Percolation connectivity in the highly supercritical regime

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Abstract

We prove that the two point finite connectivity function $\tau^{\rm f}(x,y)$ in the d dimensional Bernoulli bond percolation is analytic in p around 1. We also provide upper and lower bounds for this function in the case $d \geq 3$ and near $p = 1$. The gap between lower bound and upper bound is sufficiently narrow to conclude that the rate of exponential decay, i.e. the inverse correlation length $m(p)$, is, for p sufficiently near to 1 and for $x - y$ in the coordinate axis directions, of the form $m(p) = 2(d-1)|\ln(1-p)| + O(1-p)$, as expected by intuition based on low temperature expansion arguments.

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§1. *Introduction*

We define the d-dimensional hypercubic lattice as the infinite graph $\mathbb{L} = (\mathbb{V}, \mathbb{E})$ with vertex set $\mathbb{V} = \mathbb{Z}^d = \{x = (x_1, \ldots, x_d) : \ x_i \in \mathbb{Z}\}$ and edge set formed by all the nearest neighbor pairs, i.e., $\mathbb{E} = \{e = \{x, y\} \subset \mathbb{V} : \ |x-y| = 1\}$ where $|x - y| = \sum_{i=1}^d |x_i - y_i|$. We will suppose hereafter $\mathbb{Z}^d$ naturally merged in $\mathbb{R}^d$.

The Bernoulli bond percolation process on $\mathbb{L}$ is defined by declaring that each edge $e \in \mathbb{E}$ is either *open* with probability p or *closed* with probability $1 - p$. A *configuration* of the system is a map ω from $\mathbb{E}$ into $\{0, 1\}$. If $\omega(e) = 1$, the edge e is open, while, if $\omega(e) = 0$, the edge e is closed. Let Ω be the set of all configurations. The product measure on Ω, denoted by P_p, is the *Bernoulli probability measure*. Given $\omega \in \Omega$ we denote by $\mathbb{L}^1(\omega) = (\mathbb{V}, \mathbb{E}^1(\omega))$ the subgraph of $\mathbb{L}$ with set of edges $\mathbb{E}^1(\omega) = \{e \in \mathbb{E} : \omega(e) = 1\}$ (i.e. the set of all open edges in ω). Clearly $\mathbb{E}^0(\omega) = \mathbb{E} - \mathbb{E}^1(\omega)$ is the set of all closed edges in ω. The connected components of $\mathbb{L}^1(\omega)$ are called *open clusters*.

Given $\omega \in \Omega$ and $x \in \mathbb{V}$, we denote by $a_x(\omega)$ the *open cluster through* x, i.e. $a_x(\omega) = (C_x(\omega), E_x(\omega))$ is the (unique) connected sub-graph of $\mathbb{L}^1(\omega)$ such that $x \in C_x(\omega)$ (note that if $C_x(\omega) = \{x\}$ then $E_x(\omega) = \emptyset$). A key quantity in the bond percolation process is the probability

theta

$$\theta(p) = P_p(C_x(\omega) \text{ is infinite}), \tag{1.1}$$

which of course, by translational invariance, does not depend on x. This function is known to be zero for p below a critical value p_c and to be strictly greater than zero above p_c. An other important quantity, especially in order to study the percolation process in the supercritical regime ($p > p_c$), is the two-point finite connectivity function $\tau^{\rm f}_{x,y}(p)$, defined as the probability that x and y belong to the same open cluster and this cluster is finite, i.e.

tau

$$\tau^{\rm f}_p(x, y) = P_p(y \in C_x(\omega) \text{ and } C_x(\omega) \text{ is finite}). \tag{1.2}$$

In a previous paper [1] we have shown, via cluster expansion, that the percolation probability $\theta(p)$ is analytic in a neighborhood of $p = 1$. In the present paper we first show, using a technique based

mainly on probabilistic arguments (and no cluster expansion), that the two point connectivity function is also analytic in a neighborhood of $p = 1$. Such analyticity results agree with the stronger conjecture that both the percolation probability and the finite connectivity function admit an analytic continuation on an open set containing the interval $(p_c, 1]$. We also stress that the analyticity of the $\tau_p^{\rm f}(x, y)$ is an important information to look for further properties on its behavior at large distances. In particular a representation in terms of a convergent polymer expansion, as (3.10) below, seems to be a key ingredient in order to show that $\tau_p^{\rm f}(x, y)$ has an asymptotic Ornstein-Zernike (OZ) decay at large distance (i.e a polynomial correction to the exponential decay), see [3],[4], [5]. As a matter of fact, the lack of such a "polymer expansion" representation for $\tau_p^{\rm f}(x, y)$ in the supercritical phase could have something to do with the fact that OZ decay for the Bernoulli percolation process in this regime, thought widely conjectured, was still an open question *even for p near 1.* We will show in a future paper [2] the OZ asymptotic decay for $\tau_p^{\rm f}(x, y)$ in the (highly) supercritical phase, using the analyticity results obtained here. Beyond the proof of analyticity, we also provide here upper and lower bounds for the connectivity functions $\tau_p^{\rm f}(x, y)$ in the $d \geq 3$ case. Our bounds holds both in the neighborhood of $p = 1$ and $p = 0$ and allow us to give a quite detailed description of the behavior of the finite connectivity. These bounds show that the two-point connectivity function decays exponentially at large distances (this is well known since long time [6]). Moreover they imply that the inverse correlation length $m(p) = -\lim_{n\to\infty} \frac{1}{n} \ln \tau_p(0, e_n)$ (where $e_n = (n, 0, \ldots, 0)$) has, for p near 1, a leading term equal to $2(d - 1) \ln (1 - p)$ plus corrections of the order $O(1 - p)$, and satisfies a simple relation near $p = 1$, namely, $m(p) = 2(d - 1)m(1 - p) + O(1 - p)$. Such relation between inverse correlation length in supercritical and subcritical regime was known to hold exactly for all $p \neq p_c$ in the two-dimensional case, while, as far as we know, was neither known nor conjectured for $d \geq 3$, see e.g. [8]. Since in the two-dimensional case such relation was proved using crucially the duality of $\mathbb{Z}^2$, we are not able to state an analogous result in $d > 2$ for all $p \neq p_c$.
Concluding this introduction we would like to recall that similar results for the Ising model have been provided in [9] for the high temperature case and in [12], [10] for the low temperature case. Moreover, in a specific site percolation model similar problems were faced recently in [7].

§2 *Dual animals, minimal cut sets and Peierls contours*

Hereafter, if R is a finite set, then $|R|$ will denote the number of its elements. Given $U \subset \mathbb{V}$, we define $\mathbb{E}|_U = \{e \in \mathbb{E} : e \subset U\}$. The subgraph $\mathbb{L}|_U = (U, \mathbb{E}|_U) \subset \mathbb{L}$ is known as *the restriction of* $\mathbb{L}$ *to* U. Let $U \in \mathbb{V}$, then U is *connected* if $\mathbb{L}|_U$ is a connected graph. A connected and finite subgraph $a = (V_a, E_a) \subset \mathbb{L}$, with vertex set V_a and edge set E_a, will be called hereafter a *lattice animal* or simply *animal*. We will denote by $\partial_e a$ the edge boundary of a, i.e. the set $\partial_e a = \{e \in \mathbb{E} - E_a : e \cap V_a \neq \emptyset\}$. We will denote by $\mathcal{A}$ the set of all lattice animals in $\mathbb{E}$.
Let $e = \{x, y\} \in \mathbb{E}$ be an edge, we denote by $\bar{e}$ the unit straight segment in $\mathbb{R}^d$ uniquely associated to e whose end points are x and y. Draw now for each $e \in \mathbb{E}$ a $(d - 1)$-dimensional closed unit hyper-square $\varphi(e) \subset \mathbb{R}^d$ perpendicular to the unit segment $\bar{e}$, with $d - 2$ dimensional unit sides parallel to the axis of $\mathbb{R}^d$, and in such way that $\varphi(e) \cap \bar{e}$ is the center of the hyper-square $\varphi(e)$ and the middle point of the segment $\bar{e}$. The vertices of the hypersquares $\varphi(e)$ lie in a d-dimensional hypercubic lattice (the *dual lattice*) $\mathbb{V}^*$ shifted by the vector $(1/2, \ldots, 1/2)$ with respect to $\mathbb{V}$. Note that the map $\varphi : \mathbb{E} \to \varphi(\mathbb{E})$, associating to each e the corresponding $\varphi(e)$, is one-to-one from the set $\mathbb{E}$ into the set $\varphi(\mathbb{E})$ of plaquettes with vertices in $\mathbb{V}^*$. We say that the plaquette $\varphi(e)$ *is dual to the edge* e. We will make use of the shorter notation $\varphi(e) = e^*$. So, if $E \subset \mathbb{E}$, then $\varphi(E) = E^*$ *is dual to* E and in particular $\varphi(\mathbb{E}) = \mathbb{E}^*$ *is dual to* $\mathbb{E}$. If $E^* \subset \mathbb{E}^*$ then $\bar{E}^* = \cup_{e^* \in \gamma} e^* \subset \mathbb{R}^d$. We call $\bar{E}^*$ the *immersion* of E^* into $\mathbb{R}^d$.

We say that two plaquettes are *dual connected* if they share a $(d-2)$-dimensional side. In particular in $d=3$ a plaquette turns out to be simple square and two plaquettes are connected if they share a whole unit side. A set $E^* \subset \mathbb{E}^*$ is *dual connected* if for any partition $\{E_1^*, E_2^*\}$ of E^*, there is a plaquette $e_1^* \in E_1^*$ and a plaquette $e_2^* \in E_2^*$ such that e_1^* and e_2^* are dual connected. A *dual connected finite* subset of $\mathbb{E}^*$ will be called a *dual animal*. We denote by $\mathcal{D}^*$ the set of all dual animals in $\mathbb{E}^*$ and we will usually use Greek letters $\alpha, \beta, \gamma, \sigma, \tau$ to denote dual animals. Given a dual animal $\alpha \in \mathcal{D}^*$, we denote by $\bar{\alpha}$ the immersion of α in $\mathbb{R}^d$. The set of $(d-2)$-dimensional unit sides in $\bar{\alpha}$ which belong to one and only one plaquette of α is denoted by $\partial_s \alpha$ and called *the side-boundary of* α. We will say that the dual animals α and β are *compatible*, and we write $\alpha \sim \beta$, if $\alpha \cup \beta$ is not a dual animal (i.e. is not a set of pairwise connected plaquettes). We will denote by $|\alpha|$ the cardinality of α, i.e. the number of plaquettes in α.

A *cut set* in $\mathbb{E}$ is a subset $b \subset \mathbb{E}$ such that the graph $(\mathbb{V}, \mathbb{E} - b)$ is disconnected. A *minimal cut set* $c \subset \mathbb{E}$ is a cut set such that for any $e \in c$ the set $c - e$ is no more a cut set. We now show two key properties of a minimal cut set in $\mathbb{L}$.

Proposition 1. *Let $c \subset \mathbb{E}$ be a minimal cut set in $\mathbb{L}$ such that $|c| < \infty$. Then the graph $\mathbb{L}_c = (\mathbb{L}, \mathbb{E} - c)$ has only two connected components $\Gamma_c^{\rm ext} = (V_c^{\rm ext}, E_c^{\rm ext})$ and $\Gamma_c^{\rm int} = (V_c^{\rm int}, E_c^{\rm int})$, the first being infinite and second finite with $E_c^{\rm ext} = \mathbb{E}|_{V_c^{\rm ext}}$ and $E_c^{\rm int} = \mathbb{E}|_{V_c^{\rm int}}$.*

Proof. First observe that $\mathbb{L}_c$ has only one infinite component. This follows from the hypothesis that c is finite and from the fact that the d dimensional hypercubic lattice $\mathbb{L}$ is a graph with only one end. We denote by $\mathbb{L}_c^{\rm ext} = (V_c^{\rm ext}, E_c^{\rm ext})$ this unique infinite component of $\mathbb{L}_c$. Let now $\mathbb{L}_c^{\rm int}$ be the graph with vertex set $V_c^{\rm int} = \mathbb{V} - V_c^{\rm ext}$ and with edge set $E_c^{\rm int} = \mathbb{E} - (c \cup E_c^{\rm ext})$. It is now easy to prove that $\mathbb{L}_c^{\rm int}$ is connected. Indeed, by definition of minimal cut set, for any edge $e \in c$, the graph $(V_c^{\rm ext} \cup V_c^{\rm int}, E_c^{\rm ext} \cup E_c^{\rm int} \cup e) = (\mathbb{V}, \mathbb{E} - (c - e))$ is connected, and this would not be possible if $\mathbb{L}_c^{\rm int}$ is done by more than one connected component. Finally, the identities $E_c^{\rm ext} = \mathbb{E}|_{V_c^{\rm ext}}$ and $E_c^{\rm int} = \mathbb{E}|_{V_c^{\rm int}}$ follow from the fact that $E_c^{\rm int} \cup E_c^{\rm ext} \cup c = \mathbb{E}$. $\square$

The set $V_c^{\rm ext}$ is called *the exterior of the minimal cut set* c and the set $V_c^{\rm int}$ is called *the interior of* c. Note that c coincides with the *edge boundary* in $\mathbb{L}$ of the connected set $V_c^{\rm int}$. I.e. $c = \partial_e V_c^{\rm int} = \{e \in \mathbb{E} : e = \{x, y\} : x \in V_c^{\rm int}, y \notin V_c^{\rm int}\}$.

Proposition 2. *Let c be a minimal cut set in $\mathbb{L}$ such that such that $|c| < \infty$. Then $c^* = \gamma$ is a dual animal.*

Proof. First, we prove that every $(d-2)$-dimensional side is shared by an even number of plaquettes in γ. Suppose that $\partial_s \gamma \neq \emptyset$. Then there exists at least one plaquette, say e^*, in γ with a side not connected with any other plaquette in γ. e^* is the dual plaquette of an edge $e = \{x, y\} \in c$, with $x \in V_c^{\rm int}$ and $y \in V_c^{\rm ext}$. Observe now that in any d the possible plaquettes connected with the free side of e^* are three, one of them, say $\tilde{e}^*$, parallel to e^*, and the other two perpendicular to e^*. Let now $\tilde{e}$ be the edge dual to $\tilde{e}^*$. By construction, $\tilde{e} = \{x', y'\}$ with $f = \{x, x'\} \in \mathbb{E}$ and $g = \{y, y'\} \in \mathbb{E}$. Note that $\tilde{e}^*$, f^* and g^* are the three plaquettes connected with the free edge in e^*. Thus $\tilde{e} \notin c$, $f \notin c$ and $g \notin c$. Thus we have a path $\ell = \{f, \tilde{e}, g\}$, connecting x to y and such that $\ell \cap c = \emptyset$, in contradiction with the hypothesis that c is minimal cut set. Suppose now that this side is connected with exactly two plaquettes (so that the side is shared by three plaquettes). In this case the path $\ell = \{f, \tilde{e}, g\}$, would cross two times the minimal cut set starting from a point $x \in V_c^{\rm int}$ and arriving at $y \in V_c^{\rm ext}$, which is impossible. Now, since $\partial_s \gamma = \emptyset$, either γ is connected, or γ is the disjoint union of connected components $\gamma_1, \dots, \gamma_n$ such that $\partial_s \gamma_i = \emptyset$. If γ is connected the proposition is proved. Let us suppose that γ

is not connected. Then a connected component γ_i of γ, is such that $\partial_s \gamma_i = \emptyset$ and such that every $(d-2)$-dimensional side is shared an even number of plaquettes. The immersion $\bar{\gamma}_i$ of γ_i into $\mathbb{R}^d$ is a surface with no edge formed by a finite set of pairwise connected plaquettes. Namely, $\bar{\gamma}_i$ is either a topological surface of $\mathbb{R}^d$ with no self intersections, or it can be turned, as usual, into the disjoint union of topological surface, by slightly deforming sides or corners shared by more than two plaquettes, see e.g. [11]. Then, by the Jordan-Brouwer separation theorem, $\bar{\gamma}_i$ separates $\mathbb{R}^d$ into (at least) two disjoint subsets A_1 and A_2 with common boundary, and any path connecting a vertex $x \in \mathbb{V} \cap A_1$ to a vertex $y \in \mathbb{V} \cap A_2$ must cross some plaquette in $\bar{\gamma}_i$. Then we have that $\varphi^{-1}(\gamma_i)$ surely disconnects $\mathbb{V} \cap A_1$ from $\mathbb{V} \cap A_2$. So each γ_i is the dual of a cut set contained in c, contradicting the assumption the c is a minimal cut set. Then γ is connected. $\square$

We are now ready to give the definition of Peierls contours and of contours.

Definition 3. *A set of plaquettes $\gamma \subset \mathbb{E}^*$ is called a Peierls contour if $c = \varphi^{-1}(\gamma)$ is a minimal cut set.*

By proposition 1, a Peierls contour γ separates $\mathbb{V}$ into two disjoint sets: $I_\gamma = V_c^{\text{int}}$ is called *the inside of* γ and $O_\gamma = V_c^{\text{ext}}$ is called *the outside of* γ. Moreover, by proposition 2, a Peierls contour γ is a dual animal with empty side boundary, and each side of γ can be shared only by an even number of plaquettes. Note finally that a Peierls contour γ also induces a partition of $\mathbb{E}^*$ in *three* disjoint subsets: $O^*_\gamma = (E_c^{\text{ext}})^*$, which is called the *dual outside of* γ, $I^*_\gamma = (E_c^{\text{int}})^*$, which is called *dual inside of* γ, and $\gamma = c^*$. We denote by $\mathcal{C}^*$ the set of all Peierls contours in $\mathbb{E}^*$.

Definition 4. *A dual animal $\sigma \in \mathcal{D}^*$ is called a contour if there exists $\gamma \subset \sigma$ such that γ is a Peierls contour.*

Note that if σ is a contour then $c = \varphi^{-1}(\sigma)$ is a cut set. A contour σ induces a partition of $\mathbb{V}$ into two disjoint subsets: the *inside of a contour* is the union of the interiors of all Peierls contours contained in σ, i.e. is set $I_\sigma = \bigcup_{\gamma \subset \sigma : \gamma \in \mathcal{C}^*} I_\gamma$. The *outside of the contour* σ is the set $O_\sigma = \mathbb{V} - I_\sigma$. Moreover σ also induces a partition of $\mathbb{E}^*$ into three disjoint subsets: $O^*_\sigma = [\mathbb{E}|_{O_\sigma}]^* - \sigma$, which is called the *dual outside of* σ, $I^*_\sigma = [\mathbb{E}|_{I_\sigma}]^* - \sigma$, which is called *dual inside of* σ, and σ. We denote by $\mathcal{S}^*$ the set of all contours in $\mathbb{E}^*$. Clearly $\mathcal{C}^* \subset \mathcal{S}^*$.

We conclude this section by showing that it is possible to associate to an animal a a Peierls contour whose interior contains a.

Proposition 5. *Let $a = (V_a, E_a)$ be an animal in $\mathbb{L}$. Then there is a unique Peierls contour γ_a such that $\gamma_a \subset (\partial_e a)^*$ and $I_{\gamma_a} \supset V_a$. Moreover, if $\gamma \subset (\partial_e a)^*$ is also a Peierls contour different from γ_a, then $I_\gamma \cap V_a = \emptyset$.*

Proof. Consider the graph $\mathbb{L}_a = (\mathbb{V} - V_a, \mathbb{E} - E_a)$. Note that $\partial_e \mathbb{L}_a = \partial_e a$. Moreover, since $\mathbb{L}$ has one end and V_a is finite, then only one among the connected components of $\mathbb{L}_a$ can be infinite. Let $\Gamma_a^{\text{ext}} = (V_a^{\text{ext}}, E_a^{\text{ext}})$ be this unique infinite connected component of $\mathbb{L}_a$ and let $\Gamma_a^{\text{int}} = (V_a^{\text{int}}, E_a^{\text{int}})$ be the subgraph of $\mathbb{L}_a$ containing all the finite components of of $\mathbb{L}_a$; note that $V_a^{\text{int}} = \mathbb{V} - (V_a \cup V_a^{\text{ext}})$ and $E_a^{\text{int}} = \mathbb{E} - (E_a \cup E_a^{\text{ext}})$. Consider now the graph $\bar{a} = (\bar{V}_a, \bar{E}_a)$ where $\bar{V}_a = \mathbb{V} - V_a^{\text{ext}}$ and $\bar{E}_a = \mathbb{E} - E_a^{\text{ext}}$. $\bar{a}$ is an animal: $\bar{V}_a$ is finite since it is the union of two finite sets $\bar{V}_a = V_a \cup V_a^{\text{int}}$; $\bar{a}$ is connected since a is connected, and since each connected component of Γ_a^{int} is connected to a through its edge boundary $\partial_e \Gamma_a^{\text{int}} \subset \partial_e a$. By definition $c_a = \partial_e \bar{a} = \partial_e \Gamma_a^{\text{ext}} \subset \partial_e a$ is a minimal cut set and its interior is $\bar{V}_a \supset V_a$. Now $\gamma_a = c_a^*$ is the Peierls contour requested.

Let now $\Gamma^1, \ldots, \Gamma^k$ be the connected components of Γ_a^{int}. For any connected component $\Gamma^j =$

(V^j, E^j), the set E^j can be written as $\mathbb{E}|_{V^j} \cup c^j$ where $c_j = \partial_e \Gamma^j$, $\mathbb{E}|_{V^j} \cap c^j = \emptyset$ and $c^j \subset \partial_e a$. Moreover, if $i \neq j$, then $c_j \cap c_i = \emptyset$ and thus $\partial_e a$ is the disjoint union $\partial_e a = c_a \cup [\cup_j c^j]$. By construction, for any $j = 1, \dots k$, c^j is a minimal cut set with interior V^j and exterior $V_a \cup [\cup_{i \neq j} V^i] \cup V_a^{\text{ext}}$. Now we prove that the sets $c_a, c^1, \dots, c^k$ are the unique cut sets in $\partial_e a$. Suppose e.g. that $c \subset \partial_e a$ is a cut set containing a pair $\{e, e'\} \subset c$ such that $e \in c_a$ and $e' \in c_i$ for some $i = 1, \dots, k$, i.e. $e = \{x, y\}$ with $x \in V^{\text{ext}}$ and $y \in V_a$, and $e' = \{x', y'\}$ with $x' \in V^i$ and $y' \in V_a$. Then the only possibility is that $y, y' \in V_c^{\text{int}}$ and $x, x' \in V_c^{\text{ext}}$. But then $V_a \subset V_c^{\text{int}}$, since, V_a being connected, no vertex in V_a can be in V_c^{ext}. This yields that x and x' are disconnected (since V^{ext} and V^i are disconnected, i.e. they are connected only through points in V_c^{int}) in contradiction with the hypothesis that c is a cut set and thus its exterior is connected. In a similar manner one can treat the case $e \in c_i$ and $e' \in c_j$. $\square$

We call $\gamma_a = c_a^*$ *the Peierls contour enclosing* A.

Given a Peierls contour $\gamma \subset \mathbb{E}^*$ and a vertex set $X \subset \mathbb{V}$, we say that γ *surrounds* X and we write $\gamma \bigodot X$ if there exists an animal $a = (V_a, E_a)$ such that $X \subset V_a$ and $\gamma_a = \gamma$. We say that γ *separates* X and we write $\gamma \bigotimes X$, if for any animal $a = (V_a, E_a)$ such that $X \subset V_a$, $E_a \cap \varphi^{-1}(\gamma) \neq \emptyset$. We say that γ *avoids* X if $X \in O_\gamma$ and we write $\gamma \bigcirc X$.

§3 *Dual animal expansion in the supercritical phase*

Consider a *finite* set $\Lambda \subset \mathbb{V}$; we denote shortly $\mathbb{E}_\Lambda = \mathbb{E}|_\Lambda$. Ω_Λ will denote the set of all functions from $\mathbb{E}_\Lambda$ in $\{0,1\}$ (configurations in Λ) and P_p^Λ will denote the restriction of the Bernoulli measure to $\mathbb{E}_\Lambda$. The vertex set $\partial\Lambda = \{x \in \Lambda : \exists y \in \mathbb{V} - \Lambda : |x - y| = 1\}$ is called the *internal vertex boundary of* Λ. The edge set $\partial_e \Lambda = \mathbb{E}|_{\partial\Lambda}$ is called the *internal edge boundary of* Λ. The set $\mathbb{E}_\Lambda^* = \mathbb{E}_\Lambda - \mathbb{E}|_{\partial\Lambda}$ is called the *edge interior of* Λ. We denote by $\mathcal{D}_\Lambda^*$ the set of all dual animals in Λ. Namely, $\gamma \in \mathcal{D}_\Lambda^*$ if $\gamma \in \mathcal{D}^*$ and $\gamma \subset \mathbb{E}_\Lambda^*$. A Peierls contour γ is said to be a *Peierls contour relatively to* Λ if $\gamma \subset \mathbb{E}_\Lambda^*$. Note that if γ is a Peierls contour relatively to Λ, then any animal $a = (V_a, E_a)$ such that $\gamma_a = \gamma$ satisfies $V_a \cap \partial\Lambda = \emptyset$. We denote by $\mathcal{C}_\Lambda^*$ the set of all Peierls contours relatively to Λ. A contour $\sigma \subset \mathbb{E}_\Lambda^*$ is said to be a *contour relatively to* Λ if it exists $\gamma \in \mathcal{C}_\Lambda^*$ such that $\sigma \supset \gamma$.

The *finite volume truncated connectivity* $\tau_p^\Lambda(x, y)$ is defined as the probability that the vertices $x, y \in \Lambda$ belongs to the same open cluster and this open cluster does not intersect the boundary. Namely, recalling the notation of section 1,

confin
$$\tau_p^\Lambda(x, y) = P_p^\Lambda(y \in C_x(\omega) \text{ and } C_x(\omega) \cap \partial\Lambda = \emptyset) \tag{3.1}$$

By definition (1.2), we have that

con
$$\tau_p^{\text{f}}(x, y) = \lim_{\Lambda \to \infty} \tau_p^\Lambda(x, y) \tag{3.2}$$

Given a configuration $\omega \in \Omega_\Lambda$ we denote by $\mathbb{E}_\Lambda^1(\omega)$ the set of all open bonds of ω and with $\mathbb{E}_\Lambda^0(\omega)$ the set of all closed bonds in ω. Hence the probability assigned to ω is simply:

$$P_p^\Lambda(\omega) = p^{|\mathbb{E}_\Lambda^1(\omega)|}(1-p)^{|\mathbb{E}_\Lambda^0(\omega)|}.$$

This probability is normalized:

1
$$\sum_{\omega \in \Omega_\Lambda} P(\omega) = \sum_{\omega \in \Omega_\Lambda} p^{|\mathbb{E}_\Lambda^1(\omega)|}(1-p)^{|\mathbb{E}_\Lambda^0(\omega)|} = 1. \tag{3.3}$$

We now rewrite (3.3) as follows

2

$$1 = \sum_{\omega\in\Omega_\Lambda} p^{|\mathbb{E}^1_\Lambda(\omega)|+|\mathbb{E}^0_\Lambda(\omega)|}\left(\frac{1-p}{p}\right)^{|\mathbb{E}^0_\Lambda(\omega)|} = p^{|\mathbb{E}_\Lambda|}\sum_{\omega\in\Omega_\Lambda}\left(\frac{1-p}{p}\right)^{|\mathbb{E}^0_\Lambda(\omega)|}. \tag{3.4}$$

Hereafter we will assume that Λ is the cube $[-N,N]^d$ in $\mathbb{Z}^d$. So $\lim_{\Lambda\to\infty}$ means simply $\lim_{N\to\infty}$. Let us now define

lambda

$$\lambda = \frac{1-p}{p} \tag{3.5}$$

and the "partition function"

ZL

$$Z_\Lambda(\lambda) = \sum_{\omega\in\,\Omega_\Lambda}\lambda^{|\mathbb{E}^0_\Lambda(\omega)|} = \left(\frac{1}{p}\right)^{|\mathbb{E}_\Lambda|}. \tag{3.6}$$

The finite volume connectivity $\tau^\Lambda_p(x,y)$ can be written in terms of an expression similar to (3.6). Namely,

TL

$$\tau^\Lambda_p(x,y) = \sum_{\substack{\omega\in\Omega_\Lambda:\, y\in C_x(\omega)\\ C_x(\omega)\cap\partial\Lambda=\emptyset}} p^{|\mathbb{E}^1_\Lambda(\omega)|}(1-p)^{|\mathbb{E}^0_\Lambda(\omega)|} = \frac{1}{Z_\Lambda(\lambda)}\sum_{\substack{\omega\in\Omega_\Lambda:\, y\in C_x(\omega)\\ C_x(\omega)\cap\partial\Lambda=\emptyset}}\lambda^{|\mathbb{E}^0_\Lambda(\omega)|} \tag{3.7}$$

It is now easy to rewrite $Z_\Lambda(\lambda)$ and $\tau^\Lambda_p(x,y)$ in term of a dual animal expansion. Observe that a configuration $\omega\in\Omega_\Lambda$ is uniquely given once we specify the set $\mathbb{E}^0_\Lambda(\omega)\subset\mathbb{E}$ of its closed edges. We associate to $\mathbb{E}^0_\Lambda(\omega)$ the set of plaquettes $[\mathbb{E}^0_\Lambda(\omega)]^*\subset\mathbb{E}^*_\Lambda$ dual to the closed edges in $\mathbb{E}^0_\Lambda(\omega)$. Recalling now the definitions of previous section, let $\{\alpha_1,\dots,\alpha_n\}$ be the (dual) connected components of $[\mathbb{E}^0_\Lambda(\omega)]^*$. Namely, $\{\alpha_1,\dots,\alpha_n\}$ is an unordered n-uple of pairwise compatible dual animals in $\mathbb{E}^*_\Lambda$, i.e., each α_i is a dual connected and finite subset of $\subset\mathbb{E}^*_\Lambda$, and each pair $\{\alpha_i,\alpha_j\}$ is compatible, i.e. $\alpha_i^*\sim\alpha_j^*$ (for $1\le i<j\le n$). Any configuration $\omega\in\Omega_\Lambda$ is uniquely associated to a collection $\{\alpha_1,\dots,\alpha_n\}$ of pairwise compatible dual animals in $\mathbb{E}^*_\Lambda$. Then $[\mathbb{E}^0_\Lambda(\omega)]^* = \cup_{i=1}^n\alpha_i^*$ and $|\mathbb{E}^0_\Lambda(\omega)| = |[\mathbb{E}^0_\Lambda(\omega)]^*| = \sum_{i=1}^n|\alpha_i|$.
With these notations the partition function (3.6) can be rewritten as

OKK

$$Z_\Lambda(\lambda) = \sum_{\substack{\{\alpha\}_n:\,\alpha_i\in\mathcal{D}^*_\Lambda\\ \alpha_i\sim\alpha_j}}\lambda^{|\{\alpha\}_n|} \tag{3.8}$$

where we denoted shortly $\{\alpha\}_n = \{\alpha_1,\dots,\alpha_n\}$, $|\{\alpha\}_n| = \sum_{i=1}^n|\alpha_i|$. We put $\{\alpha\}_0=\emptyset$ and $|\{\alpha\}_0|=0$, so that the term corresponding to $n=0$ is the contribution of the configuration ω for which $\mathbb{E}^0_\Lambda(\omega)=\emptyset$ (i.e., all bonds are open).
In order to understand how we can rewrite $\tau^\Lambda_p(x,y)$ in term dual animals, observe that the sum of the numerator (3.7) runs over the configurations $\omega\in\Omega_\Lambda$ such that the open cluster $a_x(\omega)=(C_x(\omega),E_x(\omega))$ is an animal with the properties $\{x,y\}\subset C_x(\omega)$ and $C_x(\omega)\cap\partial\Lambda=\emptyset$. Since the boundary $\partial_e a_x(\omega)$ contains only closed edges, it is a subset of $\mathbb{E}^0_\Lambda(\omega)$. ¿From proposition 5, $\partial_e a\supset\gamma_a$, where γ_a is the Peierls contour enclosing a. Moreover proposition 3 implies that $[\mathbb{E}^0_\Lambda(\omega)]^*$ does not contain other Peierls contours $\gamma\neq\gamma_a$ such that $I_\gamma\subset I_{\gamma_a}$ and $\{x,y\}\cap I_\gamma\neq\emptyset$. This means that in the numerator of (3.7) the sum runs on the collections $\{\alpha\}_n$ of dual animals containing a Peierls contour $\gamma\subset\cup_{i=1}^n\alpha_i$ such that $\gamma\bigodot\{x,y\}$ and the set $I^*_\gamma\cap\cup_{i=1}^n\alpha_i$ does not contains Peierls contours γ' such that $\gamma'\bigotimes\{x,y\}$. We denote shortly this condition with the

symbol $\{\alpha\}_n \bigodot \{x,y\}$. The Peierls contour γ is called the *minimal Peierls contour surrounding* $\{x,y\}$.

Therefore in the contour notations we can rewrite the connectivity as

frac
$$\tau_p^\Lambda(x,y) = \frac{1}{Z_\Lambda(\lambda)} \sum_{\substack{\{\alpha\}_n:\, \alpha_i \in \mathcal{D}^*_\Lambda,\ \alpha_i \sim \alpha_j \\ \{\alpha\}_n \bigodot \{x,y\}}} \lambda^{|\{\alpha\}_n|} \tag{3.9}$$

§4 *Analyticity of the connectivity*

In this section we prove the analyticity of the connectivity. We start with a preliminary lemma. Let us define the function

3.0
$$\tau_p^\Lambda(x) = \sum_{\substack{\omega \in \Omega_\Lambda \\ C_x(\omega) \cap \partial\Lambda = \emptyset}} p^{|\mathbb{E}^1_\Lambda(\omega)|}(1-p)^{|\mathbb{E}^0_\Lambda(\omega)|} = \frac{1}{Z_\Lambda(\lambda)} \sum_{\substack{\{\alpha\}_n:\, \alpha_i \in \mathcal{D}^*_\Lambda,\ \alpha_i \sim \alpha_j \\ \{\alpha\}_n \bigodot \{x\}}} \lambda^{|\{\alpha\}_n|} \tag{4.1}$$

where $\Lambda \subset \mathbb{V}$, $x \in \Lambda$, and, consistently with previous notation, $\{\alpha\}_n \bigodot \{x\}$ means that there exists $\alpha_j \in \{\gamma\}_n$ such that $\alpha_j \in \mathcal{S}^*_\Lambda$ and $\alpha_j \bigodot \{x\}$. Analogously, the notation $\{\beta\}_n \bigcirc \{x\}$ means all contours eventually present in the collection $\{\beta\}_n$ are avoiding $\{x\}$, i.e. there is no contour in the collection $\{\beta\}_n$ which surrounds $\{x\}$.

The limit

3.0.1
$$\tau_p^{\mathrm{f}}(x) = \lim_{\Lambda \to \infty} \tau_p^\Lambda(x) \tag{4.2}$$

is the complement of the percolation probability (1.1).

Lemma 6. *The function $\tau_p^{\mathrm{f}}(x)$ defined in (4.2) is analytic in λ in a domain D containing a disk $|\lambda| < \lambda_0$ with λ_0 independent on x.*

Proof. First we prove that the function $\tau_p^\Lambda(x)$ is uniformly bounded in Λ in any compact domain $\bar{D}$ contained in a disk $|\lambda| < \lambda_0$ with λ_0 independent on x.

Let $U \subset \mathbb{E}^*_\Lambda$, then $\mathcal{D}^*_U$ denotes the set of dual animals in U, and recalling (3.5), (3.6) and (3.8), the partition function $Z_U(\lambda)$ is defined by the formulas:

z1
$$\frac{1}{Z_U(\lambda)} \sum_{\substack{\{\alpha\}_n:\, \alpha_i \in \mathcal{D}^*_U \\ \alpha_i \sim \alpha_j}} \lambda^{|\{\alpha\}_n|} = 1 \quad ; \qquad Z_U(\lambda) = (1+\lambda)^{|U|}. \tag{4.3}$$

Given now $\{\alpha\}_n \bigodot \{x\}$, let $\gamma \subset \cup_{i=1}^n \alpha_i$ be the minimal Peierls contour surrounding $\{x\}$, i.e. $\gamma \in \mathcal{C}^*_\Lambda$, $\gamma \bigodot \{x\}$ and if $\gamma' \subset \cup_{i=1}^n \alpha_i$ is another Peierls contour such that $\gamma' \bigodot \{x\}$, then $I_\gamma \subset I_{\gamma'}$. Let now $\sigma \in \mathcal{S}^*$ be the contour in the collection $\{\alpha\}_n$ which contains γ and let $\tilde{\sigma} = \sigma \cup \Delta\sigma$ where $\Delta\sigma \subset \mathbb{E}^*_\Lambda$ is the set of plaquettes in $\mathbb{E}^*_\Lambda$ which are not in σ but are connected to some plaquette of σ. Remark that $\Delta\sigma \cap (\cup_{i=1}^n \alpha_i) = \emptyset$, since $\{\alpha\}_n$ are pairwise compatible. Let now $\gamma' \subset \tilde{\sigma}$ be the minimal Peierls contour contained in $\tilde{\sigma}$ such that $I_{\gamma'} \subseteq I_\gamma$ and $\gamma' \bigodot \{x\}$ (this Peirels contour eventually, but not necessarily, coincides with γ). Let us define $\mathbb{I}_\sigma = I^*_{\gamma'} - \tilde{\sigma}$ and $\mathbb{O}_\sigma = O^*_{\gamma'} - \tilde{\sigma}$. Then $\{\mathbb{I}_\sigma, \mathbb{O}_\sigma, \tilde{\sigma}\}$ is a partition of $\mathbb{E}^*_\Lambda$, hence $Z_\Lambda(\lambda) = Z_{\mathbb{E}^*_\Lambda}(\lambda) = Z_{\mathbb{O}_\sigma}(\lambda) Z_{\mathbb{I}_\sigma}(\lambda) Z_{\tilde{\sigma}}(\lambda)$ and we can write:

$$\tau_p^\Lambda(x) = \sum_{\substack{\sigma \in \mathcal{S}^*_\Lambda \\ \sigma \bigodot \{x\}}} \left[\frac{\lambda^{|\sigma|}}{Z_{\tilde{\sigma}}(\lambda)} \left(\frac{1}{Z_{\mathbb{O}_\sigma}(\lambda)} \sum_{\substack{\{\alpha\}_n:\, \rho_i \in \mathcal{D}^*_{\mathbb{O}_\sigma} \\ \alpha_i \sim \alpha_j}} \lambda^{|\{\rho\}_n|} \right) \left(\frac{1}{Z_{\mathbb{I}_\sigma}(\lambda)} \sum_{\substack{\{\beta\}_n:\, \tau_i \in \mathcal{D}^*_{\mathbb{I}_\sigma} \\ \beta_i \sim \beta_j,\, \{\beta\}_n \bigcirc \{x\}}} \lambda^{|\{\tau\}_n|} \right) \right] =$$

3.1.0

$$= \sum_{\substack{\sigma\in\mathcal{S}^*_\Lambda \\ \sigma \odot \{x\}}} \frac{\lambda^{|\sigma|}}{(1+\lambda)^{|\tilde\sigma|}} \left[\frac{1}{Z_{\mathbb{I}_\sigma}(\lambda)} \sum_{\substack{\{\beta\}_n:\, \tau_i\in\mathcal{D}^*_{\mathbb{I}_\sigma} \\ \beta_i\sim\beta_j,\, \{\beta\}_n \bigcirc \{x\}}} \lambda^{|\{\tau\}_n|} \right] \tag{4.4}$$

where in the second line of (4.4) we have use the first of identities (4.3) in the case $U = \mathbb{O}_\sigma$ and the second of (4.3) in the case $U = \tilde\sigma$. (4.4) can be now rewritten as

3.1

$$\tau_p^\Lambda(x) = \sum_{\substack{\sigma\in\mathcal{S}^*_\Lambda \\ \sigma \odot \{x\}}} \frac{\lambda^{|\sigma|}}{(1+\lambda)^{|\tilde\sigma|}} \left[1 - \tau_p^{\mathbb{I}_\sigma}(x) \right] \tag{4.5}$$

where

3.1b

$$\tau_p^{\mathbb{I}_\sigma}(x) = \frac{1}{Z_{\mathbb{I}_\sigma}(\lambda)} \sum_{\substack{\{\alpha\}_n:\, \alpha_i\in\mathcal{D}^*_{\mathbb{I}_\sigma} \\ \alpha_i\sim\alpha_j,\, \{\alpha\}_n \odot \{x\}}} \lambda^{|\{\alpha\}_n|} \tag{4.6}$$

Iterating (4.5) and (4.6) we get

$$\tau_p^\Lambda(x) = \sum_{n\geq 1} \sum_{\substack{\sigma_1\in\mathcal{S}^*_\Lambda \\ \sigma_1 \odot \{x\}}} \frac{\lambda^{|\sigma_1|}}{(1+\lambda)^{|\tilde\sigma_1|}} \sum_{\substack{\sigma_2\in\mathcal{S}^*_\Lambda:\, \sigma_2\subset\mathbb{I}_{\sigma_1} \\ \sigma_2 \odot \{x\}}} \frac{\lambda^{|\sigma_2|}}{(1+\lambda)^{|\tilde\sigma_2|}} \cdots \sum_{\substack{\sigma_n\in\mathcal{S}^*_\Lambda:\, \sigma_n\subset\mathbb{I}_{\sigma_{n-1}} \\ \sigma_n \odot \{x\}}} \frac{\lambda^{|\sigma_n|}}{(1+\lambda)^{|\tilde\sigma_n|}}$$

Now we study the region of analyticity of the function

fl

$$f(\lambda) = \sum_{n=1}^{\infty} \left[\sum_{\substack{\sigma\in\mathcal{S}^* \\ \sigma \odot \{x\}}} \frac{\lambda^{|\sigma|}}{(1+\lambda)^{|\tilde\sigma|}} \right]^n \tag{4.7}$$

since any term in the sum contributing to $f(\lambda)$ is also contained in the sum contributing to $\tau_p^\Lambda(x)$. The function $f(\lambda)$ is a geometric series in powers of the function

3.10

$$g(\lambda) = \sum_{\substack{\sigma\in\mathcal{S}^* \\ \sigma \odot \{x\}}} \left[\frac{\lambda}{1+\lambda} \right]^{|\sigma|} \left[\frac{1}{1+\lambda} \right]^{|\Delta\sigma|} \tag{4.8}$$

Thus $f(\lambda)$ is analytic whenever $g(\lambda)$ is analytic and $|g(\lambda)| < 1$. For $\mathrm{Re}\lambda \geq 0$ the factor $|\frac{1}{1+\lambda}|$ can be bounded by 1((this is the region which contains the real interval $0 \leq p \leq 1$) and we get

$$|g(\lambda)| \leq \sum_{\substack{\sigma\in\mathcal{S}^* \\ \sigma \odot \{x\}}} \left| \frac{\lambda}{1+\lambda} \right|^{|\sigma|} \leq \sum_{n=2d}^{\infty} C_n \left| \frac{\lambda}{1+\lambda} \right|^n$$

where (see e.g. lemma 4.1 in [1])

$$C_n = \sum_{\substack{\sigma\subset\mathcal{S}^*:\, |\sigma|=n \\ \sigma \odot \{x\}}} 1 \leq [e + 6e(d-1)]^n$$

Therefore $g(\lambda)$ is analytic and $|g(\lambda)| < 1$ at least in the region

pp

$$D_+ = \Big\{ \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0 \Big\} \cap \Big\{ \lambda \in \mathbb{C} : \left| \frac{\lambda}{1+\lambda} \right| < \lambda_0 \Big\} \tag{4.9}$$

where $\lambda_0 = \alpha \{e[1+6(d-1)]\}^{-1}$ with α solution of $\alpha^{2d}+\alpha-1=0$. Hence we conclude that $f(\lambda) = \sum_{n=1}^{\infty}[g(\lambda)]^n$ is analytic at least in the region (4.9).
When Re$\lambda < 0$ using the rough bound $|\Delta\gamma| \le 6(d-1)|\gamma|$ and repeating the same computation, we find that $g(\lambda)$ is analytic and $|g(\lambda)| < 1$ also in the region

pps

$$D_- = \Big\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\Big\} \cap \Big\{\lambda \in \mathbb{C} : |\lambda| < \frac{\lambda_0}{3}\Big\} \tag{4.10}$$

Therefore, as Λ grows, $\tau_p^{\Lambda}(x)$ converges uniformly on any compact region contained in the region $R = D_+ \cup D_-$ which contains in its turn the open disk $|\lambda| < \lambda_0/3$.
On the other hand, for any Λ, $\tau_p^{\Lambda}(x)$ is also manifestly analytic in λ for $\lambda \neq -1$, since, by definition (4.1), is a polynomial in p. Therefore the limit $\tau_p^f(x)$ is analytic in R. $\square$
Note that this argument shows the analyticity of the percolation probability. $\theta(p) = 1 - \tau_p^f(x)$.

Turning back to the analyticity of the connectivity, we can now easily prove the following theorem.

Theorem 7. *The function $\tau_p^{\mathrm{f}}(x,y)$ is analytic in λ in a domain D containing a disk $|\lambda| < \lambda_0$ with λ_0 independent on x,y.*
Proof. Proceeding as in (4.4), we get

3.1.2

$$\tau_p^{\Lambda}(x,y) = \sum_{\substack{\sigma \in \mathcal{S}^*_{\Lambda} \\ \sigma \odot \{x,y\}}} \frac{\lambda^{|\sigma|}}{(1+\lambda)^{|\tilde\sigma|}} \left[\frac{1}{Z_{\mathbb{I}_\sigma}(\lambda)} \sum_{\substack{\{\alpha\}_n :\, \alpha_i \in \mathcal{D}^*_{\mathbb{I}_\sigma} \\ \alpha_i \sim \alpha_j,\, \{\alpha\}_n \bigcirc \{x,y\}}} \lambda^{|\{\alpha\}_n|} \right] \tag{4.11}$$

which can be rewritten, using a decomposition similar to the one presented above, as

3.1.3

$$\tau_p^{\Lambda}(x,y) = \sum_{\substack{\sigma \in \mathcal{S}^*_{\Lambda} \\ \sigma \odot \{x,y\}}} \frac{\lambda^{|\sigma|}}{(1+\lambda)^{|\tilde\sigma|}} \left[1 - \sum_{\substack{\{\alpha\}_n : \gamma_i \in \mathcal{D}^*_{\mathbb{I}_\sigma} \\ \alpha_i \sim \alpha_j,\, \{\alpha\}_n \odot \{x\}}} \frac{\lambda^{|\{\alpha\}_n|}}{Z_{\mathbb{I}_\sigma}(\lambda)} - \sum_{\substack{\{\beta\}_n :\, \beta_i \in \mathcal{D}^*_{\mathbb{I}_\sigma},\, \beta_i \sim \beta_j, \\ \{\beta\}_n \odot \{y\}, \{\beta\}_n \bigcirc \{x\}}} \frac{\lambda^{|\{\beta\}_n|}}{Z_{\mathbb{I}_\sigma}(\lambda)} \right] \tag{4.12}$$

The last addend of (4.12) can be rewritten as

$$\sum_{\substack{\{\beta\}_n :\, \beta_i \in \mathcal{D}^*_{\mathbb{I}_\sigma},\, \gamma_i \sim \gamma_j, \\ \{\beta\}_n \odot \{y\}, \{\beta\}_n \bigcirc \{x\}}} \frac{\lambda^{|\{\beta\}_n|}}{Z_{\mathbb{I}_\sigma}(\lambda)} = \sum_{\substack{\tau \in \mathcal{S}^*_{\mathbb{I}_\sigma} : \tau \odot \{y\} \\ \tau \bigcirc \{x\}}} \frac{\lambda^{|\tau|}}{(1+\lambda)^{|\tilde\tau|}} \sum_{\substack{\{\alpha\}_n :\, \alpha_i \in \mathcal{D}^*_{\mathbb{I}_\tau} \\ \{\alpha\}_n \bigcirc \{y\}}} \frac{\lambda^{|\{\alpha\}_n|}}{Z_{\mathbb{I}_\tau}(\lambda)} \sum_{\substack{\{\beta\}_n :\, \beta_i \in \mathcal{D}^*_{\mathbb{O}_\tau} \\ \{\beta\}_n \bigcirc \{x\}}} \frac{\lambda^{|\{\beta\}_n|}}{Z_{\mathbb{O}_\tau}(\lambda)}$$

where $\mathbb{I}_\tau = I^*_{\tilde\tau} \cap I^*_{\gamma''}$, with $\gamma'' \subset \tilde\tau$ being the Peierls contour in $\tilde\tau$ surrounding $\{y\}$ and $\mathbb{O}_\tau = \mathbb{I}_\sigma - (\mathbb{I}_\tau \cup \tilde\tau)$. Hence, recalling the definition (4.1), (4.12) can be rewritten as

$$\tau_p^{\Lambda}(x,y) = \sum_{\substack{\sigma \in \mathcal{S}^*_{\Lambda} \\ \sigma \odot \{x,y\}}} \frac{\lambda^{|\sigma|}}{(1+\lambda)^{|\tilde\sigma|}} \left\{ 1 - \tau_p^{\mathbb{I}_\sigma}(x) - \sum_{\substack{\tau \in \mathcal{S}^*_{\mathbb{I}_\sigma} : \tau \odot \{y\} \\ \tau \bigcirc \{x\}}} \frac{\lambda^{|\tau|}}{(1+\lambda)^{|\tilde\tau|}} \Big[1 - \tau_p^{\mathbb{I}_\tau}(y)\Big] \Big[1 - \tau_p^{\mathbb{O}_\tau}(x)\Big] \right\}$$

The l.h.s. of the equality above is again uniformly bounded in Λ as a function of λ in any compact domain contained in R. Whence it follows the analyticity of the connectivity in the region R above. $\square$

§5 *Upper bound and lower bound for the truncated connectivity*

We now get an upper bound and a lower bound for the connectivity for positive values of λ sufficiently near $\lambda = 0$ using representation (3.9).
Concerning the upper bound, we recall that we are considering only the case $d \geq 3$. Given $\{\alpha\}_n \odot \{x,y\}$, let $\gamma \subset \cup_{i=1}^n \alpha_i$ be the minimal Peierls contour surrounding $\{x,y\}$. We get immediately the following upper bound uniformly in Λ:

$$\tau_p^\Lambda(x,y) \leq \sum_{\substack{\gamma \in \mathcal{C}^*_\Lambda \\ \gamma \odot \{x,y\}}} \lambda^{|\gamma|} \frac{\sum_{\substack{\{\alpha\}_n : \alpha_i \in \mathcal{D}^*_\Lambda \\ \alpha_i \subset (\mathbb{E}^*_\Lambda - \gamma),\, \alpha_j \sim \alpha_i}} \lambda^{|\{\alpha\}_n|}}{\sum_{\substack{\{\alpha\}_n : \alpha_i \in \mathcal{D}^*_\Lambda \\ \alpha_j \sim \alpha_i}} \lambda^{|\{\alpha\}_n|}} \leq \sum_{\substack{\gamma \in \mathcal{C}^* \\ \gamma \odot \{x,y\}}} \lambda^{|\gamma|}$$

Therefore we need to bound the positive series

ser
$$\sum_{\substack{\gamma \in \mathcal{C}^* \\ \gamma \odot \{x,y\}}} \lambda^{|\gamma|} \tag{5.1}$$

Denoting as

nmin
$$N = 2(d-1)(|x-y|+1)+2, \tag{5.2}$$

any Peierls contour surrounding $\{x,y\}$ has at least N hyper-squares. We now split the series (5.1) into two terms, the first one contains the sum over "small" contours, i.e. those with cardinality at most $\frac{d}{d-1}N$ (the constant $d/(d-1)$ is so chosen for later convenience), and the second one containing the "large" contours with cardinality grater than $\frac{d}{d-1}N$. Namely, we write

split
$$\sum_{\substack{\gamma \in \mathcal{C}^* \\ \gamma \odot \{x,y\}}} \lambda^{|\gamma|} = \sum_{\substack{\gamma \in \mathcal{C}^* :\, \gamma \odot \{x,y\} \\ N \leq |\bar{\gamma}| \leq \frac{d}{d-1}N}} \lambda^{|\gamma|} + \sum_{\substack{\gamma \in \mathcal{C}^* :\ \gamma \odot \{x,y\} \\ |\gamma| > \frac{d}{d-1}N}} \lambda^{|\gamma|} \tag{5.3}$$

First consider

$$\sum_{\substack{\gamma \in \mathcal{C}^* :\ \gamma \odot \{x,y\} \\ |\gamma| > \frac{d}{d-1}N}} \lambda^{|\gamma|} = \sum_{n > \frac{d}{d-1}N} F_n(x,y)\lambda^n$$

where

$$F_n(x,y) = \sum_{\substack{\gamma \in \mathcal{C}^* :\ \gamma \odot \{x,y\} \\ |\gamma| = n}} 1$$

is the number of Peierls contours of size n surrounding $\{x,y\}$. We will use for n large the Ruelle bound [11] $F_n(x,y) < 3^n$, whence

$$\sum_{\substack{\gamma \in \mathcal{C}^* :\ \gamma \odot \{x,y\} \\ |\gamma| > \frac{d}{d-1}N}} \lambda^{|\gamma|} \leq \sum_{n > \frac{d}{d-1}N} \lambda^n 3^n = \lambda^N \sum_{n > \frac{d}{d-1}N} 3^n \lambda^{n-N} =$$

$$= \lambda^N \sum_{k > \frac{1}{d-1}N} 3^{N+k}\lambda^k \leq \lambda^N \sum_{k \geq 1} 3^{dk}\lambda^k \leq \lambda^N$$

provided $3^d\lambda \leq 1/2$. Then

7star
$$\sum_{\substack{\gamma \in \mathcal{C}^* :\ \gamma \odot \{x,y\} \\ |\gamma| > \frac{d}{d-1}N}} \lambda^{|\gamma|} \leq \lambda^N \tag{5.4}$$

The rest of the section is mainly devoted to the proof of an upper bound for the first factor in r.h.s. of (5.3) which, apart from correction small in $(1-p)$, gives an exponential behavior similar to (5.4). We first bound

nx2
$$\sum_{\substack{\gamma\in\mathcal{C}^*:\ \gamma\odot\{x,y\}\\ |\gamma|=N+k}} \lambda^{|\gamma|} \qquad \text{where } \ 0\le k\le \frac{N}{d-1} \tag{5.5}$$

Let us introduce some notations. For $x\in\mathbb{V}$ we denote by $x^{\square}$ the closed unit hypercube in $\mathbb{R}^d$ with sides parallel to the axis and center in x. Note that the faces of $x^{\square}$ are plaquettes in $\mathbb{E}^*$.
A *self avoiding walk* $w\subset\mathbb{L}$ *(SAW), joining* x *to* y, is an animal $w=(V_w,E_w)$ in $\mathbb{L}$ such that $V_w=\{x_1,\dots,x_n\}$ (all vertices x_i are distinct), with $x_1=x$, $x_n=y$, $E_w=\{e_1,\dots,e_{n-1}\}$, where $e_i=\{x_i,x_{i+1}\}$. The vertex $x_1=x$ is called *the start vertex* of w and the vertex $x_n=y$ is called *the end vertex* of w and $|E_w|=|w|$ is called the length of the SAW. A SAW $t=(V_t,E_t)$ such that $\mathbb{E}|_{V_t}=E_t$ is called a *tubular walk (TW)*. If $t\subset\mathbb{L}$ is a TW joining x to y, then it is easy to see that the dual of its edge boundary $\tau=[\partial_e t]^*$ is a Peierls contour, and we will call it a *tube from* x *to* y. Clearly, if $\tau\subset\mathbb{E}^*$ is a tube, then there is a unique TW t such that $\tau=[\partial_e t]^*$ and we say that t *generates the tube* τ. Note that if w is a SAW such that $\mathbb{E}|_{V_w}\neq E_w$, then $[\partial_e w]^*$ *is not* a Peierls contour.
If τ is a tube from x to y generated by the tubular walk $t=(V_t,E_t)$ with $V_t=\{x_1,x_2\dots,x_{n-1},x_n\}$ (with $x=x_1$ and $x_n=y$), then, denoting shortly $t^{\square}=\cup_{i=1}^n x_i^{\square}$, the immersion $\bar{t}^{\square}$ in $\mathbb{R}^d$ of $t^{\square}$ is a tubular polyhedron in $\mathbb{R}^d$ formed by $|t^{\square}|$ unit hypercubes pairwise connected through faces in such way that each $x_i^{\square}\in t^{\square}$ shares one and only one $(d-1)$-dimensional face with $x_{i-1}^{\square}$ and with $x_{i+1}^{\square}$ for all $i=2,\dots,n-1$, while $x_1^{\square}=x^{\square}$, *the start-cube of* $t^{\square}$, shares one $(d-1)$-dimensional face only with $x_2^{\square}$ and $x_n^{\square}=y^{\square}$, *the end-cube of* $t^{\square}$, shares one $(d-1)$-dimensional face only with $x_{n-1}^{\square}$. Moreover, let e_x be the unique edge in $\partial_e t$ such that $x\in e_x$ and e_x is parallel to $e_1=\{x,x_2\}$ and let e_y be the unique edge in $\partial_e t$ such that $y\in e_y$ and e_y is parallel to $e_{n-1}=\{x_{n-1},y\}$. Then $e_x^*\in\tau$ is called the *start-plaquette of the tube* and $e_y^*\in\tau$ is called *the end-plaquette of the tube.* Clearly $|\tau|=2(d-1)|t^{\square}|+2$ is the number of plaquettes in the tube. For $\{x,y\}\in\mathbb{V}$, we denote by $\mathcal{T}_{xy}^*$ the set of all tubes in $\mathbb{E}^*$ from x to y.
A tube τ from x to y generated by the tubular walk t is *minimal* when $|t^{\square}|=|x-y|+1$. Given two Peierls contours, γ and γ', we say that $\gamma\sqsubset\gamma'$ if $I_\gamma\subset I'_\gamma$. Given a Peierls contour γ, let $\{x,y\}\subset I_\gamma$. We say that a tube $\tau\sqsubset\gamma$ from x to y is *minimal relatively to* γ if there exists no tube $\tau'\sqsubset\gamma$ from x to y such that $|\tau'^{\square}|<|\tau^{\square}|$.

Lemma 8. *Let $\gamma\in\mathcal{C}^*$ be a Peierls contour such that $\{x,y\}\subset I_\gamma$ and let $\tau\sqsubset\gamma$ a minimal tube relatively to γ from x to y. Then*

5.5
$$|\gamma-\tau|\ge\frac{d+1}{d-1}|\tau-\gamma| \tag{5.6}$$

Proof. We will give the proof for $d=3$, since the higher dimensional case is a straightforward generalization of this. Then (5.6) becomes

5.6
$$|\gamma-\tau|\ge 2|\tau-\gamma| \tag{5.7}$$

We first prove that to each plaquette in $\tau-\gamma$ we can associate in a one-to-one way a plaquette in $\gamma-\tau$. Let e be an edge such that $e^*\in\tau-\gamma$ and let z be the vertex of e which belongs to the tubular path t generating τ. Consider the normal n^+ to the plaquette e^* in the direction

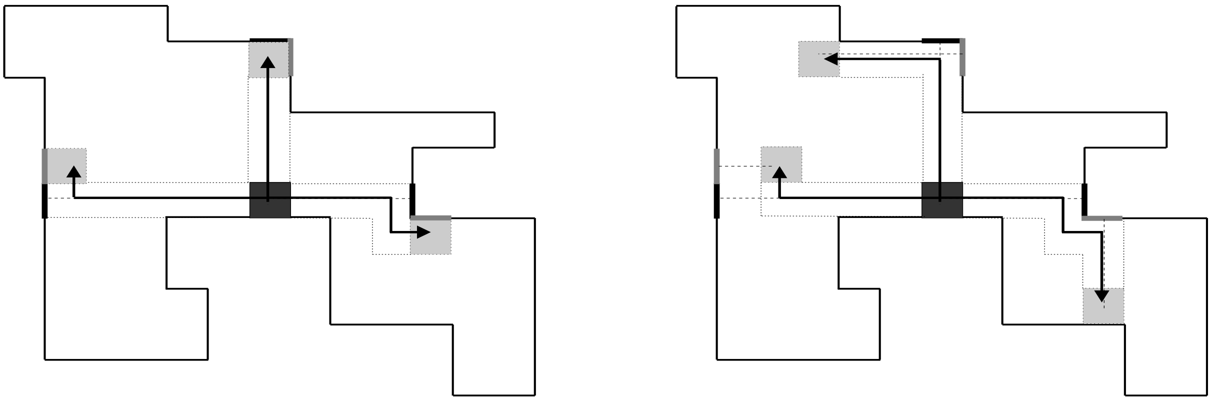

Figure 1.

external to the cube $z^{\square}$ and let r be the straight SAW with start vertex z formed by all the edges that are collinear to e and lies from e in the direction of n^+. Let e_1 be the first edge starting from e in r such that $e_1^* \in \gamma$. This edge does exist by definition of interior of γ. The corresponding plaquette e_1^* is in $\gamma - \tau$, since otherwise it would exists a straight tube inside γ connecting two cubes of $t^{\square}$, in contradiction with the hypothesis that τ is minimal in γ. For the same reason e_1^* cannot be the projection on γ of another plaquette belonging to $\tau - \gamma$. Hence the correspondence between $e^* \in \tau - \gamma$ and $e_1^* \in \gamma - \tau$ is one-to-one. The plaquette e_1^* is called the *projection of* e^* *on* γ.

Let now $V_t = \{x_1 = x, x_2, ..., x_{n-1}, x_n = y\}$ be the vertex set of the tubular walk t generating τ. For $2 \leq i \leq n-1$, we will say that x_i is *rectilinear* if the two edges in t containing x_i are parallel, i.e. collinear (the start vertex x and the end vertex y are assumed rectilinear). If x_i is not rectilinear it is an *elbow*. Let us start from the rectilinear vertices in t. For each x_i rectilinear, consider the two planes (hyperplanes in $d > 3$) π_i, π_{i+1} containing the two parallel faces of $x_i^{\square}$ that are the plaquettes dual to the two edges e_{i-1} and e_i containing x_i (if $i = 1$ then $e_{i-1} = e_x$ and if $i = n$ then $e_i = e_y$). Let $e_{i,k}^*$ ($k = 1, 2, 3, 4$) be the plaquettes belonging to $x_i^{\square}$ and different from e_i^* and e_{i+1}^*. Let us call now e^* one of the plaquettes $e_{i,k}^*$ and assume that $e^* \in \tau - \gamma$. Note that the projection e_1^* of e^* on $\gamma - \tau$ is in the region of the space between the two planes π_i and π_{i+1}.

Consider now the subset $\gamma_i \subset \gamma$ belonging to this region of the space between the two planes π_i and π_{i+1} which contains the projection e_1^* of e^*. This subset contains plaquettes of γ that are perpendicular to π_i and π_{i+1} and have the same height along the direction of e_i of the plaquettes $e_{i,k}^*$.

Now consider the two plaquettes belonging to γ_i and connected with e_1^*. These plaquettes are both in $\gamma - \tau$ and cannot be the projection on γ of some other plaquette belonging to τ, since in both these cases there would be a minimal tube inside γ, contained in the region of the space between the two planes π_i and π_{i+1}, which connects two cubes of τ, again contradicting the hypothesis that τ is minimal relatively to γ. This situation is illustrated in figure 1. In case 1 minimal tubes in γ are shown if one assumes that the plaquettes belonging to γ_i and connected with e_1^* are in τ. In case 2 minimal tubes in γ are shown if one assumes that the plaquettes belonging to γ_i and connected with e_1^* are projection on γ of some other plaquette belonging to τ.

Then we are left with the elbows. Again we can call $e^*_{i,k}$ the plaquettes belonging to $x_i^\square$ such that they are not the plaquettes dual to the two edges e_i and e_{i+1} containing x_i. Now this four plaquettes are not all perpendicular to a fixed plane. Let us again call e^* one of the plaquettes $e^*_{i,k}$ and assume that $e^* \in \tau - \gamma$. We have to choose again two planes $\pi_i(e^*)$ and $\pi_{i+1}(e^*)$. They depend now on the plaquette, and not only on the vertex. The two planes $\pi_i(e^*)$ and $\pi_{i+1}(e^*)$ are chosen in such a way that they are both perpendicular to the plane containing the two edges insisting in the elbow and to the plane containing e^*. This choice may not be unique, because some e^* may be parallel to the plane containing the two edges insisting in the elbow. The two planes, obviously, contains two opposite edges of the plaquette e^*. The construction may now be repeated in the same way as before.
Hence we have shown that to each plaquette $e^* \in \tau-\gamma$ we can associate the projection $e^*_1 \in \gamma-\tau$, plus a pair of plaquette of $\gamma - \tau$ that cannot be the projection on γ of other plaquettes of $\tau - \gamma$. Each of the plaquettes in this pair may be reached by the construction above at most $2(d-1)$ times. This gives the bound above. □

We can now bound the sum (5.5). First observe that it is possible to construct any Peierls contour $\gamma \bigodot \{x, y\}$ with $|\gamma| = N + k$, by choosing first a tube τ from x to y of size $|\tau| = N + q$ where $0 \le q \le k$, then choosing a set μ of $|\mu| = m$ plaquettes in τ (μ will coincide with $\tau-\gamma$ and $m = |\tau-\gamma|$) and attaching to the sides in $\partial_s\mu$ (the side boundary of μ, see section 1) a number of plaquettes equal to $k-q+m$ to reconstruct γ (hence $|\gamma-\tau| = k-q+m$). By the Ruelle bound the number of ways in which we can attach to the edges in $\partial_s\mu$ $k-q+m$ plaquettes to form a closed contour is less than 3^{k-q+m}. Observe also that, by lemma 8, $m \le \frac{d-1}{2}(k-q) \equiv m_{\max}$. Hence we have

star

$$\sum_{\substack{\gamma\in\mathcal{C}^* \\ \gamma \bigodot \{x,y\} \\ |\gamma|=N+k}} 1 \le \sum_{q=0}^{k} \sum_{\substack{\tau\in\mathcal{T}^*_{xy} \\ |\tau|=N+q}} \sum_{m=1}^{m_{\max}} \binom{N+q}{m} 3^{k-q+m} \le 3^{dk} \sum_{q=0}^{k} \sum_{\substack{\tau\in\mathcal{T}^*_{xy} \\ |\tau|=N+q}} \sum_{m=1}^{m_{\max}} \binom{N+q}{m} \tag{5.8}$$

Recalling that $m \le \frac{d-1}{2}(k-q) \le \frac{d-1}{2}\,k \le \frac{N}{2}$ and that $q \le k \le \frac{N}{d-1}$, one can easily check that

$$\binom{N+q}{m} \le 3^q \binom{N}{m} \le 3^k \binom{N}{\frac{d-1}{2}(k-q)}$$

Thus we get from (5.8)

2star

$$\sum_{\substack{\gamma\in\mathcal{C}^* \\ \gamma \bigodot \{x,y\} \\ |\gamma|=N+k}} 1 \le m_{\max} 3^k \sum_{q=0}^{k} \sum_{\substack{\tau\in\mathcal{T}^*_{xy} \\ |\tau|=N+q}} \binom{N}{\frac{d-1}{2}(k-q)} \le C^k \sum_{q=0}^{k} \binom{N}{\frac{d-1}{2}(k-q)} \Bigg[\sum_{\substack{\tau\in\mathcal{T}^*_{xy} \\ |\tau|=N+q}} 1 \Bigg] \tag{5.9}$$

for some constant C. In what follows we will denote with C a constant depending only on the dimension d of the lattice. The constant may vary from line to line. To give a bound of the last term in (5.9), we just note that, according to the previous definitions

$$\sum_{\substack{\tau\in\mathcal{T}^*_{xy} \\ |\tau|=N+q}} 1 = \sum_{\substack{t \text{ TW from } x \text{ to } y \\ |w|=|x-y|+\ell}} 1 \le \sum_{\substack{w \text{ SAW from } x \text{ to } y \\ |w|=|x-y|+\ell}} 1$$

where $q = 2(d-1)\ell$ and $\ell \le |x-y|/2$. As it is well known (see e.g. [Si] thm. V.6.9 p. 447) the number of SAW from x to y of (vertex) length $|x-y|+\ell$ if $\ell \le |x-y|/2$ is bounded by

nsaw

$$\sum_{\substack{w \text{ SAW from } x \text{ to } y \\ |w|=|x-y|+\ell}} 1 \le \; L(x,y)\binom{|x-y|}{\ell} C^{\ell} \tag{5.10}$$

where $C \le 6d$ and

NM

$$L(x,y) = \sum_{\substack{w \text{ SAW from } x \text{ to } y \\ |w|=|x-y|}} 1 = \frac{(|x_1-y_1|+\ldots+|x_d-y_d|)!}{|x_1-y_1|!\cdots|x_d-y_d|!} \tag{5.11}$$

is the number of SAW of (minimal) length $|x-y|$ from x to y. Since now $\binom{|x-y|}{\ell} \le \binom{N}{q}$, we have

3star

$$\sum_{\substack{\tau\in\mathcal{T}^*_{xy} \\ |\tau|=N+q}} 1 \le L(x,y)\binom{N}{q} C^q \tag{5.12}$$

Inserting now (5.12) in (5.9) we get

4star

$$\sum_{\substack{\gamma\in\mathcal{C}^* \\ \gamma \bigodot \{x,y\} \\ |\gamma|=N+k}} 1 \le L(x,y)\sum_{q=0}^{k}\binom{N}{\frac{d-1}{2}(k-q)} C^k \binom{N}{q} C^q \le L(x,y)C^k \sum_{q=0}^{k}\binom{N}{\frac{d-1}{2}(k-q)}\binom{N}{q} \tag{5.13}$$

It is now easy to check that

6star

$$\binom{N}{\frac{d-1}{2}(k-q)}\binom{N}{q} \le \binom{N}{\frac{d-1}{2}(k-q)}\binom{N}{\frac{d-1}{2}q} \le 2^{(d-1)k}\binom{N}{\frac{d-1}{2}k} \tag{5.14}$$

Inserting (5.14) in (5.13) we finally get

7bstar

$$\sum_{\substack{\gamma\in\mathcal{C}^* \\ \gamma \bigodot \{x,y\} \\ |\gamma|=N+k}} 1 \le L(x,y)C^k\binom{N}{\frac{d-1}{2}k} \tag{5.15}$$

With the aid of (5.15) we have the following upper bound of the first term in r.h.s. of (5.3)

8star

$$\sum_{\substack{\gamma\in\mathcal{C}^*:\,\gamma \bigodot \{x,y\} \\ N\le|\gamma|\le\frac{d}{d-1}N}} \lambda^{|\gamma|} = \lambda^N \sum_{k=0}^{N/(d-1)} \lambda^k \sum_{\substack{\gamma\in\mathcal{C}^*:\,\gamma \bigodot \{x,y\} \\ |\gamma|=N+k}} 1 \le L(x,y)\lambda^N \sum_{k=0}^{N/(d-1)} \lambda^k C^k \binom{N}{\frac{d-1}{2}k} \le$$

$$\le L(x,y)\lambda^N \sum_{s=0}^{N/2} \lambda^s C^s \binom{N}{s} \le L(x,y)\lambda^N(1+C\lambda)^{N/2} \tag{5.16}$$

Inserting (5.4) and (5.16) in (5.3) we get finally

9star

$$\tau^{\Lambda}_{p}(x,y) \le 2L_(x,y)\lambda^N(1+C\lambda)^{N/2} \tag{5.17}$$

Inequality (5.17), recalling definition (5.2) for N, gives our upper bound.

Concerning the lower bound we get easily

$$\tau_p^\Lambda(0,x) = Z_\Lambda^{-1}(\lambda) \sum_{\substack{\{\alpha\}_n:\, \alpha_i\in\mathcal{D}^*_\Lambda,\ \alpha_i\sim\alpha_j \\ \{\alpha\}_n \odot \{x\}}} \lambda^{|\{\alpha\}_n|} \geq Z_\Lambda^{-1}(\lambda) \sum_{\substack{\tau\in\mathcal{T}^*_{xy} \\ |\tau|=N}} \lambda^{|\tau|} \sum_{\substack{\{\alpha\}_n:\, \alpha_i\in\mathcal{D}^*_\Lambda,\, \alpha_i\sim\alpha_j \\ \alpha_i\subset[\mathbb{E}|_{\Lambda-I_\tau}]^*}} \lambda^{|\{\alpha\}_n|}$$

$$= \lambda^N L(x,y) Z_\Lambda^{-1}(\lambda) Z_{[\mathbb{E}|_{\Lambda-I_\tau}]^*}(\lambda)$$

where we have used that $\sum_{\tau\in\mathcal{T}^*_{xy},\ |\tau|=N} 1 = L(x,y)$ and that

$$\sum_{\substack{\{\alpha\}_n:\, \alpha_i\in\mathcal{D}^*_{\mathbb{E}^*_\Lambda-\gamma}, \\ \alpha_i\sim\alpha_j}} \lambda^{|\{\alpha\}_n|} = Z_{[\mathbb{E}|_{\Lambda-I_\tau}]^*}(\lambda) = Z_{\Lambda-I_\tau} = p^{-|\mathbb{E}|_{\Lambda-I_\tau}|}$$

Hence

$$\tau_p^\Lambda(0,x) \geq L_0(x)\lambda^{\bar{n}_x} p^{|\mathbb{E}_\Lambda|-|\mathbb{E}|_{\Lambda-I_\tau}|}$$

Clearly, recalling that $|\tau|=N$, i.e. τ is a minimal tube, we have $|\mathbb{E}_\Lambda|-|\mathbb{E}|_{\Lambda-I_\tau}| = N+|x-y| \leq 2N$. Thus

lower

$$\tau_p^\Lambda(x,y) \geq L(x,y)\lambda^N p^{2N} \geq L(x,y)\left[\frac{\lambda}{(1+\lambda)^2}\right]^N \tag{5.18}$$

Note that both upper and lower bounds are uniform in Λ and then they hold also in the limit $\Lambda\to\infty$.
Hence, we have shown the following theorem

Theorem 9. *The truncated connectivity function $\tau_p^{\mathrm{f}}(x,y)$ defined in (3.2) admits the following bounds for $(1-p)$ sufficiently small and for $d\geq 3$*

uplow

$$L(x,y)\left[\frac{\lambda}{(1+\lambda)^2}\right]^N \ \leq\ \tau_p^{\mathrm{f}}(x,y)\ \leq\ 2L(x,y)\lambda^N(1+C\lambda)^{N/2} \tag{5.19}$$

where $L(x,y)$ is the function defined in (5.11), N is defined in (5.2) and C is a constant depending at most on the dimension d.

The relation (5.19) gives the mass (inverse correlation length) for the exponential decay of the finite connectivity in the supercritical regime. As expected, the mass depends on the direction of the vector $x-y$ and it reaches its minimum value when $x-y$ is parallel to the coordinate axis, i.e. $x-y=e_n=(n,0,\ldots,0)$ or permutations. Hence defining as usually [8]

dmass

$$m(p) = \lim_{n\to\infty}\left\{-\frac{1}{n}\ln\tau_p^{\mathrm{f}}(0,e_n)\right\} \tag{5.20}$$

we have that, for p sufficiently near 1,

mass

$$m(p) = 2(d-1)|\ln(1-p)| + O(1-p) \tag{5.21}$$

This relation has an intuitive meaning: the mass is proportional to the minimal number of closed edges needed to surround the set $\{x,y\}$. The constant of proportionality is obviously the logarithm of the (small) probabilistic weight of a single closed edge.

§6 *Lattice animals expansion in the sub-critical phase*

With a little extra work one can get a bound for $\tau_p^{\mathrm{f}}(x,y)$ similar to (5.19) also in the sub-critical phase. In the case $p < p_c$ the percolation probability is strictly zero and the truncated connectivity function coincides with the simple connectivity function. Hence we have that $\tau_p^{\mathrm{f}}(x,y) = P_p(y \in C_x(\omega))$. Then we obtain trivially

$$\tau_p^{\mathrm{f}}(x,y) = P_p(y \in C_x(\omega)) \leq \sum_{w \text{ SAW from } x \text{ to } y} p^{|w|}$$

Using now again the estimate (5.10) and recalling definition (5.11) we get immediately the upper bound

upper2

$$\tau_p^{\mathrm{f}}(x,y) \leq L(x,y)p^{|x-y|}(1+C_2p)^{|x|/2} \tag{6.1}$$

where $C_2 \leq 6d$ is a constant depending only on the dimension d.
Concerning now the lower bound, we can use the well known representation of $\tau_p^{\mathrm{f}}(x,y)$ in terms of lattice animal and get

lower2

$$\tau_p^{\mathrm{f}}(x,y) = \sum_{\substack{a\in\mathcal{A} \\ \{x,y\}\subset V_a}} p^{|E_a|}(1-p)^{|\partial_e a|} \geq L(x,y)p^{|x-y|}(1-p)^{C_1|x-y|} \tag{6.2}$$

where we have used that $|\partial_e a| \leq \{2(d-1)(|E_a|+1)+2\} \leq C_1|E_a|$ for some suitable constant C_1 depending only on the dimension d.
Hence, we have shown that the truncated connectivity function $\tau_p^{\mathrm{f}}(x,y)$ defined in (3.2) admits the following bounds for p sufficiently small

uplowb

$$L(x,y)p^{|x-y|}[(1-p)^{C_1}]^{|x-y|} \;\leq\; \tau_p^{\mathrm{f}}(x,y) \;\leq\; L(x,y)p^{|x-y|}(1+C_2p)^{|x|/2} \tag{6.3}$$

where $L(x,y)$ is the function defined in (5.11) and C_1 and C_2 are constant depending at most on the dimension d.
In this case we get for the value of the mass $m(p)$ defined in (5.20):

mass2

$$m(p) = |\ln p| + O(p) \tag{6.4}$$

Comparing (6.4) with (5.21) we get the relation, valid for p sufficiently near 1, and for $d \geq 3$

relat

$$m(p) \approx 2(d-1)m(1-p) \tag{6.5}$$

where $\approx$ means that (6.5) is true neglecting corrections of the order of $1-p$. Such a relation has been shown to be a strict identity in the two dimensional case for any $p \neq p_c$ (see [8] theorem 11.24 and reference therein), while no extension was known for $d \geq 3$. (6.5) has the following interpretation: accepting that the behavior of the finite connectivity $\tau^{\mathrm{f}}(x,y)$ is given by the leading term in the small parameter $1-p$ for the supercritical case, and p in the subcritical one, in order to have a *finite* open cluster one has to have at least $2(d-1)|x-y|$ closed bonds in the supercritical case and only $|x-y|$ open bonds in the subcritical one. It is an open question to prove that the relation above holds exactly for all $p \neq p_c$ and for all $d \geq 3$.

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