0) = z$, and $\eta = \sum_{i=1}^{n} du_i \wedge dv_i$. In terms of these coordinates, Lemma 3 reduces to:

Lemma 3a. *Let $\varepsilon > 0$ and $\delta > 0$. There is a number $\rho > 0$ such that for any $(a, b) \in B_\rho(0,0)$, there is a symplectic diffeomorphism ϕ of $\mathbb{R}^{2n}$ which satisfies the following:*
(F1) $\phi(u, v) = (u, v)$ *if* $\|(u, v)\| \geq \delta$,
(F2) $\phi(0,0) = (a, b)$, *and*
(F3) $\|\phi - id\|_r < \varepsilon$, *where* $\|h\|_r = \sup_{0 \leq j \leq r} \sup_{p \in \mathbb{R}^{2n}} \left\|D^j_p h\right\|$, *and* $D^j_p h$ *denotes the j-derivative of h at p.*

Symplectic diffeomorphisms can be easily constructed with the aid of generating functions (see chapter 5.2 of [AbMa]). For $r \geq 1$, let $S(u, t)$ be a function of class $C^{r+1}$from $\mathbb{R}^{2n}$ to $\mathbb{R}$, such that the matrix $\left(\frac{\partial S}{\partial u_i \partial t_j}(u, t)\right)$ is non-singular for every (u, t). If we define

$$f(u,t) = (u_1, ..., u_n, \frac{\partial S}{\partial u_1}(u,t), ..., \frac{\partial S}{\partial u_n}(u,t))$$
$$g(u,t) = (\frac{\partial S}{\partial t_1}(u,t), ..., \frac{\partial S}{\partial t_n}(u,t), t_1, ..., t_n),$$

then f and g are diffeomorphisms of class C^r of $\mathbb{R}^{2n}$. Let A and B be the matrices of $f'(u, t)$ and $g'(u, t)$, respectively. Using the fact that S is C^2, a simple calculation with partial derivatives shows that $A^tJA = B^tJB$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. If we consider $\phi = g \circ f^{-1}$ and denote the matrix of $\phi'(u,v)$ by C, then $C^tJC = J$ and ϕ is a symplectic diffeomorphism of $\mathbb{R}^{2n}$. If generating functions S_1 and S_2 are C^{r+1} close, then the corresponding symplectic diffeomorphisms ϕ_1 and ϕ_2 are C^r close.

Given $(a,b) \in \mathbb{R}^{2n}$, for the generating function

$$S(u,t) = \sum_{i=1}^{n} u_i t_i + \sum_{i=1}^{n} a_i t_i - \sum_{i=1}^{n} b_i u_i$$

we have that $f(u,t) = (u, t-b)$, $g(u,t) = (u+a, t)$, and setting $t - b = v$, we have that $\phi(u,v) = (u+a, v+b)$ is a translation by (a,b).

In order to prove Lemma 3a, we consider $\sigma \in (0,\delta)$ and a C^∞ function $\alpha : \mathbb{R}^{2n} \to [0,1]$ such that $\alpha(u,t) = 1$ if $\|(u,t)\| \le \sigma$ and $\alpha(u,t) = 0$ if $\|(u,t)\| \ge \delta$. Let

$$S_{a,b}(u,t) = \sum_{i=1}^{n} u_i t_i + \alpha(u,t)\left(\sum_{i=1}^{n} a_i t_i - \sum_{i=1}^{n} b_i u_i\right).$$

Since

$$\frac{\partial^2 S_{a,b}}{\partial u_i \partial t_j}(u,t) = \delta_{ij} + \frac{\partial^2 \alpha}{\partial u_i \partial t_j}(u,t)\left(\sum_{i=1}^{n} a_i t_i - \sum_{i=1}^{n} b_i u_i\right) + a_j \frac{\partial \alpha}{\partial u_i}(u,t) - b_i \frac{\partial \alpha}{\partial t_j}(u,t),$$

if $\rho > 0$ is small enough, then $\|(a,b)\| < \rho$ implies that the matrix $\left(\frac{\partial S_{a,b}}{\partial u_i \partial t_j}(u,t)\right)$ is non-singular for every (u,t). Let $\phi_{a,b}$ be the corresponding symplectic diffeomorphism. Note that $S_{0,0}$ corresponds to $\phi_{0,0} = id$. Since

$$S_{a,b}(u,t) - S_{0,0}(u,t) = \alpha(u,t)\left(\sum_{i=1}^{n} a_i t_i - \sum_{i=1}^{n} b_i u_i\right),$$

for ρ small enough, we have that if $\|(a,b)\| < \rho$, then $S_{a,b}$ is C^{r+1} close to $S_{0,0}$, and $\phi_{a,b}$ is C^r close to id. Finally, since $\alpha(u,t) = 0$ if $\|(u,t)\| \ge \delta$, we have that $\phi_{a,b}(u,v) = (u,v)$ if $\|(u,v)\| \ge \delta$, and since $\alpha(u,t) = 1$ if $\|(u,t)\| \le \sigma$, we have that $\phi_{a,b}(u,v) = (u+a, v+b)$ if $\|(u, v+b)\| \le \sigma$.

3.2. **The volume preserving case.** The arguments are similar to those in the symplectic case, so we just indicate the changes.

In this case, η is a volume form, and there are coordinates $(\xi_1, ..., \xi_k)$ around z such that $\eta = d\xi_1 \wedge ... \wedge d\xi_k$. In terms of these coordinates, Lemma 3 translates to:

Lemma 3b. *Let $\varepsilon > 0$ and $\delta > 0$. There is $\rho > 0$ such that for any $y \in B_\rho(0)$, there is a volume preserving diffeomorphism ϕ* of $\mathbb{R}^k$ which satisfies the following:

(G1) $\phi(\xi) = \xi$ *if* $\|\xi\| \ge \delta$,

(G2) $\phi(0) = y$, and

(G3) $\|\phi - id\|_r < \varepsilon$.

When k is even, every symplectic diffeomorphism is volume preserving, and Lemma 3b follows from Lemma 3a in this case. Suppose now that $k = 2n+1$. We are going to denote the coordinates $(\xi_1, ..., \xi_k)$ around z by $(u,v,w) = (u_1, ..., u_n, v_1, ..., v_n, w)$, and write $y = (a,b,c)$, with $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$. By making an orthogonal change of coordinates, we may assume that $c = 0$.

For $r \geq 1$, let $S(u,t,w)$ be a function of class $C^{r+1}$from $\mathbb{R}^{2n+1}$ to $\mathbb{R}$, such that the matrix $\left(\frac{\partial S}{\partial u_i \partial t_j}(u,t,w)\right)$ is non-singular for every (u,t,w). If we define

$$f(u,t,w) = (u_1, ..., u_n, \frac{\partial S}{\partial u_1}(u,t,w), ..., \frac{\partial S}{\partial u_n}(u,t,w), w)$$
$$g(u,t,w) = (\frac{\partial S}{\partial t_1}(u,t,w), ..., \frac{\partial S}{\partial t_n}(u,t,w), t_1, ..., t_n, w),$$

then $\det f'(u,t,w) = \det g'(u,t,w) = \det\left(\frac{\partial S}{\partial u_i \partial t_j}(u,t,w)\right) \neq 0$, implying that f and g are diffeomorphisms of class C^r of $\mathbb{R}^{2n+1}$, and that $\phi = g \circ f^{-1}$ is a volume preserving diffeomorphism of $\mathbb{R}^{2n+1}$. If functions S_1 and S_2 are C^{r+1} close, then the corresponding volume preserving diffeomorphisms ϕ_1 and ϕ_2 are C^r close.

Given $(a,b,0) \in \mathbb{R}^{2n+1}$, for the function

$$S(u,t,w) = \sum_{i=1}^{n} u_i t_i + \sum_{i=1}^{n} a_i t_i - \sum_{i=1}^{n} b_i u_i$$

we have that $f(u,t,w) = (u, t-b, w)$, $g(u,t,w) = (u+a, t, w)$, and if $t - b = v$ then $\phi(u,v,w) = (u+a, v+b, w)$.

Let $\sigma \in (0,\delta)$ and a C^∞ function $\alpha : \mathbb{R}^{2n+1} \to [0,1]$ such that $\alpha(u,t,w) = 1$ if $\|(u,t,w)\| \leq \sigma$ and $\alpha(u,t,w) = 0$ if $\|(u,t,w)\| \geq \delta$. Let

$$S_{a,b}(u,t,w) = \sum_{i=1}^{n} u_i t_i + \alpha(u,t,w)\left(\sum_{i=1}^{n} a_i t_i - \sum_{i=1}^{n} b_i u_i\right).$$

If $\rho > 0$ is small enough, then $\|(a,b,0)\| < \rho$ implies that the matrix $\left(\frac{\partial S_{a,b}}{\partial u_i \partial t_j}(u,t,w)\right)$ is non-singular for every (u,t,w), and if we denote by $\phi_{a,b}$ the corresponding volume preserving diffeomorphism, then $\|(a,b,0)\| < \rho$ implies that $S_{a,b}$ is C^{r+1} close to $S_{0,0}$, and $\phi_{a,b}$ is C^r close to id. Finally, since $\alpha(u,t,w) = 0$ if $\|(u,t,w)\| \geq \delta$, we have $\phi_{a,b}(u,v,w) = (u,v,w)$ if $\|(u,v,w)\| \geq \delta$, and since $\alpha(u,t,w) = 1$ if $\|(u,t,w)\| \leq \sigma$, we have $\phi_{a,b}(u,v,w) = (u+a, v+b, w)$ if $\|(u, v+b, w)\| \leq \sigma$.

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