Pencils of Quadrics, Binary Forms and Hyperelliptic Curves

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To Robin Hartshorne on his 60th birthday

1 Introduction

A pencil of quadrics $(Q_\lambda)_{\lambda\in\mathbb{P}_1}$ is called *general* if its discriminant admits no multiple root. Let $\mathcal{M}^p_n$ denote the (coarse) moduli space of general pencils of quadrics in $\mathbb{P}_n$. If $\mathcal{M}^b_n$ denotes the moduli space of binary forms of degree n with no multiple root and $\mathcal{H}_g$ denotes the moduli space of smooth hyperelliptic curves of genus g, then it is a classical fact that there are canonical isomorphisms:

$$\mathcal{M}^p_{2g+1} \simeq \mathcal{M}^b_{2g+2} \simeq \mathcal{H}_g \tag{1}$$

The moduli spaces are constructed as a quotient of algebraic varieties modulo the action of a reductive group. We have, therefore, for any of the three objects above the notion of stability and semistability which leads to natural compactifications $\overline{\mathcal{M}}^p_{2g+1}, \overline{\mathcal{M}}^b_{2g+2}$, and $\overline{\mathcal{H}}_g$. So if we identify $\mathcal{M}^p_{2g+1} = \mathcal{M}^b_{2g+2} = \mathcal{H}_g$ under the isomorphism (1), we obtain 3 compactifications of this space in a natural way. It is a priori not clear that these compactifications should be isomorphic since the groups leading to them are different in each case: For $\overline{\mathcal{M}}^p_n$ the group is SL_{n+1}, for $\overline{\mathcal{M}}^b_n$ it is SL_2 and for $\overline{\mathcal{H}}_g$ it is PGL_{6g-5}. (Note that $\overline{\mathcal{H}}_g$ is usually defined as the closure of $\mathcal{H}_g$ in the moduli space of stable curves of genus g - see [12]- so embedding the curve tricanonically leads to the action of PGL_{6g-5} on the corresponding Hilbert scheme.) It is the aim of this paper to study the relations between the 3 compactifications. In the whole paper we assume the ground field to be the field $\mathbb{C}$ of complex numbers. Most results are valid for an arbitrary algebraic closed field of characteristic $\neq 2$. At one point, however, which is certainly technical, we need the analytic topology.

The contents of the paper is as follows: after giving a short review of the theory of pencils of quadrics, we determine, in Section 3, the stable and semistable pencils. A pencil in $\mathbb{P}_n$ is stable (respectively semistable) if and only if its discriminant admits no multiple root (respectively no root of multiplicity $> n/2$). In Section 4, we show that the discriminant induces an isomorphism between the moduli spaces $\overline{\mathcal{M}}^p_n \longrightarrow \overline{\mathcal{M}}^b_{n+1}$. This was already pointed out in [2], but in this paper we give a more direct proof. In particular, it gives a better understanding of the closed orbits. Notice, moreover, that

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the open subset $\mathcal{M}_n^p$ of stable pencils does not map onto the open subset of stable binary forms (see Theorem 3.1).

The relation between $\overline{\mathcal{M}}_{2g+2}^b$ and $\overline{\mathcal{H}}_g$ is more complicated and we have a result only for $g = 2$. Note first that for every n the space $\overline{\mathcal{M}}_n^b$ is a one-point compactification of the moduli space of stable binary forms of degree n. Thus there is exactly 1 point $p_{ss} \in \overline{\mathcal{M}}_n^b$ representing all semistable, not stable binary forms. In section 5 we show, for $g = 2$ and $n = 2g + 2 = 6$, that the point p_{ss} is a smooth point of $\overline{\mathcal{M}}_6^b$ and $\overline{\mathcal{H}}_2$ is canonically isomorphic to the blowup of $\overline{\mathcal{M}}_6^b$ at p_{ss}. It follows that there is a canonical map associating to every stable curve of genus 2 a semistable pencil of quadrics in $\mathbb{P}_5$, namely a pencil in the uniquely determined closed orbit representing the moduli point. Conversely, since many pencils of quadrics are represented by the same point in the moduli space $\overline{\mathcal{M}}_5^p$, it seems not clear that one can associate to every semistable pencil a stable curve of genus 2. In the final section, we will see that this is the case: to every semistable pencil of quadrics in $\mathbb{P}_5$, one can associate a curve of arithmetic genus 2 in a canonical way (see Theorem 6.6). This curve will not be stable in many cases. However applying stable reduction we obtain a stable curve (see Corallary 6.8).

The motivation for writing this paper was the following: for every smooth hyperelliptic curve C there is a canonical way of describing the Jacobian and certain moduli spaces of vector bundles on C via the corresponding pencil of quadrics (see [15] and [7]). Moreover there is some progress extending this construction to stable hyperelliptic curves in some cases (see [3],[4] and [5]). In order to extend this construction further we thought it necessary to investigate first which curves belong to which pencil of quadrics.

There are some obvious problems left open by this paper:
1. The first problem is to extend to arbitrary genus the construction of Section 5 of $\overline{\mathcal{H}}_2$ out of $\overline{\mathcal{M}}_6^b$.
2. Extend the map associating to a semistable pencil of quadrics in $\mathbb{P}_5$ a curve of arithmetic genus 2 to a pencil of quadrics in $\mathbb{P}_{2g+1}$ for $g \geq 3$.
3. The curve C is defined as the set of lines in the base locus X of the pencil intersecting a general line of X. In the case of a general pencil the set of all lines in X forms the Jacobian of the curve C (see e.g. [9], Chapter 6). In the semistable cases a component of the set of all lines in X should be a compactification of the generalized Jacobian of C.
4. The variety X may be considered as a quadratic line complex. These complexes have been studied extensively for general pencils. Some of the results should be generalized to semistable pencils. For example one should study the singular surface of a semistable pencil.

We will come back to these questions in a subsequent paper.

2 Pencils of quadrics

Let $N := \frac{1}{2}n(n+3)$ and consider $\mathbb{P}_N$ as the space of quadrics in $\mathbb{P}_n$. A *pencil of quadrics* in $\mathbb{P}_n$ is by definition a line in $\mathbb{P}_N$. So the space of pencils of quadrics is by definition the Grassmannian $Gr(2, N+1)$. A pencil $\mathcal{P} \in Gr(2, N+1)$ is usually given as

$\{\lambda A+\mu B|(\lambda:\mu)\in \mathbb{P}_1\}$ where A and B are symmetric $(n+1)\times(n+1)$ matrices. By abuse of notation we write

$$\mathcal{P}=\lambda A+\mu B.$$

For any $(\lambda_0:\mu_0)\in\mathbb{P}_1$, we denote by $Q_{(\lambda_0,\mu_0)}$ or inhomogeneously (with a slight abuse of notation) Q_{λ_0} the quadric in $\mathbb{P}_n$ given by the matrix $\lambda_0 A+\mu_0 B$.

The *discriminant of the pencil* $\mathcal{P}$ is by definition the binary form in (λ,μ) :

$$\Delta=\Delta(\lambda,\mu):=\det(\lambda A+\mu B)$$

Note that the discriminant $\Delta(\mathcal{P})$ depends on the choice of the matrices A and B. The roots of Δ, however, are uniquely determined up to an isomorphism of $\mathbb{P}_1$. In particular the multiplicities of the roots are uniquely determined. Note, moreover, that Δ is not identically zero if and only if Δ is a binary form of degree $n+1$ if and only if the general quadric of the pencil is nondegenerate.

Suppose Δ is not identically zero and $(\lambda_0:\mu_0)$ is a root of Δ. It may also happen that all the subdeterminants of $\lambda_0 A+\mu_0 B$ of a certain order vanish. Suppose that all subdeterminants of order $n+1-d$ vanish for some $d\geq 0$, but not all subdeterminants of order $n-d$. This means that the quadric $Q_{(\lambda_0,\mu_0)}$ is a d-cone with vertex a linear space of dimension d and directrix a smooth quadric in a linear space of dimension $n-1-d$.

Let l_i denote the minimum multiplicity of the root $(\lambda_0:\mu_0)$ in the subdeterminants of order $n+1-i$, for $i=0,1,\dots,d$. Then $l_i>l_{i+1}$ for all i so that $e_i:=l_i-l_{i+1}>0$, and we have :

$$\Delta(\lambda,\mu)=(\lambda\mu_0-\lambda_0\mu)^{e_0}\dots(\lambda\mu_0-\lambda_0\mu)^{e_d}\Delta_1(\lambda,\mu),$$

with $\Delta_1(\lambda_0,\mu_0)\neq 0$.

The numbers e_i are called the *characteristics numbers* of the the root $(\lambda_0:\mu_0)$ and the factors $(\lambda\mu_0-\lambda_0\mu)^{e_i}$ are called the *elementary divisors* of the pencil $\mathcal{P}$. If $(\lambda_i:\mu_i)$ for $i=1,\dots,r$ are the roots of Δ and $e_0^j,\dots,e_{d_j}^j$ the characteristic numbers associated to the root $(\lambda_j:\mu_j)$ and $d_1\geq d_2\geq\ldots\geq d_r$, then $[(e_0^1,\dots,e_{d_1}^1),(e_0^2,\dots,e_{d_2}^2),\dots,(e_0^r,\dots,e_{d_r}^r)]$ is called the *Segre symbol* of the pencil $\mathcal{P}$. The parentheses are omitted if $d_i=1$.

It is a classical fact that 2 pencils of quadrics $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{P}_n$, whose discriminants have roots exactly at $(\lambda_i^1:\mu_i^1)$ and $(\lambda_i^2:\mu_i^2)$, are isomorphic, that is, projectively equivalent in $\mathbb{P}_n$, if and only if they have the same Segre symbol and there is an automorphism of $\mathbb{P}_1$ taking $(\lambda_i^1:\mu_i^1)$ to $(\lambda_i^2:\mu_i^2)$ for all i. This can be used to define a normal form for the pencils $\mathcal{P}$ whose discriminant is not identically zero (see [13], p. 280). Let $(\lambda-a_1)^{e_1},(\lambda-a_2)^{e_2},\dots,(\lambda-a_r)^{e_r}$ denote the elementary divisors of $\mathcal{P}$. Then the coordinates can be chosen in such a way that $\mathcal{P}=\lambda A+\mu B$ with $A=diag(P_1,\dots,P_r)$ and $B=diag(Q_1,\dots,Q_r)$ are diagonal block matrices with blocks:

$$P_i=\begin{pmatrix} 0 & 0 & \dots & 0 & a_i\\ 0 & \dots & 0 & a_i & 1\\ \dots & \dots & \dots & \dots & \dots\\ 0 & a_i & 1 & \dots & 0\\ a_i & 1 & 0 & \dots & 0\end{pmatrix} \quad\text{and}\quad Q_i=\begin{pmatrix} 0 & 0 & \dots & 0 & 1\\ 0 & 0 & \dots & 1 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 1 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0\end{pmatrix}$$

of size e_i. The parentheses in the Segre Symbol correspond 1-1 to the cones in the pencil. Suppose Q_λ is a cone in the pencil corresponding to $(e_0,\dots,e_d)$. We call Q_λ a cone *of type* $(e_0,\dots,e_d)$. The quadric Q_λ is then a d-cone and the corresponding root in the discriminant Δ is a root of multiplicity $e := \sum_{i=0}^{d} e_i$. By a slight abuse of notation we call Q_λ a *d-cone of multiplicity* e.

Let X denote the *base locus* of the pencil $\mathcal{P}$. By definition X is the intersection of any two different quadrics of the pencil. If Q_λ is a cone of type $(e_0, \dots, e_d)$, its vertex is a d-plane whose intersection with X depends heavily on the characteristic numbers $e_0, \dots, e_d$. We need only these intersections in the case of semistable pencils of quadrics in $\mathbb{P}_5$ (in Section 6). The cones in this case are of type $1, 2, 3, (1,1), (2,1)$ or $(1,1,1)$. A cone of type 1 in $\mathbb{P}_5$ is a 0-cone with vertex away from X. In this case, we call the cone an *ordinary* cone. Otherwise the cone is called *exceptional.* The cones of types 2, 3 are also 0-cones but the vertices are singular points of X. A cone of type $(1,1)$ (respectively $(2,1)$) is a 1-cone with vertex a line intersecting X in two different singular points (respectively one singular point) of X. A cone of type $(1,1,1)$ is a 2-cone in $\mathbb{P}_5$ with vertex a plane intersecting X in a smooth conic along which X is singular.

3 (Semi-) Stability of pencils of quadrics in $\mathbb{P}_n$

As in the last section consider $Gr(2, N+1)$, $N = \frac{1}{2}n(n+3)$ as the space of pencils of quadrics in $\mathbb{P}_n$. The group $SL_{n+1} := SL_{n+1}(\mathbb{C})$ acts on $Gr(2, N+1)$ by

$$T \circ \mathcal{P} = \lambda T A T^t + \mu T B T^t$$

for any $T \in SL_{n+1}$, where $\mathcal{P} = \lambda A + \mu B$. Certainly this action does not depend on the choice of A and B. Thinking of $Gr(2, N+1)$ as embedded into projective space via the Plücker mapping we have the notions of stability and semistability of Geometric Invariant Theory for pencils of quadrics in $\mathbb{P}_n$ (see [14]).

Theorem 3.1 (a) *A pencil* $\mathcal{P} \in Gr(2, N+1)$ *is semistable if and only if its discriminant admits no root of multiplicity* $> n/2$.
(b) *A pencil* $\mathcal{P} \in Gr(2, N+1)$ *is stable if and only if its discriminant admits no multiple root.*

Remark 3.2 Part of this result can also be deduced from Wall's results on actions of products (see [17]). Notice, however, that Wall has a different terminology: he considers stability of pairs of quadrics rather than pencils of quadrics.

For the proof we need some preliminaries. In order to turn the discriminant into a morphism, we have to pass to a principal $PGL_2(\mathbb{C})-$bundle over $Gr := Gr(2, N+1)$. Consider the affine space $V = \mathbb{C}^{N+1}$ of symmetric $(n+1) \times (n+1)$ matrices and let $\mathbb{P}(V \times V)$ denote the projective space of lines in $V \times V$ and define

$$S_n := \{(A, B) \in \mathbb{P}(V \times V) |\ A \text{ and } B \text{ are linearly independent}\}.$$

The group SL_2 (or $PGL_2 := PGL_2(\mathbb{C})$) acts on S_n by:

$$\alpha \circ (A, B) := (\alpha_{11} A + \alpha_{21} B, \alpha_{12} A + \alpha_{22} B)$$

for $\alpha = (\alpha_{ij}) \in SL_2$. It is easy to see that the projection

$$\pi : S_n \longrightarrow Gr$$

has the structure of a principal PGL_2-bundle. In particular we have $Gr \cong S_n/PGL_2$. The action of SL_{n+1} on Gr lifts to an action on S_n which commutes with the action of SL_2. Let $\mathbb{P}_{n+1}$ denote the space of binary forms of degree $n+1$, $f_{n+1}(\lambda, \mu) = \sum_{i=0}^{n+1} a_i \lambda^i \mu^{n+1-i}$. The group SL_2 acts on $\mathbb{P}_{n+1}$ via:

$$\alpha \circ f_{n+1}(\lambda, \mu) := \sum_{i=0}^{n} a_i (\alpha_{11}\lambda + \alpha_{12}\mu)^i (\alpha_{21}\lambda + \alpha_{22}\mu)^{n+1-i}$$

A pencil of quadrics is called *nondegenerate,* if it contains a nondegenerate quadric or equivalently if the general quadric of the pencil is nondegenerate. Let W_n denote the open subset of elements of S_n whose corresponding pencil is nondegenerate, that is,

$$W_n := \{(A, B) \in S_n | rk(\lambda A + \mu B) = n+1 \text{ for general } (\lambda : \mu)\}$$

Define $D(A, B) := \det(\lambda A + \mu B)$, the discriminant of $(A, B) \in S_n$. Then we have:

Lemma 3.3 *The map $D : W_n \longrightarrow \mathbb{P}_{n+1}$ is a proper, surjective morphism equivariant with respect to the action of $SL_{n+1} \times SL_2$ where SL_{n+1} acts trivially on $\mathbb{P}_{n+1}$.*

PROOF: The equivariance of the action is easy to check. That D is surjective follows immediately using the theory of normal forms of pencil of quadrics. In order to see that D is proper, note that the inverse image of a compact set (with respect to the analytic topology) is compact. In fact, the preimage of a point or a closed set in $\mathbb{P}_{n+1}$ is given by equations in $\mathbb{P}(V \times V)$ and hence is closed. On the other hand, it is automatically contained in W_n. □

PROOF OF THE THEOREM: Note that W_n is a principal PGL_2-bundle over an open set of Gr. It follows that the (semi-)stability of a point $(A, B) \in W_n$ with respect to the group $SL_{n+1}(\mathbb{C}) \times SL_2(\mathbb{C})$ is equivalent to the (semi-)stability of the pencil $\mathcal{P} = \pi(A, B)$ with respect to the group SL_{n+1}. (See also [17].) Hence we can prove the theorem on the level of W_n.

(a): It is clear that if (A, B) is unstable in W_n, so is $D(A, B) \in \mathbb{P}_{n+1}$, since D is equivariant and instability means that 0 is in the closure of the orbit. But the converse implication is also valid. In order to see this one uses a result of Adamovich and Golovina (see [1]) saying that a system of generators of the ring of invariants of pairs of symmetric matrices A, B under the action of $SL_{(n+1)}$ is given by the coefficients of the discriminant $\det(\lambda A + \mu B)$. For details see [2], Section 3. Hence $(A, B) \in S_n$ is unstable if and only if $D(A, B)$ is unstable and it is well known that this the case if and only if $D(A, B)$ admits a root of multiplicity $> n/2$ (see [14], p.78).

(b) STEP I *The pair $(A, B) \in W_n$ is stable if $D(A, B)$ admits no multiple roots:* According to [14], loc.cit., such a $D(A, B)$ is stable. Hence the preimage of the orbit $O(D(A, B))$ is closed in W_n. But in this case the pencil is uniquely determined by its discriminant. Hence the orbit $O(A, B)$ is closed in S. Since in this case the stabilizer of (A, B) is finite, this implies the assertion.

For the converse implication it suffices to show that pencils of type $[2, 1, \dots, 1]$ and $[(1, 1), 1, \dots, 1]$ are not stable, since the remaining pencils are in the closure of the spaces of pencils of these types.

Step II *A pencil of type* $[2, 1, \dots, 1]$ *is not stable:*

The normal form of such pencil is (See §1):

$$\begin{aligned} XAX^t &= \lambda_0 X_0^2 + \lambda_1 X_1^2 + \dots + \lambda_{n-2} X_{n-2}^2 + X_n^2 + 2\lambda_{n-1} X_{n-1} X_n \\ XBX^t &= X_0^2 + X_1^2 + \dots + X_{n-2}^2 + 2X_{n-1} X_n \end{aligned}$$

with $\lambda_i \neq 0, 1, \lambda_j$ for $i \neq j$. Applying the Hilbert-Mumford criterion for stability it suffices to give a 1-parameter group $\lambda : \mathbb{C}^* \longrightarrow SL_{n+1}$ all of whose weights with respect to the Plücker embedding $Gr \longrightarrow \mathbb{P}_M$ are nonnegative. The nonzero Plücker coordinates of $\mathcal{P} = \lambda A + \mu B$ are the nonzero 2×2 subdeterminants m_{ij}, $(0 \leq i < j \leq n,)$ of the matrix:

$$\begin{pmatrix} \lambda_0 X_0^2 & \lambda_1 X_1^2 & \dots & \lambda_{n-2} X_{n-2}^2 & X_{n-1}^2 & 2\lambda_{n-1} X_{n-1} X_n \\ X_0^2 & X_1^2 & \dots & X_{n-2}^2 & 0 & 2X_{n-1} X_n \end{pmatrix}.$$

Define $\lambda(t)$ acting on $\mathbb{C}^{n+1} = \mathbb{C}^{n+1}(X_0, \dots, X_n)$ by:

$$\begin{aligned} \lambda(t)(X_i) &= X_i \text{ for } 0 \leq i \leq n-2 \\ \lambda(t)(X_{n-1}) &= tX_{n-1} \\ \lambda(t) &= t^{-1} X_n \end{aligned}$$

Then the nonzero Plücker coordinates are

$$\begin{aligned} m_{ij} &= (\lambda_i - \lambda_j) X_i^2 X_j^2 && \text{with weight 0 for } 0 \leq i < j \leq n-2 \\ m_{i,n-1} &= -X_i^2 X_{n-1}^2 && \text{with weight 2 for } 0 \leq i \leq n-2 \\ m_{i,n} &= 2(\lambda_1 - \lambda_{n-1}) X_i^2 X_{n-1} X_n && \text{with weight 0 for } 0 \leq i \leq n-2 \text{ and} \\ m_{n-1,n} &= 2X_{n-1}^3 X_n && \text{with weight 2.} \end{aligned}$$

This implies the assertion.

Step III *Any pencil of type* $[(1, 1), 1, \dots, 1]$ *is not stable:*

A normal form of such a pencil is (see §1):

$$\begin{aligned} XAX^t &= \lambda_0 X_0^2 + \lambda_1 X_1^2 + \dots + \lambda_{n-2} X_{n-2}^2 \\ XBX^t &= X_0^2 + X_1^2 + \dots + X_{n-2}^2 + X_{n-1}^2 + X_n^2 \end{aligned}$$

We claim the stabilizers are not finite, for instance the matrices

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & cost & sint \\ 0 & 0 & \dots & -sint & cost \end{pmatrix}.$$

lie in SL_{n+1} and stabilize A and B. This completes the proof of the theorem. □

Remark 3.4 According to [14], loc.cit., a binary form od degree $n+1$ with n odd and no root of multiplicity $\geq n/2$ is stable. Hence the preimage $D^{-1}(O(f_{n+1}))$ of the orbit of such a form f_{n+1} of degree $n+1$ contains closed orbits. One can show that the closed orbits in Gr are the pencils of type $[(1, \dots, 1), (1, \dots, 1), 1, \dots, 1]$. These are exactly the pencils, admitting no root of multiplicity $\geq n/2$, which can be simultameously diagonalized.

4 The moduli spaces of pencils of quadrics and binary forms

Let $\mathcal{M}_n^p$ (respectively $\overline{\mathcal{M}}_n^p$) denote the moduli space of stable (respectively semistable) pencils of quadrics in $\mathbb{P}_n$. Similarly let $\mathcal{M}_n^b$ (respectively $\overline{\mathcal{M}}_n^b$) denote the moduli space of stable (respectively semistable) binary forms of degree n. In this section we show that there is a canonical isomorphism $\overline{\mathcal{M}}_n^p \cong \overline{\mathcal{M}}_{n+1}^b$ under which $\mathcal{M}_n^p$ maps to a proper open subset of $\mathcal{M}_{n+1}^b$.

We call a binary form of degree N *of type* $\{m_1, \ldots, m_r\}$ $(m_1 \geq m_2 \geq \ldots \geq m_r)$ if it has exactly r roots with multiplicities $m_1, m_2, \ldots, m_r$. According to [14], p.78, a binary form of odd degree is semistable if and only if it is stable. There exists, however, strictly semistable binary forms of even degree.

Proposition 4.1 *Suppose $n = 2m$ even. Then there is exactly one closed orbit of strictly semistable binary forms of degree n in $\mathbb{P}_n$, namely the orbit of type $\{m, m\}$. In particular $\overline{\mathcal{M}}_n^b$ is a one-point compactification of $\mathcal{M}_n^b$.*

PROOF: According to [14], loc.cit., a binary form f_n of degree $n = 2m$ is strictly semistable if and only if the highest multiplicity of a root of f_n is m. We may assume that the root of multiplicity m is ∞. Hence f_n is of the form

$$f_n(x, y) = y^m(a_0 x^m + a_1 x^{m-1} y + \ldots + a_m y^m), \; a_0 \neq 0.$$

Consider the 1-parameter group $\lambda : \mathbb{C}^* \longrightarrow SL_2$ defined by $\lambda(t)(x) = tx$ and $\lambda(t)(y) = t^{-1}y$. Then

$$\lim_{\lambda \to \infty} \lambda(t) f_n(x, y) = a_0 x^m y^m.$$

This means that the orbit of $a_0 x^m y^m$ is in the closure of the orbit of f_n. It is easy to see that this orbit is closed. $\square$

Theorem 4.2 (1) *The discriminant map $D : W_n \longrightarrow \mathbb{P}_{n+1}$ induces an isomorphism*

$$\overline{D} : \overline{\mathcal{M}}_n^p \longrightarrow \overline{\mathcal{M}}_{n+1}^b.$$

(2) *The map $\overline{D}$ maps the open subset $\mathcal{M}_n^p$ of $\overline{\mathcal{M}}_n^p$ into $\mathcal{M}_{n+1}^b$. The complement of $\overline{D}(\mathcal{M}_n^p)$ in $\mathcal{M}_{n+1}^b$ is the divisor $d = 0$ where d denotes the discriminant of binary forms of degree $n + 1$.*

For the proof we need the following lemma.

Lemma 4.3 *Let $(A, B) \in W_n$ be a pair of matrices such that $\mathcal{P} = \lambda A + \mu B$ is a pencil of quadrics of type $[(e_1, \ldots, e_{r_1}), (e_{r_1+1}, \ldots, e_{r_2}), \ldots, (e_{r_{s-1}+1}, \ldots, e_{r_s})]$. Suppose one of the numbers $e_i \geq 2$. Then the orbit $O((A, B))$ under the action of $SL_{n+1} \times SL_2$ is not closed.*

PROOF: It suffices to show that the orbit $O((A, B)))$ is not closed under the action of SL_{n+1}. Without loss of generality we may assume $e_1 \geq 2$. Let $(\lambda - a_1)^{e_1}(\lambda - a_2)^{e_2} \ldots (\lambda - a_{r_s})^{e_{r_s}}$ denote the elementary divisors of the pencil $\mathcal{P}$. According to §2, we may assume that

A and B are of the form: $A = diag(P(a_1), \ldots, P(a_{r_s}))$ and $B = diag(Q(a_1), \ldots, Q(a_{r_s}))$, where A and B are diagonal block matrices with blocks:

$$P(a_i) = \begin{pmatrix} 0 & 0 & \ldots & 0 & a_i \\ 0 & \ldots & 0 & a_i & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & a_i & 1 & \ldots & 0 \\ a_i & 1 & 0 & \ldots & 0 \end{pmatrix} \text{ and } Q(a_i) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

of size e_i.

We assume $e_1 = 2l_1$ is even, the proof of the odd case being similar. The associated quadratic forms are:

$$\begin{aligned} q_A &= 2a_1X_1X_{e_1} + \ldots + 2a_1X_{l_1}X_{l_1+1} + 2X_2X_{e_1} + \ldots + 2X_{l_1}X_{l_1+2} + X_{l_1+1}^2 + f_A \\ q_B &= 2X_1X_{e_1} + \ldots + 2X_{l_1}X_{l_1+1} + f_B \end{aligned}$$

with quadratic forms $f_A = f_A(X_{e_1+1}, \ldots, X_{n+1})$ and $f_B = f_B(X_{e_1+1}, \ldots, X_{n+1})$. Consider the 1-parameter group $\lambda : \mathbb{C}^* \longrightarrow SL_{n+1}$ defined by:

$$\begin{aligned} \lambda(t)(X_i) &:= t^{-1}X_i && \text{for } 1 \le i \le l_1 \\ \lambda(t)(X_i) &:= tX_i && \text{for } l_1 + 1 \le i \le e_1 \\ \lambda(t)(X_i) &:= X_i && \text{for } e_1 + 1 \le i \le n+1 \end{aligned}$$

Then

$$\begin{aligned} \lambda(t)q_A &= 2a_1X_1X_{e_1} + \ldots + 2a_1X_{l_1}X_{l_1+1} + \ldots + 2X_{l_1}X_{l_1+2} + t^2X_{l_1+1}^2 + f_A \\ \lambda(t)q_B &= q_B \end{aligned}$$

Hence if we put

$$A_0 := \lim_{t \to 0} q_A = 2a_1X_1X_{e_1} + \ldots + 2a_1X_{l_1}X_{l_1+1} + \ldots + 2X_{l_1}X_{l_1+2} + f_A$$

we get that $(A_0, B) = \lim_{t\to 0}(A, B)$ is a pencil with one more elementary divisor than (A, B) which is not in the orbit of (A, B). $\square$

Proof of Theorem 4.2 (1): According to Lemma 3.3 the map $D : W_n \longrightarrow \mathbb{P}_{n+1}$ is equivariant with respect to the actions. Moreover by Theorem 3.1 and [14], loc.cit., a point $(A, B) \in W_n$ is semistable if and only if $D(A, B)$ is semistable. Hence D induces a morphism $\overline{D}$ such that the following diagram is commutative:

$$\begin{array}{ccc} (W_n)_{ss} & \xrightarrow{D} & (\mathbb{P}_{n+1})_{ss} \\ \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_b} \\ \overline{\mathcal{M}}_n^p & \xrightarrow{\overline{D}} & \overline{\mathcal{M}}_{n+1}^b \end{array}$$

where π_p and π_b denote the natural maps. The ring of invariants I_n of binary forms of degree n is known to be normal (see [8], p.65). Hence applying Zariski's Main Theorem and since $\overline{\mathcal{M}}^b_{n+1} = Proj(I_{n+1})$ it suffices to show that the map $\overline{D} : \overline{\mathcal{M}}^p_n \longrightarrow \overline{\mathcal{M}}^b_{n+1}$ is injective the surjectivity being clear by Lemma 3.3.

Let f be a binary form of degree $n+1$ of type $\{m_1, \ldots, m_r\}$, $m_1 \geq \ldots \geq m_r$. According to the theory of normal forms of pencils of quadrics (see §1) for every Segre symbol $[(e_1, \ldots, e_{r_1}), (e_{r_1+1}, \ldots, e_{r_2}), \ldots, (e_{r_{s-1}+1}, \ldots, e_{r_s})]$, with $\sum_{i=1}^{r_1} e_i = m_1$, $\sum_{i=r_1+1}^{r_2} e_i = m_2, \ldots, \sum_{i=r_{s-1}+1}^{r_s} e_i = m_r$, there is exactly one orbit with this Segre symbol mapping to the orbit of f. Suppose first that f is a stable binary form. Then Lemma 4.3 applies and we get that in the preimage $D^{-1}(O(f))$ there is exactly one closed orbit, namely the corresponding orbit of type $[(1, 1, \ldots, 1), (1, \ldots, 1), \ldots, (1, \ldots, 1)]$, where the ith pair of parentheses consists of m_i entries 1. Since the points of $\overline{\mathcal{M}}^p_n$ parametrize the closed orbits of W_n this means that $\overline{D^{-1}}(\pi_b(f))$ consists of one point. Now if $n+1$ is odd , then there are no strictly semistable binary forms of degree $n+1$ thus the proof of (1) in this case is complete. If $n+1 = 2m$ is even, there is exactly one strictly semistable point p_{ss} in $\overline{\mathcal{M}}^b_{2m}$. Hence it remains to show that there is only one point in $\overline{\mathcal{M}}^p_{2m}$ mapping to p_{ss}, that is, that the only closed orbit mapping to an orbit over p_{ss} is the orbit of pencils of quadrics of type $[(1, \ldots, 1), (1, \ldots, 1)]$ where the parentheses $(1, \ldots, 1)$ consists of m entries each equal to 1. According to Lemma 4.3 every closed orbit of pencils mapping to an orbit lying over p_{ss} has Segre symbol $[(1, \ldots, 1), (1, \ldots, 1), \ldots, (1, \ldots, 1)]$, where the ith parenthesis consists of e_i entries 1 and $e_1 = m \geq e_2 \geq \ldots \geq e_r$ and $e_1 + e_2 + \ldots + e_r = m$. Assume there is a closed orbit of this type with $r \geq 3$ or equivalently $e_2 < m$. Then its image under $\overline{D}$ is the corresponding orbit of binary forms of type $\{m, e_1, \ldots, e_r\}$ which is not closed in $(\mathbb{P}_{n+1})_{ss}$ by Theorem 3.1. This contradicts the fact that the map $D : W_n \longrightarrow \mathbb{P}_{n+1}$ is proper(see Lemma 3.3). This completes the proof of (1).

Statement (2) follows from the fact (see Theorem 3.1 and [14]) that a pencil of quadrics is strictly semistable if and only if its discriminant binary form admits a multiple root. □

Remark 4.4 Wall writes in [17] p.61, l. 5-7 that the closed orbits in W_n are exactly the semistable orbits with Segre symbol of type $[(1, \ldots, 1), \ldots, (1, \ldots, 1)]$, with r parentheses where the i^{th} pair of parentheses consists of e_i entries 1. Of course we may assume that $e_1 \geq \ldots \geq e_r$. Then the above proof shows that the closed orbits are those of above type with either n odd or $n = 2m$ even and $e_1 < m$ or $e_1 = e_2 = m$. For pencils of quadrics in $\mathbb{P}_3$ one can see this also as follows: The stable orbits of pencils of quadrics in $\mathbb{P}_3$ are those with Segre symbol $[1, 1, 1, 1]$. The corresponding moduli space $\mathcal{M}^p_3$ is isomorphic to the affine line $\mathbb{C}$, being isomorphic to the moduli space of stable binary quartics. Hence if the semistable orbits $[(1, 1), 1, 1]$ and $[(1, 1), (1, 1)]$ both would be closed, the moduli space $\overline{\mathcal{M}}^p_4$ would be a complete connected curve, a completion of the affine line by 2 points and this is impossible.

5 The moduli space of binary sextics and curves of genus 2

In this section we describe the moduli space $\overline{\mathcal{M}}^b_6$ of semistable binary sextics explicitly and moreover show that the moduli space $\overline{\mathcal{H}}_2$, of stable curves of genus 2, is a blow up of a smooth point of $\overline{\mathcal{M}}^b_6$.

Let $I = \oplus_{i=0}^{\infty} I_n$ denote the graded ring of invariants of binary sextics. According to Geometric Invariant Theory

$$\overline{\mathcal{M}}_6^b = Proj(I)$$

There is a canonical isomorphism (see [10], Prop. 2.4.7):

$$Proj(I) \cong Proj(\oplus_{i=0}^{\infty} I_{2n}),$$

Hence we may think of $\overline{\mathcal{M}}_6^b$ as the projective variety associated to the ring of even degree invariants of binary sextics. In order to to define some invariants consider the binary sextic:

$$\begin{aligned} f(x,y) &= \prod_{i=1}^{6} (\mu_i x - \nu_i y) \\ &= \mu_1 \ldots \mu_6 \prod_{i=1}^{6} (x - \frac{\nu_i}{\mu_i} y) \\ &= \sum_{i=0}^{6} u_i x^{n-i} y^i, \end{aligned}$$

where μ_i, ν_i are independent variables over $\mathbb{C}$. Then $u_0 = \mu_1 \ldots \mu_6$ and $\frac{u_1}{u_0}, \ldots, \frac{u_6}{u_0}$ are up to sign the elementary symmetric functions of the roots ν_i/μ_i for $i = 1, \ldots, 6$. Denote, for abbreviation

$$(ij) := \mu_i \nu_j - \nu_i \mu_j$$

and consider the expressions:

$$\begin{aligned} A &:= \text{the monomial symmetric polynomial generated by } u_0^2 (12)^2 (34)^2 (56)^2 \\ B &:= \text{the monomial symmetric polynomial generated by } u_0^4 (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 \\ C &:= \text{the monomial symmetric polynomial generated by} \\ &\quad\ u_0^6 (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 (14)^2 (25)^2 (36)^2 \\ D &:= u_0^{10} \prod_{i<j} (ij)^2 \end{aligned}$$

According to the fundamental theorem on symmetric functions A, B, C, D can be expressed as homogeneous polynomials of degree $2, 4, 6$ and 10 respectively in the coefficients $u_0, \ldots, u_6$. They are invariant under the action of SL_2, as a straightforward computation shows. Since we are only interested in semistable sextics, and every semistable sextic has at least 2 different roots, we may assume that one of the roots is ∞ and another is 0 (inhomogeneously written). In other words, we have only to compute the invariants of binary sextics of the form (inhomogeneously written):

$$g(x) = x^5 - a_1 x^4 + a_2 x^3 - a_3 x^2 + a_4 x.$$

Doing so we obtain applying a linear transformation and using maple:

$$\begin{aligned} A(g) &= 40a_4 - 16a_1a_3 + 6a_2^2 \\ B(g) &= 4a_1a_3a_4 + 36a_2^2a_4 - 80a_4^2 + 4a_1^2a_3^2 - 12a_2a_3^2 - 12a_1^2a_2a_4 \\ C(g) &= 8a_1^2a_2^2a_3^2 - 36a_1^4a_4^2 - 24a_1^2a_2^3a_4 + 76a_1^3a_2a_3a_4 - 24a_1^3a_3^3 + 26a_2a_3^2a_4 + 26a_1^2a_2a_4^2 \\ &\quad +28a_1^2a_3^2a_4 + 72a_2^4a_4 - 238a_1a_2^2a_3a_4 + 76a_1a_2a_3^3 + 64a_1a_3a_4^2 + 176a_2^2a_4^2 \\ &\quad -320a_4^3 - 36a_3^4 - 24a_2^3a_3^2 \end{aligned}$$

Notice the invariants do not appear in homogeneous form, since we put the coefficient of x^5 equal to 1. The invariant D is just the discriminant of a sextic, of which we do not need the explicit form, but just the fact that $D(g) = 0$ if and only if the sextic g admits a double root.

It is well known (see, e.g. [16], p.92) that the invariants A, B, C, D generate the ring of invariants of even degree and moreover are independent. We conclude that $\overline{\mathcal{M}}_6^b$ is the 3-dimensional weighted projective space:

$$\overline{\mathcal{M}}_6^b = Proj(\mathbb{C}[A, B, C, D])$$

with weights $(2, 4, 6, 10)$, since A, B, C and D are homogeneous invariants of degree $2, 4, 6$ and 10. Since there is a canonical isomorphism between the 3-dimensional weighted projective spaces with weights $(2, 4, 6, 10)$ and $(1, 2, 3, 5)$ (see, [6], p.35), we conclude:

Theorem 5.1 *There is a canonical isomorphism*

$$\overline{\mathcal{M}}_6^b \cong Proj(\mathbb{C}[y_1, y_2, y_3, y_5]),$$

where the y_i are independent variables of weight i respectively.

This result allows us to find certain geometric subspaces of $\overline{\mathcal{M}}_6^b$.

Corollary 5.2 (a) *The subspace of $\overline{\mathcal{M}}_6^b$ of binary sextics admitting a double root is the divisor $y_5 = 0$, that is, the weighted projective space $Proj(\mathbb{C}[y_1, y_2, y_3])$.*
(b) *The moduli space of smooth projective curves of genus 2 is canonically isomorphic to $Spec(\mathbb{C}[\frac{y_1^5}{y_5}, \frac{y_2^5}{y_5^2}, \frac{y_3^5}{y_5^3}])$*

PROOF: (a) This is a consequence of the fact that the binary sextic f admits a double root if and only if $D(f) = 0$.
(b) Since the complement of the divisor $y_5 = 0$ is the basic open subset $Spec(\mathbb{C}[\frac{y_1^5}{y_5}, \frac{y_2^5}{y_5^2}, \frac{y_3^5}{y_5^3}])$, this follows from the fact that the moduli space of smooth curves of genus 2 and binary sextics without multiple root are canonically isomorphic. $\square$

Let p_{ss} denote the (uniquely determined) semistable point of $\overline{\mathcal{M}}_6^b$ (see 3.1).

Corollary 5.3 *The point $p_{ss} = (1 : 0 : 0 : 0)$ is a smooth point of $\overline{\mathcal{M}}_6^b$.*

In terms of the invariants A, B, C and D this implies that a binary sextic f admits a triple root if and only if $B(f) = C(f) = D(f) = 0$.

PROOF: In order to show that $p_{ss} = (1 : 0 : 0 : 0)$, it suffices to compute the invariants B and C for binary sextics with $(12) = (23) = (13) = 0$. But any summand in the definition of B and C contains either (12) or (23) or (13) as a factor. For a generic sextic f with a triple point, $A(f)$ is different from zero since it is semistable and hence yields a point in $\overline{\mathcal{M}}_6^b$. That $(1 : 0 : 0 : 0)$ is nonsingular follows from the fact that the variable y_1 has weight 1 (see [6],p.38). $\square$

Remark 5.4 According to the theory of weighted projective spaces (see [6], loc.cit.) the only singularities of $Proj(\mathbb{C}[y_1, y_2, y_3, y_5])$ are the double point $p_2 = (0 : 1 : 0 : 0)$, the triple point $p_3 = (0 : 0 : 1 : 0)$ and the fivefold point $p_5 = (0 : 0 : 0 : 1)$. The point p_5 represents the binary sextic $y(x^5 - y^5)$ with an automorphism of order 5, p_2 and p_3 represents the sextics $y^2(x^4 - y^4)$ with an automorphism of order 4 and $x^2y(x^3 - y^3)$ with an automorphism of order 3.

One can also determine the locus of binary sextics with two double roots.

Corollary 5.5 *The subspace of $\overline{\mathcal{M}}_6^b$ of binary sextics admitting two double roots is the curve in the weighted projective plane $Proj(\mathbb{C}[y_1, y_2, y_3]) \subset \overline{\mathcal{M}}_6^b$ given by the equation*

$$y_1^2y_2^2 - y_2^3 - 6y_1y_2y_3 + 9y_3^2$$

Proof: We claim that a binary sextic f admits 2 double roots if and only if $A(f) = -4ab + 6a^2 + 6b^2$, $B(f) = 4a^2b^2$ and $C(f) = 8a^4b^2 - 8a^3b^3 + 8a^2b^4$ with $a, b \in \mathbb{C}$. To see this it suffices to compute the invariants A, B and C of the binary sextic $X^2(X - a)(X - b)$ which gives the result since the values of A, B, C and D determine the orbit of the sextic. Setting $D = 0$ we get the weighted projective plane $Proj(\mathbb{C}[A, B, C])$. Using Macaulay to eliminate a and b from the equations $A = -4ab + 6a^2 + 6b^2, B = 4a^2b^2$ and $C = 8a^4b^2 - 8a^3b^3 + 8a^2b^2$ and applying the canonical isomorphism $Proj(\mathbb{C}[A, B, C]) \xrightarrow{\sim} Proj(\mathbb{C}[y_1, y_2, y_3])$ we obtain the above equation. $\square$

Finally we use these results to construct the moduli space $\overline{\mathcal{H}}_2$ of stable curves of genus 2 out of $\overline{\mathcal{M}}_6^b$.

Theorem 5.6 *The moduli space $\overline{\mathcal{H}}_2$ of stable curves of genus 2 is canonically isomorphic to the blow up of $Proj(\mathbb{C}[y_1, y_2, y_3, y_5])$ at the semistable point $p_{ss} = (1 : 0 : 0 : 0)$.*

Proof: The first step is to construct a morphism $\varphi : \overline{\mathcal{H}}_2 \longrightarrow \overline{\mathcal{M}}_6^b$. On the open set $\mathcal{H}_2$ of smooth curves of genus 2 the map φ is the canonical isomorphism of Corollary 5.2. We have to extend this map to the boundary components. According to [11] the boundary $\overline{\mathcal{H}}_2 \setminus \mathcal{H}_2$ consists of two irreducible divisors Δ_1 and Ξ_0 intersecting transversely in a curve.

The divisor Δ_1 is the space of double covers of 2 copies of $\mathbb{P}_1$ intersecting in a point p ramified over p and over 3 other points on each component $\mathbb{P}_1$. A typical member of Δ_1 is shown in Figure 1, where E_1 and E_2 are elliptic curves intersecting in q. The branch points over x_1, x_2, x_3 and p (resp. x_1', x_2', x_3' and p) are smooth points of E_1 (respectively E_2). A generic member of Ξ_0 is shown in Figure 2, where E_1 is an elliptic curve and $E_0 \cong \mathbb{P}_1$ intersecting in 2 points q_1 and q_2. At a generic point of Ξ_0 the map $\varphi|_{\Xi_0} : \Xi_0 \longrightarrow \overline{\mathcal{M}}_6^b$ is defined as follows: contract the component $\mathbb{P}_1$ with two branch points. The double covering of the contracted curve will be an elliptic curve with an ordinary double point given by an equation:

$$z^2 = (x - x_0)^2(x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

Define $\varphi(E_1 \cup E_0)$ to be the moduli point of the binary sextic $(x - x_0y)^2(x - x_1y)(x - x_2y)(x - x_3y)(x - x_4y)$.

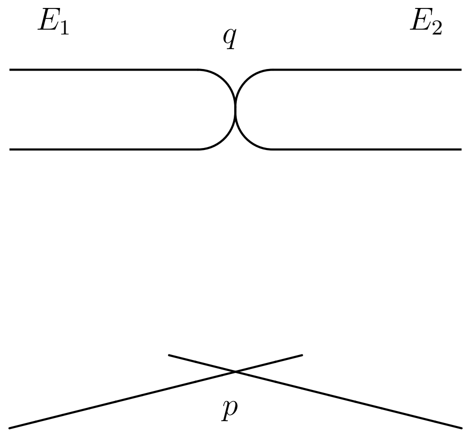

Figure 1: A typical member of Δ_1

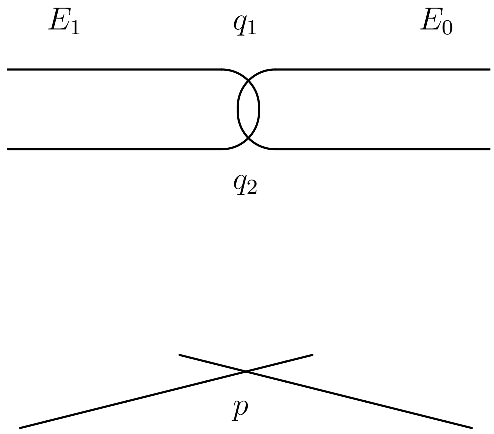

Figure 2: A typical member of Ξ_0

It is clear how to define the map φ on the degenerations in $\Xi_0 \setminus \Delta_1$ of the typical curve $E_0 \cup E_1$. Finally define the map φ on the component Δ_1 to be the constant map with image the semistable point $p_{ss} \in \overline{\mathcal{M}}_6^b$.

We have to show that the map $\varphi : \overline{\mathcal{H}}_2 \longrightarrow \overline{\mathcal{M}}_6^b$ is holomorphic. It certainly is holomorphic on $\overline{\mathcal{H}}_2 \setminus \Delta_1$. According to Riemann's Extension Theorem it suffices to show that φ is continuous. So let $X(t)$ be a curve on $\overline{\mathcal{H}}_2 \setminus \Delta_1$ with $\lim_{t \to \infty} X(t) \in \Delta_1$. Then $\varphi(X(t))$ moves to the boundary of $\overline{\mathcal{M}}_6^b$. But this consists of one point, namely p_{ss}, which implies the assertion.

It remains to show that $\varphi : \overline{\mathcal{H}}_2 \longrightarrow \overline{\mathcal{M}}_6^b$ is just the blowup of the point $p_{ss} \in \overline{\mathcal{M}}_6^b$. But φ is bijective on $\overline{\mathcal{M}}_6^b - \Delta_1$. For this we only note that if f is a sextic with only double points (and no triple point), consider the curve C given by the equation $y^2 = f(x)$. Let $\pi : C \longrightarrow \mathbb{P}_1$ denote the obvious double covering and blow up the images of the double points of C in $\mathbb{P}_1$. One obtains an admissible double covering (see [12], Chapter 3, §G) which is a member of Ξ_0. Certainly it is uniquely determined by f.

Since the semistable point p_{ss} is a smooth point of $\overline{\mathcal{M}}_6^b$, it suffices to show that $\Delta_1 \cong \mathbb{P}_2$. For this we may assume that the point q at which the two elliptic curves E_1 and E_2 meet is the zero point of both elliptic curves. Since the moduli space of semistable elliptic curves is $\mathbb{P}_1$, we obtain

$$\Delta_1 = \mathbb{P}_1 \times \mathbb{P}_1 / \{\pm 1\} \cong \mathbb{P}_2$$

where the group $\{\pm 1\}$ acts interchanging the two elliptic curves. This completes the proof of the theorem. $\square$

6 The curve of genus two associated to a semistable pencil

In the last section, we saw that the moduli space $\overline{\mathcal{H}}_2$ of stable curves of genus 2 is the blow up of the semistable point of the moduli space $\overline{\mathcal{M}}_5^p$ of semistable pencils of quadrics in $\mathbb{P}_5$. This implies that to every stable curve C in $\overline{\mathcal{H}}_2$ one can associate a semistable pencil in a canonical way (up to isomorphism): just associate to C the pencil in the corresponding unique closed orbit. It is by no means clear that there is also a converse map. In this section we show that conversely any semistable pencil of quadrics in $\mathbb{P}_5$ determines a stable curve of genus 2 in a canonical way.

In order to define the map, we need some preliminaries. In this section a pencil will always denote a pencil of quadrics in $\mathbb{P}_5$. So let $\mathcal{P} = \lambda A + \mu B$ denote an arbitrary pencil. Since A and B are linearly independent, the base locus of $\mathcal{P}$ is a 3-dimensional variety $X = X_{\mathcal{P}} \subset \mathbb{P}_5$. Note that a pencil contains 3 cones of multiplicity 2 if and only if it is of the type: $[2,2,2], [2,2,(1,1)], [2,(1,1),(1,1)]$ or $[(1,1),(1,1),(1,1)]$.

Proposition 6.1 *Let $\mathcal{P}$ be a semistable pencil in $\mathbb{P}_5$.*
(a) *If $\mathcal{P}$ does not contain three cones of multiplicity 2, its base locus X does not contain a plane.*
(b) *If $\mathcal{P}$ is of type $[2,2,2]$ (respectively $[2,2,(1,1)], [2,(1,1),(1,1)], [(1,1),(1,1),(1,1)]$), its base locus X contains exactly 1(respectively 2,4,8) planes.*

PROOF: (a) Note first that if $\mathcal{P}(t)$ is a family of pencils such that the base locus $X(t)$ contains a plane for each $t \neq 0$, then also the special fibre $X(0)$ contains a plane. Since moreover it is easy to see using the theory of normal forms that the property of the base locus containing a plane depends only on the Segre symbol of the pencil, it is enough to prove the proposition for some families of pencils, such that every semistable pencil is a generization of one of these families. Hence for (a) it is enough to show that the base locus of any pencil of type $[(1,1,1,1,1),1]$ or $[(1,1,1),(1,1,1)]$ does not contain a plane.

(i) Type $[(1,1,1,1,1),1]$: In this case X is the intersection of a double hyperplane $2P$ in $\mathbb{P}_5$ with a cone Q_0 with vertex a point v outside P. Hence X is a double structure of a smooth quadric in $\mathbb{P}_4$. But a smooth quadric in $\mathbb{P}_4$ does not contain any plane.

(ii) Type $[(1,1,1),(1,1,1)]$: Here X is the intersection of 2 cones Q_1 and Q_2 with vertices planes V_1 and V_2 and directrix a smooth conic. From the normal form one sees that $V_1 \cap V_2 = \emptyset$. Moreover $V_i \cap X$ is a smooth conic C_i in V_i. Hence Q_1 (respectively Q_2) is the cone with vertex V_1 (respectively V_2) and directrix C_2 (respectively C_1). It follows that X is the join of the 2 conics C_1 and C_2. But this does not contain a plane.

(b): Suppose x_1, x_2 and x_3 are singularities of X arising from three different cones Q_1, Q_2, Q_3 of the pencil $\mathcal{P}$. So x_i is contained in the vertex of Q_i for $i = 1,2,3$. This implies the line $\overline{x_1 x_2}$ is in $Q_1 \cap Q_2 = X$. Similarly for the other lines $\overline{x_i x_j}$. Since the vertices are independent in $\mathbb{P}_5$ it follows that $\overline{x_1 x_2 x_3}$ is a plane. It is contained in Q_1 and Q_2 and thus in X.

A 0-cone of multiplicity 2 yields 1 singular point of X namely its vertex. A 1-cone of multiplicity 2 yields two singular points of X, the intersection of the vertex with a second quadric of the pencil. Hence X contains as many planes as claimed. It suffices to show that these are the only ones. We give the proof for case $[(1,1),(1,1),(1,1)]$, the others being similar. So let P be a plane contained in X. It must be contained in the three cones Q_1, Q_2 and Q_3. Let the vertices of the Q_i be V_i and x_i, $x_i' \in X$ be the intersection of V_i with a second quadric of the pencil. Since these cones are cones over a non-singular quadric of dimension 2, it follows that P must intersect the vertex of the three cones but since it is in X it must go through either x_i or x_i' for $i = 1,2$ or 3 and therefore P is one of the above mentioned planes. $\square$

Corollary 6.2 *Let $\mathcal{P} = (Q_\lambda)_{\lambda \in \mathbf{P_1}}$ be a semistable pencil in $\mathbb{P}_5$. For every plane $P \subset \mathbb{P}_5$ not passing through three singularities of X arising from three different cones of $\mathcal{P}$ there is at most one $\lambda \in \mathbb{P}_1$ with $P \subset Q_\lambda$.*

In order to associate a curve to a pencil $\mathcal{P}$, we fix a "general" line on X. The following proposition points out what this means and shows that such a line exists.

Proposition 6.3 *Let $\mathcal{P} = (Q_\lambda)_{\lambda \in \mathbf{P_1}}$ be a semistable pencil in $\mathbb{P}_5$ with base locus X.*
(a) *Suppose X contains only finitely many singularities.Then there exists a line L on X avoiding the vertices of all cones of $\mathcal{P}$ and not contained in any plane of X.*
(b) *Suppose X contains a curve S of singularities. Every line on X intersects S. If $\mathcal{P}$ is not of type $[(1,1,1),(1,1,1)]$ then there is a line on X avoiding the vertices of the 0-cones and 1-cones of X.*

Proof: (a) Assume first that $\mathcal{P}$ contains two 0-cones, Q_1 and Q_2 say. Suppose further that $\mathcal{P}$ contains an ordinary cone or is of type $[3,3]$. In this case the base locus X does not contain a plane. Choose a hyperplane $H \subset \mathbb{P}_5$ avoiding the singularities of X as well as the vertices of the 0-cones. The intersection $\widetilde{Q_i} = Q_i \cap H$ is a quadric in $H \cong \mathbb{P}_4$. The intersection $\widetilde{Q_1} \cap \widetilde{Q_2}$ is a Del Pezzo surface. The number of lines it contains depends on the Segre symbol of the pencil of quadrics in $\mathbb{P}_4$ having $\widetilde{Q_1} \cap \widetilde{Q_2}$ as base locus. If the Segre symbol is $[1,1,1,1,1]$, $\widetilde{Q_1} \cap \widetilde{Q_2}$ contains 16 lines. In the worst possible case, with the above hypotheses, the pencil has Segre symbol $[(2,1),1,1]$ and the base locus contains 4 lines. (see [2]). Let L be any of these lines. By construction L does not pass through the vertices of the 0-cones nor does it intersect the vertices of the 1-cones, since the intersection would be a singularity of X. Still assuming that $\mathcal{P}$ contains two 0-cones, Q_1 and Q_2 it remains to consider pencils of type $[2,2,2]$ and $[2,2,(1,1)]$. Then X contains at most 2 planes. Choose a hyperplane $H \subset \mathbb{P}_5$ avoiding the singularities of X as well as the vertices of the 0-cones. Then $H \cong \mathbb{P}_4$ determines a pencil of quadrics in $\mathbb{P}_4$ of type $[2,2,1]$ or $[2,(1,1),1]$. In this case the intersection $\widetilde{Q_1} \cap \widetilde{Q_2}$ is a Del Pezzo surface containing 9 lines in the first case and 6 lines in the second case (see [2]). The intersection of the 2 planes of X with (a general) H consists of at most 2 lines. Let L be any of the above lines not in a plane of X.

We are left with the pencils containing at least two 1-cones, Q_1 and Q_2 say, and at most one 0-cone. Note first of all that through every point of X there passes a line of X, since this is the case for every pencil of type $[1,1,1,1,1,1]$ and any $\mathcal{P}$ is a specialization of pencils of such type. Suppose every line on X passes through a singular point of X. Then X is the union of 0-cones (not necessarily quadric cones) with vertex a singular point and directrix a surface in $\mathbb{P}_4$. It cannot be just one 0-cone since in this case the line joining its vertex to one other singularity would be a line of singularities, a contradiction. Therefore X is the union of at least two 0-cones. The cones cannot be two quadric cones since X is a complete intersection of two quadrics in $\mathbb{P}_5$. But then the degree of X would be bigger than 4, a contradiction. So there is a line L not passing through any singular point of X and not contained in any of the finitely many planes of X.

(b) The pencil $\mathcal{P}$ contains at least one 2-cone, say Q_1 with vertex a plane V_1. The base locus X is singular along the smooth conic $S_1 = X \cap V_1$. We claim first that any line on X intersects S_1. For suppose a line L on X does not intersect the plane V_1. Then there is a plane P containing L not intersecting V_1. The intersection $Q_1 \cap P$ is a directrix of the cone Q_1 and hence a smooth conic in P. Since $L \subset Q_1$, this conic would have to contain L, a contradiction. This shows L intersects V_1 but since it is contained in X it also intersects S_1.

If $\mathcal{P}$ is not of type $[(1,1,1),(1,1,1)]$, it contains a 0-cone, Q_2 say, with vertex v_2. We have to show that X contains a line avoiding v_2 and the vertices of the remaining cones. Choose a plane P passing through v_2 and not intersecting V_1. As a directrix of Q_1, $\widetilde{Q_1} = Q_1 \cap P$ is a smooth conic. The intersection $X \cap P = X \cap \widetilde{Q_1}$ consists of 4 points counted with multiplicities. Apart from v_2 there is at least one other point, p_3 say, in $X \cap \widetilde{Q_1}$. Let q_1 be a point in S_1. The line $\overline{p_3 q_1}$ is contained in X and does not contain v_2 nor the other vertices of the cones of $\mathcal{P}$ since the vertices of the cones of $\mathcal{P}$ are independent in $\mathbb{P}_5$. $\square$

Now given $\mathcal{P}$ with base locus X we fix a *general* line $L \subset X$. By this we mean one of the lines of Proposition 6.3. Define:

$$B_L := \{ \text{ planes } P \subset \mathbb{P}_5 |\ L \subset P \subset Q_\lambda \text{ for some } \lambda \in \mathbb{P}_1 \}$$

Obviously B_L is an analytic subvariety of the Grassmannian $Gr(3,6)$. We will see that in case $\mathcal{P}$ is semistable, B_L is a curve which can be determined explicitly.

Remark 6.4 If P is a plane in $\mathbb{P}_5$ with $L \subset P \subset Q_\lambda$ then $X \cap P$ is a conic in P containing L. Hence $X \cap P = L \cup L'$, for a unique line $L' \subset P$. So using Levi's Extension Theorem one sees that one could equivalently define B_L to be the closure in the Grassmannian $Gr(4,6)$ of the set of lines L' in X, $L' \neq L$ meeting L. This is the point of view of [9], Chapter 6. For our purposes the above definition seems more appropriate.

Corollary 6.2 gives an obvious holomorphic map:

$$\pi : B_L \longrightarrow \mathbb{P}_1$$

associating to $P \in B_L$ the unique $\lambda \in \mathbb{P}_1$, with $P \subset Q_\lambda$. We will use π to analize B_L. Recall that a cone Q_λ is *ordinary* if its vertex is a point outside X, that is if it corresponds to a 1 in the Segre symbol of the pencil. Otherwise the cone is called *exceptional.*

Lemma 6.5 *Let $Q_{\lambda_1}, \dots, Q_{\lambda_r}$ be the exceptional and $Q_{\lambda_{r+1}}, \dots, Q_{\lambda_{r+s}}$ be the ordinary cones of the pencil $(Q_\lambda)_{\lambda \in \mathbb{P}_1}$. The map*

$$\pi : B_L \setminus \pi^{-1}(\{\lambda_1, \dots, \lambda_r\}) \longrightarrow \mathbb{P}_1 \setminus \{\lambda_1, \dots, \lambda_r\}$$

is a smooth double covering ramified exactly at the points $\lambda_{r+1}, \dots, \lambda_{r+s}$.

Locally the set up is exactly the same as in the case of a stable pencil, that is, a pencil of type $[1,1,1,1,1,1]$. So for the proof of the lemma we can refer to [9], Chapter 6 or [15].

Hence $B_L \setminus \pi^{-1}(\{\lambda_1, \dots, \lambda_r\})$ is connected or consists of 2 components. We call their closures the *ordinary components* and the other components the *exceptional components* of B_L. We denote by C, C_1 and C_2 the ordinary components. The exceptional components which are always isomorphic to $\mathbb{P}_1$ are denoted by P_i. In the following theorem the genus always means the geometric genus of a curve. A *node* (respectively a *cusp*) means an ordinary double point (respectively a cusp with multiplicity 2).

Theorem 6.6 *Let $\mathcal{P} = (Q_\lambda)_{\lambda \in \mathbf{P_1}}$ be a semistable pencil in $\mathbb{P}_5$ with base locus X and L a general line of X. Then B_L is a connected curve of arithmetic genus 2. To be more precise here is the list of the possible curves B_L according to the Segre symbol of $\mathcal{P}$:*

(1) $[1,1,1,1,1,1]$: *The curve B_L is smooth.*

(2) $[2,1,1,1,1]$: *The curve B_L is irreducible of genus 1 with one node.*

(3) $[2,2,1,1]$: *The curve B_L is an irreducible rational curve with two nodes.*

(4) $[2,2,2]$: *The curve B_L consists of 2 ordinary components isomorphic to $\mathbb{P}_1$ and intersecting each other transversely in 3 points.*

(5) $[(1,1),1,1,1,1]$: *The curve B_L consists of 1 ordinary component C, a smooth elliptic curve, and 1 exceptional component intersecting C in a pair of points conjugate under π.*

(6) $[(1,1),(1,1),1,1]$: *The curve B_L consists of 1 ordinary component C isomorphic to $\mathbb{P}_1$ and two exceptional components intersecting C in two pairs of points conjugate under π.*

(7) $[(1,1),(1,1),(1,1)]$: *The curve B_L consists of two ordinary components C_1 and C_2, isomorphic to $\mathbb{P}_1$, and 3 exceptional components intersecting C_1 and C_2 in 3 pairs of points each pair conjugate under π.*

(8) $[2,(1,1),1,1]$: *The curve B_L consists of 1 ordinary component C, a rational curve with 1 node, and 1 exceptional component intersecting C in a pair of points conjugate under π.*

(9) $[2,2,(1,1)]$: *The curve B_L consists of two ordinary components C_1 and C_2, isomorphic to $\mathbb{P}_1$, intersecting each other transversely in 2 points and 1 exceptional component intersecting C_1 and C_2 in two points conjugate under π.*

(10) $[2,(1,1),(1,1)]$: *The curve B_L consists of 2 ordinary components C_1 and C_2 isomorphic to $\mathbb{P}_1$ intersecting each other in 1 point and two exceptional components each intersecting C_1 and C_2 in a pair of points conjugate under π.*

(11) $[3,1,1,1]$: *The curve B_L is an irreducible curve of genus 1 with a cusp.*

(12) $[3,2,1]$: *The curve B_L is an irreducible rational curve with a cusp and a node.*

(13) $[3,(1,1),1]$: *The curve B_L consists of 1 ordinary component C, a rational curve with a cusp, and 1 exceptional component intersecting C in a pair of points conjugate*

under π.

(14) $[3,3]$: *The curve* B_L *is an irreducible rational curve with 2 cusps.*

(15) *and* (16) $[(1,1,1),1,1,1]$ *and* $[(2,1),1,1,1]$: *The curve* B_L *consists of 1 ordinary component* C *a smooth elliptic curve and 1 exceptional component touching* C *in a branch point of* π *which is not a flex of the elliptic curve.*

(17) *and* (18) $[(1,1,1),2,1]$ *and* $[(2,1),2,1]$: *The curve* B_L *consists of 1 ordinary component* C, *a rational curve with a node, and 1 exceptional component touching* C *in a branch point of* π.

(19) *and* (20) $[(1,1,1),(1,1),1]$ *and* $[(2,1),(1,1),1]$: *The curve* B_L *consists of 1 ordinary component* C *isomorphic to* $\mathbb{P}_1$ *and 2 exceptional components,* P_1 *intersecting* C *transversely in two points conjugate under* π, *and* P_2 *touching* C *in a branch point of* π.

(21) *and* (22) $[(1,1,1),3]$ *and* $[(2,1),3]$: *The curve* B_L *consists of 1 ordinary component* C, *a rational curve with a cusp and 1 exceptional curve touching* C *in the smooth branch point of* π.

(23), (24) *and* (25) $[(1,1,1),(1,1,1)]$, $[(2,1),(1,1,1)]$ *and* $[(2,1),(2,1)])$: *The curve* B_L *consists of 1 ordinary component* C, *isomorphic to* $\mathbb{P}_1$ *and 2 exceptional components touching* C *in the 2 branch points of* π.

Proof: We give only the proof of some typical cases, the proofs of the remaining cases being similar. Fix in any case a line L as in Proposition 6.3.

Case (2): The pencil has one exceptional cone Q_{λ_1} which is a cone with vertex a (singular) point $v_1 \in X$ and directrix a smooth three dimensional quadric. The line $L \subset X$ does not pass through v_1. Choose a hyperplane $H \subset \mathbb{P}_5$ containing L but not v_1 We may consider the quadric $\widetilde{Q_{\lambda_1}} = Q_{\lambda_1} \cap H$ as the directrix of the cone Q_{λ_1}. Since $\widetilde{Q_{\lambda_1}}$ is a nonsingular quadric in H, there is only one plane of Q_{λ_1} containing L, namely the plane $\overline{Lv_1}$. Hence $\pi^{-1}(\lambda_1)$ consists of one point and B_L is an irreducible curve. Since the double covering $\pi : B_L \setminus \{\pi^{-1}(\lambda_1)\} \longrightarrow \mathbb{P}_1 \setminus \{\lambda_1\}$ is ramified in exactly four points corresponding to the 4 ordinary cones of $\mathcal{P}$, the normalization of B_L is an elliptic curve and $\pi^{-1}(\lambda_1)$ is a node since otherwise the normalization of B_L would be a double cover of $\mathbb{P}_1$ ramified in 5 points, which contradicts Hurwitz formula.

Case (4): Only the argument that there are 2 ordinary components of B_L is worth mentioning: the pencil $\mathcal{P}$ has no ordinary cone and 3 exceptional cones, Q_{λ_1}, Q_{λ_2}, Q_{λ_3}. As in case (2), $\pi^{-1}(\lambda_i)$ consists of 1 point for $i = 1,2,3$. Moreover the singularity type of the point $\pi^{-1}(\lambda_i)$ in B_L is the same for $i = 1,2,3$ since the pencil admits an automorphism permuting the cones Q_{λ_i}. Since the normalization of B_L cannot be ramified in 3 points, it is unramified over $\mathbb{P}_1$, and hence consists of two components, isomorphic to $\mathbb{P}_1$.

Case (5): The pencil has one exceptional cone Q_{λ_1} which is a cone with vertex a line V_1 intersecting X exactly in 2 (singular) points p_1 and p_2 and directrix a smooth 2-dimensional quadric. The line $L \subset X$ does not intersect the line V_1. Choose a 3-plane

$T \subset \mathbb{P}_5$ containing L but not intersecting V_1. We may consider the quadric $\widetilde{Q_{\lambda_1}} = Q_{\lambda_1} \cap T$ as the directrix of the cone Q_{λ_1}. Since $\widetilde{Q_{\lambda_1}}$ is a non singular quadric in T, the planes of Q_{λ_1} containing L are exactly the planes $\overline{Lp}$ where p is point on the line V_1. Hence we have that $\pi^{-1}(\lambda_1) \cong V_1 \cong \mathbb{P}_1$. The normalization of the ordinary component C of B_L is an elliptic curve (see Remark 6.10 below), being a double covering of $\mathbb{P}_1$, ramified in 4 points. It is also shown in Remark 6.10 that the intersection of C and the above $\mathbb{P}_1$ consists of two points namely the points corresponding to the planes $\overline{Lp_1}$ and $\overline{Lp_2}$. This implies the assertion.

Case (11): This case is similar to case (2). The exceptional cone Q_{λ_1} is also a 0-cone with vertex a (singular) point $v_1 \in X$. Again the line L does not pass through v_1 and $\pi^{-1}(\lambda_1)$ consists only of one point. But this time the normalization is ramified over λ_1, since otherwise it would be a double covering of $\mathbb{P}_1$ with 3 ramification points. If $\pi^{-1}(\lambda_1)$ were a smooth ramification point of $\pi : B_L \longrightarrow \mathbb{P}_1$, the curve B_L would be an elliptic curve. But as a degeneration of a smooth curve of genus 2 it is of arithmetic genus 2. Hence $\pi^{-1}(\lambda_1)$ is a cusp of the curve B_L of genus 1.

Case (15): Suppose that $\mathcal{P}$ is of type $[(1,1,1),1,1,1]$ the proof of the other case being similar. The pencil has 1 exceptional cone, a 2-cone Q_{λ_1} with vertex a plane V_1 intersecting X in a smooth conic S_1. The line $L \subset X$ intersects S_1 transversely in one point and does not pass through the vertices of the other three 0-cones. The ordinary component C is ramified over λ_1. Hence its normalization is a double covering of $\mathbb{P}_1$ ramified in 4 points. This is an elliptic curve. The planes in Q_{λ_1} are parametrized by the curve S_1 which implies that $\pi^{-1}(\lambda_1) \cong \mathbb{P}_1$. It intersects C in one point p. The point p cannot be a cusp of C, since then we would have $p_a(B_L) \geq 3$. Hence C is a smooth elliptic curve. Nor can C intersect $\pi^{-1}(\lambda_1)$ transversely since then $p_a(B_L) = 1$. Hence $\pi^{-1}(\lambda_1)$ is tangent to C at p. The point p cannot be a flex since then $p_a(B_L) = 3$. □

Proposition 6.7 *Let $\mathcal{P}$ be a semistable pencil in $\mathbb{P}_5$. The curve B_L is uniquely determined (up to isomorphism) by $\mathcal{P}$, that is, it does not depend on the choice of the line L.*

PROOF: Let $Q_{\lambda_1}, \dots, Q_{\lambda_r}$ denote the exceptional cones of $\mathcal{P}$. For every $\lambda \in \mathbb{P}_1$, $\lambda \neq \lambda_1, \dots, \lambda_r$ let C'_λ denote the set of rulings of the quadric Q_λ. It follows that C'_λ consists of two elements if Q_λ is smooth and consists of 1 element if Q_λ is an ordinary cone. As in the case of general pencils (see [15]) one sees that the C'_λ glue together to form a smooth double covering $\pi' : C' \longrightarrow \mathbb{P}_1 \setminus \{\lambda_1, \dots, \lambda_r\}$. Let C_L denote the open part of the ordinary components of B_L over $\mathbb{P}_1 \setminus \{\lambda_1, \dots, \lambda_r\}$. Clearly the map $\varphi' : C_L \longrightarrow C'$ associating to each plane $P \subset Q_\lambda$ containing L the ruling defined by the line L' such that $P \cap X = L \cup L'$ is an isomorphism. Now the structure of the fibres $\pi^{-1}(\lambda_i)$ is independent of the choice of the line L, as long as it is chosen to be general. Also the way of gluing these fibers to C_L does not depend on L. Hence we can glue them in the same way to the completion of the curve C' to obtain a complete curve B'. Certainly the isomorphism $\varphi : C_L \longrightarrow C'$ can be extended to an isomorphism $\varphi : B_L \longrightarrow B'$. This implies the assertion. □

The following corollary is an immediate consequence of Theorem 6.6 proven by just working out the stable reduction of the curve B_L.

Corollary 6.8 *Let $\mathcal{P}$ be a pencil in $\mathbb{P}_5$. Let B_s denote the stable reduction of the curve B_L associated to $\mathcal{P}$ according to Theorem 6.6. The dual graph of B_s depends only on the multiplicities of the discriminant $D(\mathcal{P})$ of $\mathcal{P}$. More precisely, suppose $D(\mathcal{P})$ is of type:*

(a) $\{1,1,1,1,1,1\}$, *then B_s is smooth of genus 2.*

(b) $\{2,1,1,1,1\}$, *then B_s is a curve of genus 1 with one node.*

(c) $\{2,2,1,1\}$, *then B_s is a curve of genus* 0 *with two nodes.*

(d) $\{2,2,2\}$, *then B_s consists of two curves isomorphic to $\mathbb{P}_1$ intersecting transversally in 3 points.*

(e) $\{3,1,1,1\}$ *or* $\{3,3\}$, *then B_s consists of 2 elliptic curves intersecting transversally.*

(f) $\{3,2,1\}$, *then B_s is a rational nodal curve with an elliptic tail.*

Remark 6.9 Concerning the moduli of the stable reduction B_s one can be more precise: For example, if $D(\mathcal{P})$ is of type $\{3,1,1,1\}$ it would be better to consider B_s as an elliptic curve E with an elliptic tail E_0. E is an elliptic curve with arbitrary moduli whereas E_0 has j-invariant 0. This comes from the fact that in the stable reduction process E_0 appears as a triple covering of $\mathbb{P}_1$ ramified in 3 points. Similarly if $D(\mathcal{P})$ is of type $\{3,3\}$, B_s should be considered a $\mathbb{P}_1$ with 2 elliptic tails, contracted to a node. Notice that whereas pencils of types (11) to (25) in Theorem 6.6 belong to the same point in the moduli space $\overline{\mathcal{M}}_6^p$ of semistable pencils in $\mathbb{P}_5$ the corresponding stable curves B_s form a curve in the moduli space $\overline{\mathcal{H}}_2$ of stable curves of genus 2.

Remark 6.10 In order to compute the intersection of the ordinary and exceptional components of B_L in the proof of Theorem 6.6 we used either Hurwitz formula or the fact that B_L is a degeneration of a curve of genus 2 and hence of arithmetic genus 2. One can avoid this and determine the intersection explicitly using normal forms. We will work this out in case $\mathcal{P}$ is of type $[(1,1),1,1,1,1]$, that is, case (5) of the above mentioned theorem. The other cases are similar. So, according to §1, let $\mathcal{P} = \lambda A + \mu B$ with

$$\begin{aligned} XAX^t &= a_1(X_1^2+X_2^2) + a_3X_3^2 + a_4X_4^2 + a_5X_5^2 + a_6X_6^2 \\ XBX^t &= X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 \end{aligned}$$

The pencil $\mathcal{P}$ admits one 1-cone and four 0-cones. The 1-cone is Q_{λ_1} with $\lambda_1 = (1 : -a_1) \in \mathbb{P}_1$ with equation $(a_3-a_1)X_3^2 + \ldots + (a_6-a_1)X_6^2$ and vertex the line V_1 given by $X_3 = X_4 = X_5 = X_6 = 0$. The intersection of the line V_1 with any other quadric of $\mathcal{P}$ gives the 2 singular points of $X : p_{1,2} = (1 : \pm 1 : 0 : 0 : 0 : 0)$. The line L was a line not intersecting any of the vertices of the cones in $\mathcal{P}$ and the exceptional curve $P = \pi^{-1}(\lambda_1)$ was given by the curves of planes $\overline{Lp}$ with $p \in V_1$. We claim that the ordinary component C of B_L intersects P transversely exactly at the points p_1 and p_2. For this it suffices to show that the plane $\overline{Lp_1}$ is the limit of a family of planes P_t containing L and contained in smooth quadrics Q_{λ_t}. The tangent hyperplane to all quadrics of $\mathcal{P}$ at p_1 is T_{p_1} given by $X_0 + iX_1 = 0$. Since $X \not\subset T_{p_1}$, we may choose L not to be contained in T_{p_1}. Hence there is a 3-plane $P \subset T_{p_1}$ not intersecting the lines V_1 and L. Intersecting the quadrics A and B with T_{p_1} we get 2 cones Q_A and Q_B with vertex p_1 and directrices smooth quadrics $\widetilde{Q_A} = Q_A \cap P$ and $\widetilde{Q_B} = Q_B \cap P$. The intersection $\widetilde{Q_A} \cap \widetilde{Q_B}$ is an elliptic curve E in T_{p_1} parametrizing all lines in X passing through p_1. Let L_1 denote the second line of X in the plane $\overline{Lp_1}$. The line L_1 intersects E in a point q. For any point $x \in E, x \neq q$ consider the plane $\overline{Lx}$. The plane $\overline{Lx}$ is not contained in the cone Q_{λ_1} and hence is a plane of the ordinary component of B_L. Taking the limit $x \to q$ shows that that the plane $\overline{Lq} = \overline{Lp_1}$ is a limit of planes $\overline{Lx}$ in the ordinary component. Certainly the intersection has to be transversal which completes the proof of the claim.

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