Stability of Pencils of Quadrics in $\mathbf{P}^4$

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Abstract

In this article the stability (in the sense of Geometric Invariant Theory) is analyzed for pencils of quadrics in $\mathbf{P}^4$. The base locus of such a pencil is in general a Del Pezzo surface of degree 4, and we relate the description of the unstable and strictly semi-stable pencils to the alternate descriptions of the Del Pezzo surfaces as blowups of the plane at a subscheme of length five.

Keywords: Algebraic Geometry, Pencils, Quadrics, Del Pezzo surfaces, Stability, Segre Symbols

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1 Introduction

In this article we present the Geometric Invariant Theory related to pencils of quadrics in $\mathbf{P}^4$, Del Pezzo surfaces of degree 4, and finite subschemes of the plane of length 5. Although the stability issues for linear systems of curves and quadrics has been studied before (see for example [12], [6], [13] and [14]) to our knowledge this has not been directly related to the invariant theory for Del Pezzo surfaces and the finite subschemes of the plane.

These three problems are, as is well-known, three aspects of the same classical subject. Given a general pencil of quadrics in $\mathbf{P}^4$, its base locus is a Del Pezzo surface of degree four; and conversely, the quadrics through such a Del Pezzo surface form a pencil. Given a finite subscheme S of the plane of length 5, the cubics through S embed the plane birationally as a Del Pezzo surface of degree 4 in $\mathbf{P}^4$; and conversely, every Del Pezzo surface arises this way. Thus to a planar subscheme of length 5 there is a naturally associated pencil of quadrics in $\mathbf{P}^4$, the quadrics through the embedded Del Pezzo surface.

We begin this paper by reviewing the classical material about pencils of quadrics, as treated by Corrado Segre (see [10]) and exposed for example in [5]. Pencils of quadrics are classified by their *Segre Symbol*, which we review. Using results of Wall, we give an explicit description of the ring of invariants of pencils of quadrics in $\mathbf{P}^4$, leading to an explicit construction of its two-dimensional moduli space; it is a double cover of a weighted projective plane, with weights $(4, 8, 12)$. We also determine the orbit structure of the moduli space; since it is only a coarse moduli space, the strictly semistable orbits are generally not represented by distinct points in moduli, and we determine the orbits corresponding to strictly semistable points.

We then relate this to the subschemes of length 5, and classify all such subschemes leading to stable and semi-stable pencils of quadrics. Since our analysis of the stability of the pencils of quadrics relies on the knowledge of the Segre Symbol for the pencil, it is convenient to have a direct relationship between the geometry of the Del Pezzo surface (which is the base locus of the pencil) and the Segre Symbol. We have found it convenient and indeed fascinating to determine the Segre Symbol from the number and nature of the singularities of the Del Pezzo surface, as well as the number of lines lying on the Del Pezzo surface. Both of these geometrically defined invariants can be readily obtained from the planar subscheme of length 5 which induces the Del Pezzo surface. We therefore present this analysis next and the paper culminates in the presentation of the precise relationships between the Segre Symbol, the number of singularities and lines on the Del Pezzo surface, and the stability properties.

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2 Segre's Classification of Pencils of Quadrics

For the reader's convenience, we review, in this section, the classical classification of pencils of quadrics as originally done by C. Segre in [10]. A more recent and accessible account although missing some of the geometric flavour of the original paper can be found in [5].

Let $(X_0, X_1, \ldots, X_n)$ be coordinates for $\mathbf{C^{n+1}}$ and write the equation of a quadric in $\mathbf{P}^n = \mathbf{P}(\mathbf{C^{n+1}})$ as:

$$F = \sum_{i,j=0}^{n} a_{ij} X_i X_j,$$

where $A = (a_{ij})$ is a symmetric matrix. Let $F = \sum_{i,j=0}^{n} a_{ij} X_i X_j, G = \sum_{i,j=0}^{n} b_{ij} X_i X_j$, be two quadrics where $A = (a_{ij}), B = (b_{ij})$ are symmetric matrices. Consider the pencil $\lambda F + \mu G$ generated by them that will be also referred to as $\lambda A + \mu B$ when we want to consider the associated symmetric matrix. Form the polynomial $\Delta(\lambda, \mu)$ of degree $n+1$ by taking the determinant of $\lambda A + \mu B$. We assume in what follows that $\Delta(\lambda, \mu)$ is not identically zero, so that in general it has $n+1$ roots. This means that the general element of the pencil is a nonsingular quadric. To the $n+1$ roots there correspond in general $n+1$ cones in the pencil. But there can be exceptions to this when a root $(\lambda_0 : \mu_0)$ of $\Delta(\lambda, \mu)$ appears with multiplicity e, say. There can also happen that $(\lambda_0 : \mu_0)$ makes not only the determinant vanish but also all its subdeterminants of order $n - h + 2$ say, $h > 0$. This means that the corresponding cone has as vertex a linear space of dimension $h - 1$. We say that this cone is an $(h-1)-$cone. When we say cone, we mean a 0-cone.

Indicating by l_i the minimum multiplicity with which the root $(\lambda_0 : \mu_0)$ appears in the subdeterminants of order $n+1-i$, $i = 0, 1, \ldots, h-1$ we see that $l_i \geq l_{i+1}$ for all i so that

$e_i = l_i - l_{i+1} \geq 0$. We have therefore:

$$\Delta(\lambda,\mu) = (\lambda\mu_0 - \lambda_0\mu)^e \Delta_1(\lambda,\mu) = (\lambda\mu_0 - \lambda_0\mu)^{e_0} \dots (\lambda\mu_0 - \lambda_0\mu)^{e_{h-1}} \Delta_1(\lambda,\mu),$$

where $\Delta_1(\lambda_0,\mu_0) \neq 0$.

The factors $(\lambda\mu_0 - \lambda_0\mu)^{e_i}$ are called the **elementary divisors** and the exponents e_i are called the **characteristics numbers**, associated to the root $(\lambda_0 : \mu_0)$. Their study goes back to Weierstrass. For each root $(\lambda_j : \mu_j)$ of $\Delta(\lambda,\mu)$ there is a sequence of exponents $e_0^j, \dots, e_{h_j-1}^j$ associated to it as above, where $e_j = \sum_{i=0}^{h_j-1} e_i^j$. C. Segre introduced the following notation to denote the various characteristic numbers associated to the r cones that appear in a pencil $\lambda A + \mu B$:

$$[(e_0^0, \dots, e_{h_0-1}^0), (e_0^1, \dots, e_{h_1-1}^1), \dots, (e_0^r, \dots, e_{h_r-1}^r)].$$

These are known today as the **Segre symbol** for the pencil. The following theorem was proved by Weierstrass:

Theorem 1 *([5], p.278) A necessary and sufficient condition for two quadratic forms F, G of $n+1$ variables to be transformed into two other forms F_1, G_1 respectively by a change of coordinates in $\mathbf{C^{n+1}}$ is that the polynomials $\Delta(\lambda,\mu) = \det(\lambda A + \mu B)$ and $\Delta_1 = \det(\lambda A_1 + \mu B_1)$ have the same elementary divisors, if $\Delta(\lambda,\mu)$ and $\Delta_1(\lambda,\mu)$ are not identically zero and A, B, A_1, B_1 are the symmetric matrices corresponding respectively to F, G, F_1, G_1.*

This theorem can be stated in a more geometrical form:

Theorem 2 *Consider two pencils of quadrics in $\mathbf{P}^n$:*

$$P_1 : \lambda_1 F_1 + \mu_1 G_1 \text{ and } P_2 : \lambda_2 F_2 + \mu_2 G_2,$$

with singular elements $(\lambda_{1,i} : \mu_{1,i}), (\lambda_{2,i} : \mu_{2,i})$ respectively. Suppose that $\Delta_1(\lambda,\mu) = det(\lambda A_1 + \mu B_1)$ and $\Delta_2(\lambda,\mu) = det(\lambda A_2 + \mu B_2)$ are not identically zero, where A_1, B_1, A_2, B_2 are the symmetric matrices corresponding to F_1, G_1, F_2, G_2 respectively. Then P_1, P_2 are projectively equivalent if and only if they have the same Segre symbol and there is an automorphism of $\mathbf{P}^1$ taking $(\lambda_{1,i} : \mu_{1,i})$ to $(\lambda_{2,i} : \mu_{2,i})$.

Theorem 1 enables us to find the following useful canonical form for a pencil of quadrics when $\Delta(\lambda,\mu)$ is not identically zero as follows. Suppose that the elementary divisors of the pencil are:

$$(\lambda - a_1)^{e_1}, (\lambda - a_2)^{e_2}, \dots, (\lambda - a_r)^{e_r}.$$

Then we can find generators for the pencil as follows([5], p. 280):

$$A_1 = \begin{pmatrix} P(a_1) & 0 & \dots & 0 \\ 0 & P(a_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & P(a_r) \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} Q(a_1) & 0 & \dots & 0 \\ 0 & Q(a_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & Q(a_r) \end{pmatrix},$$

where

$$P(a_i) = \begin{pmatrix} 0 & 0 & \dots & 0 & a_i \\ 0 & \dots & 0 & a_i & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a_i & 1 & \dots & 0 \\ a_i & 1 & 0 & \dots & 0 \end{pmatrix} \text{ and } Q(a_i) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Before we get into further detail on characteristic numbers, let us recall one more classical concept.

Definition 1 *Given a quadric* $F = \sum_{i,j=0}^{n} a_{ij} X_i X_j \subset \mathbf{P}^n$ *the polar hyperplane of a point* $y = (y_0, \ldots, y_n) \in \mathbf{P}^n$ *in relation to* F *is:*

$$H = \sum_{i,j=0}^{n} a_{ij} X_i y_j \in \check{\mathbf{P}}^n,$$

where $\check{\mathbf{P}}^n$ *denotes the dual projective* n*-space.*

Note that H is also given by:

$$H = \sum_{i=0}^{n} \frac{\partial F}{\partial X_i}(y) X_i.$$

It follows from the definition that H is the tangent hyperplane at a point y if $y \in F$.

We have the following proposition:

Proposition 1 *If* $x \in F_0$ *is a double point of a quadric belonging to the pencil* $\lambda F_1 + \mu F_2 (F_1$ *and* F_2 *nonsingular) then all polar hyperplanes of* x *in relation to the quadrics of the pencil coincide. Conversely if a point* x *lying on a quadric of a pencil has this property then it is a singular point of that quadric.*

Proof: Let $x = (x_0 : \ldots : x_n) \in F_0$ be a double point of a quadric given by the symmetric matrix $C = \lambda_0 A + \mu_0 B$, where $A = (a_{ij}), B = (b_{ij})$ are symmetric $(n+1) \times (n+1)$ matrices corresponding to F_1, F_2 respectively. Then we have:

$$F_0 = \sum c_{ij} X_i X_j = \lambda_0 \sum a_{ij} X_i X_j + \mu_0 \sum b_{ij} X_i X_j.$$

and therefore:

$$\frac{\partial F_0}{\partial X_i}(x) = \lambda_0 \frac{\partial F_1}{\partial X_i}(x) + \mu_0 \frac{\partial F_2}{\partial X_i}(x).$$

If α is the polar hyperplane of x in relation to an arbitrary quadric of the pencil, then we have:

$$\alpha = (\lambda \frac{\partial F_1}{\partial X_0}(x) + \mu \frac{\partial F_2}{\partial X_0}(x) : \ldots : \lambda \frac{\partial F_1}{\partial X_n}(x) + \mu \frac{\partial F_2}{\partial X_n}(x)) \in \check{\mathbf{P}}^n.$$

Now since x is a double point $\frac{\partial F_0}{\partial X_i}(x) = 0$ for all i and so $\frac{\partial F_1}{\partial X_i}(x) = -\frac{\lambda_0}{\mu_0} \frac{\partial F_2}{\partial X_i}(x)$. Therefore α is given by:

$$((\mu\mu_0 - \lambda\lambda_0) \frac{\partial F_2}{\partial X_0}(x) : \ldots : (\mu\mu_0 - \lambda\lambda_0) \frac{\partial F_2}{\partial X_n}(x)),$$

and therefore the coordinates of $\alpha \in \check{\mathbf{P}}^n$ do not depend on λ and μ.

Conversely, let us look for the points $x \in \mathbf{P}^n$ having the same polar hyperplane with respect to a pencil of quadrics $\lambda F_1 + \lambda F_2$. For this to happen it is enough that the coordinates $(\frac{\partial \Phi_1}{\partial X_i}(x)), (\frac{\partial \Phi_2}{\partial X_i}(x))$ of the polar hyperplanes with respect to two different quadrics Φ_1, Φ_2 of the pencil be proportional, i.e., that there exist a value (λ, μ) such that $\lambda \frac{\partial \Phi_1}{\partial X_i}(x) + \mu \frac{\partial \Phi_2}{\partial X_i}(x) = 0, i = 0, \ldots, n$. These $n+1$ linear equations alow us to eliminate x and we get:

$$\mid \lambda a_{ij} + \mu b_{ij} \mid = 0.$$

Where $A = (a_{ij}), B = (b_{ij})$ are the symmetric matrices corresponding to Φ_1, Φ_2 respectively. This shows that the quadric $\lambda A + \mu B$ is singular, the point x being its singularity. Q.E.D.

Now let x, x' be two double points lying in two different cones Φ_1, Φ_2 belonging to a pencil of quadrics $\lambda F_1 + \mu F_2$. Suppose $(\lambda_s, \mu_s), (\lambda_t, \mu_t)$ are the values of the parameters corresponding to the two cones. We have:

$$\lambda_s \frac{\partial F_1}{\partial X_i}(x) + \mu_s \frac{\partial F_2}{\partial X_i}(x) = 0 \text{ and } \lambda_t \frac{\partial F_1}{\partial X_i}(x') + \mu_t \frac{\partial F_2}{\partial X_i}(x') = 0,$$

and we get:

$$\lambda_s F_1(x, x') + \mu_s F_2(x, x') = 0 \text{ and } \lambda_t F_1(x, x') + \mu_t F_2(x, x') = 0.$$

(Here we are using the notation that if F is defined by the symmetric matrix A, so that $F(x) = x^\top Ax$, then $F(x, y) = y^\top Ax$.) Therefore we get that:

$$\lambda F_1(x, x') + \mu F_2(x, x') = 0$$

We make the following definition:

Definition 2 *Two points x, x' are said to be* **conjugate** *with respect to a quadric F if the polar hyperplane of x with respect to the quadric goes through x', and conversely the polar hyperplane of x' goes through x.*

The equation in the last paragraph then shows that if x, x' are two double points lying in two different quadrics of a pencil then they are conjugate with respect to any quadric of the pencil. We use this to understand what one of the Segre symbols means:

Proposition 2 *Let $\lambda A + \mu B$ be a pencil of quadrics in $\mathbf{P}^n$ with Segre symbol $[1, 1, \ldots, 1]$, i.e., such that the roots of $\Delta(\lambda, \mu)$ are all distinct. Then the pencil can be written as $\lambda A_1 + \mu B_1$ where A_1, B_1 are both diagonal. In particular the hypersurface $S \subset \mathbf{P}^n$, which is the base locus of the pencil, is nonsingular.*

Proof: Consider the $n + 1$ cones of the pencil and their vertices $x_0, \ldots, x_n$. Each x_i is conjugate to any $x_j, i \neq j$, i.e., the polar hyperplane of x_i in relation to any quadric of the pencil goes through the other $x_j's, j \neq i$. We get in this way $n + 1$ hyperplanes with the property that each goes through all x_i except one. Taking these $n + 1$ points as fundamental points of a coordinate system and referring the pencil to it we get the first statement. Since none of the x_i lies in S, it follows from the next lemma that the base locus is nonsingular.Q.E.D.

Lemma 1 *Let $S \subset \mathbf{P}^4$ be a singular surface, base locus of the pencil of quadrics $P : \lambda F + \mu G$, with singular point M. Then M is a singularity of a cone C belonging to the pencil P.*

Proof: A point M is singular for S, if and only if the matrix

$$\left(\begin{array}{c} \frac{\partial F}{\partial X_i}(M) \\ \frac{\partial G}{\partial X_i}(M) \end{array} \right),$$

has rank 1, i.e, if there exist λ_1, μ_1, such that:

$$\lambda_1 \left(\frac{\partial F}{\partial X_i}(M) \right) + \mu_1 \left(\frac{\partial G}{\partial X_i}(M) \right) = 0 \quad \text{for every } i.$$

It follows that: $\frac{\partial(\lambda_1 F + \mu_1 G)}{\partial X_i}(M) = 0$ for all i, so M is a singular point of the cone $\lambda_1 F + \mu_1 G$. Q.E.D.

Suppose that, conversely, the pencil of quadrics $\lambda A + \mu B$ can be written as $\lambda A_1 + \mu B_1$ with A_1, B_1 diagonal matrices. Without loss of generality we can assume that B_1 is the identity. Then besides the case where the Segre symbol is $[1, 1, \ldots, 1]$ we can also have the case where some of the diagonal entries of the other generator, A_1, are repeated so that a certain root, a_j say, has multiplicity e_j. In this case a_j will appear in the subdeterminants of order $n \times n$ with multiplicity $e_2^j = e_1^j - 1$. We go on considering the subdeterminants of order $n+1-i$ and we note that at each step the order of the root in those subdeterminants will decrease by one until we reach the subdeterminants of order $n+1-e^j$ where a_j will not be a root of these subdeterminants. We get that $l_i = e_i^j - e_{i+1}^j = 1$ and the corresponding Segre symbol for the pencil will be:$[(1, \ldots, 1), \ldots, (1, \ldots, 1)]$, where each pair of parenthesis corresponds to a distinct root and the number of 1's corresponding to a root a_j is its multiplicity. We also see that corresponding to the root a_j of multiplicity e_j there will be a cone whose vertex has dimension $e_j - 1$.

It remains to consider the case where we cannot find generators for the pencil in diagonal form.We make the following definition:

Definition 3 *Let $\lambda A + \mu B$ be a pencil of quadrics in $\mathbf{P}^n$ and consider $\Delta(\lambda, \mu) = \det(\lambda A + \mu B)$. If a root (λ_0, μ_0) is a root of $\Delta(\lambda, \mu)$ of multiplicity e and the cone $\lambda_0 A + \mu_0 B$ is a $0-$cone then we say that the cone* **has multiplicity e** *for the pencil.*

We state the following proposition for further reference.

Proposition 3 *Let $\lambda A + \mu B$ be a pencil of quadrics in $\mathbf{P}^n$ and consider $\Delta(\lambda, \mu) = \det(\lambda A + \mu B)$. If $(\lambda_0 : \mu_0)$ is a double root of $\Delta(\lambda, \mu)$, then we have two possibilities:*

1. *The root $(\lambda_0 : \mu_0)$ corresponds to a cone in the pencil whose vertex is a line.*
2. *The root $(\lambda_0 : \mu_0)$ corresponds to a cone in the pencil whose vertex is a point but appears with multiplicity two in the pencil.*

3 Stability of Pencils of Quadrics

In this section we will collect the definitions and results of Geometric Invariant Theory needed in what follows and review some results on stability of pencils of quadrics that have appeared in [13],[14].

Fix an algebraically closed field k and a reductive algebraic group G defined over k. Let V be a n-dimensional representation of G and let x be a vector in V. Let $G \cdot x$ denote the orbit and G_x the stabilizer of x.

Definition 4 1. *x is* **unstable** *if* $0 \in \overline{G \cdot x}$.

2. *x is* **semi-stable** *if* $0 \notin \overline{G \cdot x}$.

3. *x is* **stable** *if $G \cdot x$ is closed and G_x is finite.*

4. *x is* **strictly semi-stable** *if x is semi-stable but not stable.*

A point $p \in \mathbf{P}(V)$ will be called **unstable** if any non-zero vector x lying over p is unstable. Analogous definitions are made for stability and semi-stability.

Let $X \subset \mathbf{P}^n(V)$ be a projective variety and consider a linear action of a reductive group G on X. We denote by X^{ss} the set of semi-stable points of X. Then we have([7]):

Theorem 3 *There exists a projective variety Y and a morphism $\Phi : X^{ss} \longrightarrow Y$ such that (Y, Φ) is a categorical quotient of X^{ss} by G.*

Write the equation of a quadric in $\mathbf{P}^n = \mathbf{P}(\mathbf{C^{n+1}})$ as:

$$F = \sum_{i,j=0}^{n} a_{ij} X_i X_j,$$

where $A = (a_{ij}) \in S$, the set of symmetric $(n+1) \times (n+1)$ matrices. Given quadrics $F_1, \ldots, F_r$ as above, a linear system of quadrics of dimension r can be represented by:

$$\sum_{k=1}^{r} \lambda_k A_k,$$

where A_k is the symmetric matrix corresponding to F_k. Consider the natural action of $G = SL_{n+1}(\mathbf{C})$ on S, given by:

$$g \cdot A = (g^{\top})^{-1} A g^{-1},$$

and the action of $H = SL_r(\mathbf{C})$ on the linear system given by linear substitutions on the λ_k. Then $G \times H$ acts on $S \otimes \mathbf{C^r}$; G changes coordinates in $\mathbf{P}^n$ and H changes coordinates in the linear system. The linear systems are classified by the (categorical) quotient $S \otimes \mathbf{C^r}/(G \times H)$.

In our application with pencils ($r = 2$), if we consider the linear system $\lambda A + \mu B$ where $A, B \in S$, we let:

$$\Delta(\lambda, \mu) = \det(\lambda A + \mu B).$$

Then $\Delta(\lambda, \mu)$ is a polynomial of degree $n+1$ in the two variables, which is invariant under the above action of G. In fact:

Theorem 4 *([1]) The generators for the ring of invariants of pairs of symmetric $(n+1) \times (n+1)$ matrices are the coefficients of the polynomial $\Delta(\lambda, \mu)$.*

Taking the categorical quotient in two steps, first by G, and then by H, we see that the ring of invariants under $G \times H$ is the subring generated by the coefficients of the polynomial $\Delta(\lambda, \mu)$ which are invariant under H. This is a re-formulation of Theorem 2 in fact. Hence an unstable pencil is one for which the polynomial Δ is an unstable polynomial for the act ion of H. Is is a well known fact ([8], p.110) that a polynomial in two variables is unstable if and only if it has a root of multiplicity greater than half the degree. Hence:

Theorem 5 *The unstable pencils of quadrics in* $\mathbf{P}^n$ *are those for which either* $\Delta(\lambda, \mu) \equiv 0$ *or* $\Delta(\lambda, \mu)$ *has a root of multiplicity greater than* $(n+1)/2$.

Next we determine the semi-stable pencils that are not stable. So let $\lambda A + \mu B$ be a pencil of quadrics in $\mathbf{P}^n$ that is semi-stable. Suppose that $\Delta(\lambda, \mu) = \det(\lambda A + \mu B)$ has a root a of multiplicity $e \geq 1$ and that the other roots are distinct. Then from Theorem 5, it follows that $e \leq (n+1)/2$. From Section 1, we know that the root a corresponds to either a cone whose vertex has dimension ≥ 1 or to an ordinary cone with multiplicity e. In the latter case the vertex is contained in every quadric of the pencil.

Suppose that a corresponds to a cone C whose vertex V has positive dimension d. Let F be any nonsingular quadric of the pencil and consider: $V \cap F = Q$. Let $x_0, \ldots, x_d \in Q$ be linearly independent points and consider $\alpha_0, \ldots, \alpha_d$ the polar hyperplanes of the x_i, $i = 0, \ldots, d$ in relation to all quadrics of the pencil. Complete the system $x_0, \ldots, x_d$ to a coordinate system of $\mathbf{P}^n$. Referring $\lambda A + \mu B$ to this coordinate system we see that the elements of the pencil are represented by matrices of the following type:

$$\begin{pmatrix} 0 & B & C \\ B^\top & E & F \\ C^\top & F^\top & G \end{pmatrix},$$

where the upper left block of zeros is $(d+1) \times (d+1)$ and comes from the fact that we have taken the $x_0, \ldots, x_d$ as coordinate points and that each $x_i \in \alpha_j, i = 0, \ldots, d;\ j = 0, \ldots, d$.

If, on the other hand, the root a corresponds to a cone C with multiplicity e, its vertex x_0 lies in all other quadrics of the pencil. Let α_0 be the polar hyperplane to x_0 in relation to all quadrics of the pencil. Let $x_1, \ldots, x_r$ be the vertices of the other cones that appear in $\lambda A + \mu B$ $(r - d \geq 1)$. Let $\alpha_1, \ldots, \alpha_r$ be the polar hyperplanes of the x_i in relation to the quadrics of the pencil. Complete the system $x_0, \ldots, x_r$ to a coordinate system of $\mathbf{P}^n$ such that α_0 is a coordinate hyperplane and there is only one coordinate hyperplane α_{n+1} such that $x_0 \notin \alpha_{n+1}$. Referring $\lambda A + \mu B$ to this coordinate system we see that the elements of the pencil are represented by matrices of the following type:

$$\begin{pmatrix} 0 & 0 & B \\ 0 & C & D \\ B^t & D^t & E \end{pmatrix},$$

where the upper left 1×1 block of zeros comes from the fact that x_0 lies in all quadrics of the pencil and the upper middle $1 \times n-1$ block of zeros comes from the fact that the polar hyperplane to x_0 is a coordinate hyperplane.

Now given any linear system of quadrics $\sum_{r=1}^{k} \lambda_r S_r$, where S_r are quadrics and λ_r parameters we have the following result:

Theorem 6 *([14], p.232) The system* $\sum_{r=1}^{k} \lambda_r S_r$ *is not stable if and only if for some coordinates* x *and some integer* s *with* $1 \leq s \leq \frac{1}{2}(n+1)$ *we have* $a_{ij}^{(r)} = 0$ *whenever* $0 \leq r \leq k, 1 \leq i \leq s, 1 \leq j \leq n+1-s$, *where* $a_{ij}^{(r)}$ *are the symmetric matrices corresponding to* S_r.

So we have:

Corollary 1 *A pencil of quadrics* $\lambda A + \mu B$ *is not stable if and only if* $\Delta(\lambda, \mu) = \det(\lambda A + \mu B)$ *has a root of multiplicity greater than* 1.

4 Ring of Invariants of a Pencil of Quadrics in $\mathbf{P}^4$

Let us consider first the ring of invariants of binary quintics. This was known classically. We follow [3]. A binary quintic,

$$a_0x^5 + 5a_1x^4y + 10a_2x^3y^2 + 10a_3x^2y^3 + 5a_4xy^4 + a_5y^5$$

with no pure cube factor is equivalent under SL_2 to one of the form:

$$ax'^5 + 5bx'^4y' + 5ex'y'^4 + fy'^5.$$

This was known classically as Hammond's form. With this notation the ring of invariants for the binary quintics is generated by the following four invariants:

$$I_4 = a^2f^2 - 10abef + 9b^2e^2,$$

$$I_8 = a^2b^2e^2f^2 - 2a^3e^5 - 2b^5f^3 + 27b^4e^4,$$
$$I_{12} = b^2e^2(a^2b^2e^2f^2 - 4a^3e^5 - 4b^5f^3 + 18ab^3e^3f - 27b^4e^4),$$
$$I_{18} = (a^3e^5 - b^5f^3)\{(af - 5be)(a^3e^5 + b^5f^3) - 10a^2b^3e^3f^2 + 90ab^4e^4f - 216b^5e^5\}.$$

There is a syzygy connecting the four invariants :

$$2^4I_{18}^2 = I_4I_8^4 + 2^3I_8^3I_{12} - 2I_4^2I_8^2I_{12} - 2^33^2I_4I_8I_{12}^2 - 2^43^3I_{12}^3 + I_4^3I_{12}^2. \tag{1}$$

We want to consider the ring of invariants for pencils of quadrics in $\mathbf{P}^4$. Since the theorem below is true in greater generality we state it for pencils of quadrics in arbitrary $\mathbf{P}^n$. So let $\mathcal{P} = S \otimes \mathbf{C}^2$ be the space of pencils of quadrics in $\mathbf{P}^n$ with the action of $G \times H$. Let $\mathcal{V}_n$ be the space of polynomials of degree $n+1$ in two variables with the action of $SL_2(\mathbf{C})$. Then we have the following theorem:

Theorem 7 *The ring of invariants of $\mathcal{P}$ is isomorphic to the the ring of invariants of $\mathcal{V}_n$.*

Proof: Let $\mathcal{T} \subset \mathcal{P}$ be the open set of pencils that can be generated by two matrices A and B in diagonal form with no repeated entries. Let $\mathcal{W}_n \subset \mathcal{V}_n$ be the open set formed by polynomials with no repeated roots. Since invariants are already determined in an open set it will be enough to prove that:

$$\mathcal{T}^{G\times H} \cong \mathcal{W}_n^{GL_2}$$

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\Delta"] \arrow[d] & \mathcal{W}_{n+1} \arrow[d] \\
\frac{\mathcal{T}}{G\times H} \arrow[r, "\Lambda"] & \frac{\mathcal{W}_{n+1}}{GL_2}
\end{tikzcd}$$

where the map Δ sends the pencil $\lambda A+\mu B$ to the polynomial of degree $n+1$ in two variables given by $det(\lambda A+\mu B)$. Now Δ takes orbits to orbits (Theorem 2) and is surjective.

Furthermore Λ is injective, for suppose $\Lambda(\overline{x_1}) = \Lambda(\overline{x_2})$ where $x_1, x_2 \in \mathcal{T}$. It follows that $\Delta(x_1) = \Delta(x_2)$, up to the GL_2 action. But we can write $x_1 = \lambda A_1 + \mu B_1$ and $x_2 = \lambda A_2 + \mu B_2$, where A_1, A_2, B_1, B_2 are diagonal matrices with no repeated entries. Let $a_i^j(\text{resp.}b_i^j), i = 1, \ldots, n+1; j = 1, 2$ be the entries of $A_j(\text{resp.}B_j)$. It follows that:

$$\prod_1^{n+1}(\lambda a_i^1 + \mu b_i^1) = \prod_1^{n+1}(\lambda a_i^2 + \mu b_i^2).$$

This means that $\overline{x_1} = \overline{x_2}$.

Now since $\mathcal{P}$ and $\mathcal{W}_{n+1}$ are smooth the quotient spaces $\mathcal{P}/(G \times H)$ and $\mathcal{W}_{n+1}/GL_2$ are normal. So Λ is a birational regular map between normal varieties which is a bijection. By Zariski's Main Theorem it is therefore an isomorphism. This concludes the proof of the theorem. Q.E.D.

Corollary 2 *The ring of invariants of pencils of quadrics in* $\mathbf{P}^4$ *is given by:*

$$Y = \mathbf{C}[I_4, I_8, I_{12}, I_{18}]/(R(I_{18}, I_{12}, I_8, I_4)),$$

where the generator R is given by the equation (1) above. Geometrically it means that $Proj(Y) = X^{ss}/(G \times H)$ *is a double cover of a weighted* $\mathbf{P}^2$ *with weights* $(4, 8, 12)$.

5 The Orbit Structure

In **Corollary 2** of the last section, we saw that if we let $\mathcal{S} = Proj(Y) = X^{ss}/(G \times H)$ then $\mathcal{S} = Proj(Y)$ is a surface, a double cover of a weighted $\mathbf{P}^2$. Let $\Phi : X^{ss} \longrightarrow \mathcal{S}$ be the natural projection. Let us now determine how the fibers of Φ, the orbits under the action of the group, fit together in terms of the Del Pezzo surfaces they parametrize. In the next section we will prove that all the base loci of semi-stable pencils are really Del Pezzo surfaces. More precisely given such a surface S we will prove that it is possible to find a linear system of cubics with five base points in $\mathbf{P}^2$, and to blow $\mathbf{P}^2$ up at those five points to get a surface S', such that the linear system maps S' isomorphically to the given surface S. Accordingly we will call a Del Pezzo surface in $\mathbf{P^4}$ stable (resp. semi-stable) if it is the base locus of a stable (resp. semi-stable) pencil of quadrics in $\mathbf{P^4}$. First of all we know that the stable pencils are the ones such that $\Delta(\lambda, \mu)$ has no multiple roots and the Segre symbol for those pencils is $[1, 1, \ldots, 1]$ so that they correspond to nonsingular Del Pezzo surfaces (Proposition 2). The image, under Φ, of those pencils form an open set of $U \subset \mathcal{S}$. Each point of U has as its fibre under Φ a unique orbit.

Let us now look at the complement of U, the points that parametrize orbits that are semi-stable but not stable. A pencil $\lambda A + \mu B$ that is semi-stable but not stable is such that $\Delta(\lambda, \mu)$ has at least a double root so that the discriminant D of $\Delta(\lambda, \mu)$ is zero. Now the discriminant D of a quintic is given in terms of the other invariants as:([3], p.302)

$$D = I_4^2 - 2^7 I_8,$$

where we keep the notation of the last section. Imposing the condition that $D = 0$ and substituting in equation 1 we get:

$$2^{32}I_{18}^2 = Q^3 - 31.2^{10}Q^2I_{12} + 7.2^{24}QI_{12}^2 - 2^{32}.3^3I_{12}^3,$$

where $Q = I_4^3$. We can write this as:

$$2^{32}I_{18}^2 = (Q - 2^{11}I_{12})^2(Q - 2^{10}3^3I_{12}).$$

So the locus $\mathcal{S}' \subset \mathcal{S}$ where the discriminant vanishes is a singular cubic in weighted projective $\mathbf{P}^2$ with weights $(12, 12, 18)$. Since each point $s \in \mathcal{S}'$, parametrizes stable orbits of quintic equations, it follows that it corresponds to a unique choice of a double root and three simple roots of a quintic equation, up to an automorphism of $\mathbf{P}^1$. Therefore over each general point of this cubic there are two orbits of pencils of quadrics one of each of the two types below:

1. One type are the pencils with Segre symbol $[2, 1, 1, 1]$. As we saw in Section 1, Proposition 3, this means that one of the cones of the pencil, C_1 say, has multiplicity two. In terms of the corresponding Del Pezzo surface it means that the vertex of C_1 lies on the surface being its only ordinary (conic) double point. Note that a Segre symbol $[3, 1, 1]$ means that the corresponding Del Pezzo surface has an ordinary (biplanar) double point and this corresponds to an unstable pencil.

2. The other type of pencils are the ones with Segre symbol $[(1, 1), 1, 1, 1]$. This pencil is characterized by the property that it has a cone whose vertex is a line. This line intersects any other quadric of the pencil in two (distinct) ordinary double points of the corresponding Del Pezzo surface. This is a closed orbit.

The cubic $\mathcal{S}'$ has one singularity where $I_4^3 = 2^{11}I_{12}$. This is the condition for the quintic equation to have two double roots ([9], p.335). There are three orbits lying over this point with Segre symbols: $[(1, 1), (1, 1), 1], [(1, 1), 2, 1], [2, 2, 1]$.

In the first case the pencil has two cones whose vertex is a line. Those two lines intersect any other quadric of the pencil in four points all ordinary double points for the corresponding Del Pezzo surface. This orbit is a closed orbit. In the second case there is a cone in the pencil whose vertex is a line that intersects any other quadric in the pencil in two ordinary double points for the surface and there is a further singularity coming from the vertex of a cone of multiplicity two also belonging to the pencil. In the last case there are two cones in the pencil of multiplicity two whose vertices are on the surface that has therefore two ordinary double points. The line joining those two vertices is contained in every quadric of the pencil and therefore also in the surface. This distinguishes this case from case 2) above.

Let us return briefly to the case of stable pencils, where as we have noted above the discriminant $\Delta(\lambda, \mu)$ has all distinct roots and the base locus is a smooth Del Pezzo surface of degree 4 in $\mathbf{P}^4$. There is an interesting point-set equivalence arising here in the following way. The pencil of quadrics has exactly five singular members (corresponding to the roots of Δ) and the Del Pezzo surface is the blow up of five points in $\mathbf{P}^2$. These five points must lie on a unique conic; and so we have in two ways the construction of five points on a smooth rational curve.

We claim that these two sets of five points are projectively equivalent. We start with five points on a $\mathbf{P}^1$, where we may choose coordinates and call the five points $[0 : 1]$, $[1 : 0]$, $[1 : 1]$, $[a : 1]$, and $[b : 1]$. We then take the Veronese embedding of the $\mathbf{P}^1$ into $\mathbf{P}^2$, sending $[r : s]$ to $[r^2 : rs : s^2]$; this give us the five points $[0 : 0 : 1]$, $[1 : 0 : 0]$, $[1 : 1 : 1]$, $[a^2 : a : 1]$, and $[b^2 : b : 1]$ in $\mathbf{P}^2$ lying on the conic $xz = y^2$. The Del Pezzo surface corresponding to blowing up these five points is obtained by taking five independent cubics through these five points; these cubics are easily seen to be $F_1(x, y, z) = x^2z - xy^2$, $F_2 = xyz - y^3$, $F_3 = xz^2 - y^2z$, $F_4 = xy^2 - (m + 1)y^3 + (n + m)y^2z - nyz^2$, and $F_5 = x^2y - (m^2 - n + m + 1)y^3 + (mn + m^2 + m)y^2z - (mn + n)yz^2$, where $m = a + b$ and $n = ab$. The rational map from $\mathbf{P}^2$ to $\mathbf{P}^4$ embedding the blowup at these five points as the Del Pezzo surface of degree 4 is then given by the formula $[F_1 : F_2 : F_3 : F_4 : F_5]$.

Now we seek the pencil of quadrics through this Del Pezzo surface. Take the general quadric in five variables, $Q = X^\top AX$, where $X = (x_1, x_2, x_3, x_4, x_5)^\top$ and A is a symmetric matrix filled with 15 indeterminates. Plug in the F_i's for the x_i's, and one obtains a sextic polynomial in the variables x, y, z. In order for the quadric Q to contain the Del Pezzo, which is the image of the five cubics, we have to have each coefficient of this sextic identically zero. There are 28 monomials, and setting each to zero gives 28 linear conditions on the 15 indeterminate entries of the symmetric matrix A. Doing this with Maple , we then formed the 28×15 matrix and asked for the kernel. It gave us as a basis for the kernel two symmetric matrices Q_1 and Q_2, with entries being expressions in a and b. Now form the u-matrix $Q_1 - uQ_2$, and ask Maple for the determinant Δ of this; it turns out to be exactly $u(u-1)(u-a)(u-b)$, the quintic with a root at infinity and four other roots at 0, 1, a, and b, just as the original starting point for the construction.

We have been informed by I. Dolgachev that this phenomenon is a very special case of a well-known and beautiful construction related to the intermediate Jacobian of the intersection of two quadrics in general. C. Ciliberto has also provided to us a rather geometric proof which does not require machine computations.

6 Linear Systems of Cubics with 5 Base Points

Consider a linear system of cubics in $\mathbf{P}^2$, with five base points, coincident or not and not all collinear. Then they impose independent conditions on cubics([4], p.481). Suppose further that at most three of the base points lie on a line, so that a general element of the linear system is irrreducible.[1]

Blow $\mathbf{P}^2$ up enough number of times at the base points, so that the corresponding linear system at the blown up surface will not have any base points and map the resulting surface $\widehat{\mathbf{P}^2}$ to $\mathbf{P}^4$, by the induced linear system, getting a surface a Del Pezzo surface $S \subset \mathbf{P}^4$. It is well known that if the five points are distinct and in general position then S is nonsingular and is the base locus of a pencil of quadrics with Segre symbol $[1, 1, 1, 1, 1]$.([2]) If the P_i are not in general position anymore we still get a surface S base locus of a pencil of quadrics as long as S has a non singular hyperplane section since the same proof as in [2], pg. 61, goes through. It is natural to ask what are the stability properties of this surface if the five base points in $\mathbf{P}^2$ are not in general position any more. Now given a Del Pezzo surface $S \subset \mathbf{P}^4$, we can compute the number of lines, their configuration and

[1]Notice that, even though, we require that at most three points are collinear, four or even all the five points may be supported in just one point.

the number of singularities as a function of the Segre symbol of the surface. We rely on this computation to determine what are the configurations of linear systems of cubics with 5 base points that correspond to strictly semi-stable Del Pezzo surfaces. Indeed, we have determined the correspondence between all possible configurations of 5 base points of linear systems of cubics as long as at most three are collinear and the general element is irreducible; and Del Pezzo surfaces with a finite number of lines and isolated singularities. For reasons of space we postpone the presentation of the complete picture to another paper and describe here the configurations corresponding to strictly semi-stable pencils. This is a classical subject and the number of lines in the following table was already known to Corrado Segre.([11])

Segre Symbol	No. of Lines	No. of Singularities	Stability
[1,1,1,1,1]	16	0	stable
[2,1,1,1]	12	1	semistable
[3,1,1]	8	1	unstable
[4,1]	5	1	unstable
[5]	3	1	unstable
[(2,1),1,1]	4	1	unstable
[(3,1),1]	2	1	unstable
[(4,1)]	1	1	unstable
[2,2,1]	9	2	semistable
[3,2]	6	2	unstable
[(1,1),1,1,1]	8	2	semistable
[(2,1),2]	3	2	unstable
[2,(1,1),1]	6	3	semistable
[3,(1,1)]	4	3	unstable
[(2,1),(1,1)]	2	3	unstable
[(1,1),(1,1),1]	4	4	semistable

The proof is elementary and follows from considering normal forms for the pencils. Therefore we omit it.

Here is the precise result for strictly semi-stable configurations.

Theorem 8 *Consider a linear system of cubics in $\mathbf{P}^2$, with five base points. The configurations that correspond to strictly semi-stable Del Pezzo surfaces are:*

1. *Two points infinitely near and three other points in general position.*
2. *Three points on a line l and two other points in general position.*
3. *Two points P_1, P_2 infinitely near, two other points P_3, P_4 infinitely near and a fifth point P_5 in general position.*
4. *Three points P_1, P_2, P_3 lying on a line l and two other points P_4, P_5 so that the line m through P_4, P_5 goes through either P_1, P_2 or P_3.*
5. *Three points P_1, P_2, P_3 lying on a line l, two of them infinitely near and two other points in general position.*

6. *Three points P_1, P_2, P_3 lying on a line l and two other points P_3, P_4 infinitely near, with tangent direction s so that $P = s \cap l \neq P_i, i = 1, \ldots, 5$.*

7. *Three points P_1, P_2, P_3 lying on a line l, two of them infinitely near. Two other points P_4, P_5 infinitely near,with tangent direction s so that $P = s \cap l \neq P_i, i = 1, \ldots, 5$.*

8. *Three points P_1, P_2, P_3 lying on a line l, two of them P_1, P_2 say, infinitely near. Two other points P_4, P_5 such that the line determined by them intersects l in P_3*

9. *Two points P_1, P_2 infinitely near with corresponding tangent direction t, two other points P_3, P_4 infinitely near with corresponding tangent direction s and a fifth point P_5 in the intersection of t and s.*

Proof: Consider 5 points P_1, P_2, P_3, P_4, P_5 in $\mathbf{P}^2$, not all collinear and the complete linear system of cubics $|C|$, having the P_i as base points. Suppose further that the P_i's are not in general position. Since the P_i impose independent conditions on the cubics it follows that $dim(|C|) = 4$. Choose a basis c_0, c_1, c_2, c_3, c_4 for $|C|$ and suppose that $dim_{\mathbf{C}}(\mathcal{O}_{P_i}/(c_0, c_1, c_2, c_3, c_4)) = l_i$, where $dim_{\mathbf{C}}(\mathcal{O}_{P_i}/(c_0, c_1, c_2, c_3, c_4))$ denotes the dimension of the quotient of the local ring $\mathcal{O}_{P_i}$ as a vector space over $\mathbf{C}$. If $l_i \geq 2$ we say the l_i points are infinitely near (or concentrated at P_i). If $l_i = 2$ (resp. 3) the cubics in the linear system have a common first (resp. second) order tangent at P_i. The method of the proof is to blow up $\mathbf{P}^2$ at each P_i, l_i times to get a surface $\mathcal{P}$ so that the induced linear system $\mathcal{C}$ has no base points, then embed $\mathcal{P}$ in $\mathbf{P}^4$ by $\mathcal{C}$, getting a Del Pezzo surface S. We can then determine the singularities and the number of lines of S as well as their configuration determining thus the Segre symbol of S. The proof is long and involves many cases so we divide it up in lemmas according roughly to the number of times we have to blow up. Since the method is always the same we consider in detail only the case where we have to blow up twice(Lemma 2). Since we already know the Segre symbols of semi-stable pencils, once we establish the correspondence between configuration of five points and Segre symbols we get the result.

So, consider the blow up of $\mathbf{P}^2$ once at each of the base points P_i, to get a surface $\mathcal{P}_1$. Let $p_1 : \mathcal{P}_1 \longrightarrow \mathbf{P}^2$, be the blow up map and E_i the exceptional divisor corresponding to P_i. Consider the linear system $|C_1|$, on $\mathcal{P}_1$, induced by p_1. An element C_1 of $|C_1|$ being given by:

$$C_1 = p_1^*(C) - \sum_{i=1}^{r} E_i,$$

where r is the number of distinct base points P_i. Then we have two cases to consider:

1. The complete linear system $|C_1|$ in $\mathcal{P}_1$ has no base points. It is straightforward to check that in this case we get two Del Pezzo surfaces with Segre symbols $[2, 1, 1, 1]$ and $[2, 2, 1]$ corresponding respectively to the following configurations:

   (a) Three of the points P_i lie on a line and the other two are in general position.

   (b) Three of the points $P_i, (P_1, P_2, P_3,$ say) lie on a line and the other two are collinear with either P_1, P_2 or P_3.

2. The linear system $|C_1|$ in $\mathcal{P}_1$, has base points. Call them P_{ii}.

In this last case we blow the P_{ii} up to get a surface $\mathcal{P}_2$. Let E_{ii} be the exceptional divisor corresponding to the P_{ii}. Let $p_2 : \mathcal{P}_2 \longrightarrow \mathcal{P}_1$, be the second blow up map and $|C_2|$ be the complete linear system induced by p_2, as before. The base points P_{ii} correspond to points P_i in the original complete linear system $|C|$, such that:

$$dim_{\mathbf{C}}(\mathcal{O}_{P_i}/(c_0, c_1, c_2, c_3, c_4)) \geq 2,$$

where c_0, c_1, c_2, c_3, c_4 denote a basis for $|C|$. We will assume, for the moment, that $|C|$ has only one such base point. We then have:

Lemma 2 *Suppose that the complete linear system $|C|$, has exactly one base point of length two, so that the complete linear system $|C_2|$ in $\mathcal{P}_2$ has no base points. Then the corresponding Del Pezzo surface has Segre symbol of one of the following types:*

$$[2, 1, 1, 1], [2, 2, 1], [(1, 1), 1, 1, 1], [2, (1, 1), 1], [3, 1, 1] \text{ or } [3, 2].$$

Proof: Suppose that $P_1 = P_2$ is the point of length 2 and let t be the tangent to all cubics in $|C|$ at P_1. When we perform the first blow up at P_1, P_3, P_4 and P_5, we get a base point P_{11} of the linear system $|C_1|$, at the point of E_1, corresponding to t. When we do the second blow at P_{11}, we get:

$$p_2^*(E_1) = F_1 + E_{11},$$

where F_1 denotes the proper transform of E_1 in $\mathcal{P}_2$. We then have that $F_1^2 = p_2^*(E_1)^2 + E_{11}^2 = -2$. We also have that: $p_1^*(t) = t_1 + E_1$, where t_1 is the proper transform of t in $\mathcal{P}_1$. It follows that: $t_1^2 = 1 - 1 = 0$. When we perform the second blow up at P_{11}, we get $p_2^*(t_1) = t_{11} + E_{11}$, where t_{11} is the proper transform of t_1 in $\mathcal{P}_2$. We get that $t_{11}^2 = -1$.

Let us consider now the other three base points of $|C|$. If these three points are in general position every line l_{ij} through two of them is such that $m_{ij}^2 = -1$. We also have a unique nonsingular conic K going through the four points P_1, P_3, P_4, P_5 and tangent at P_1 to t. Consider $p_1^*(K) = F + E_1 + E_3 + E_4 + E_5$, where F is the proper transform of K in $\mathcal{P}_1$. Then we have that $F^2 = 4 - 4 = 0$. Now let $p_2^*(F) = G + E_{11}$, where G is the proper transform of F in $\mathcal{P}_2$. We get that $G^2 = -1$.

Consider now the mapping j from $\mathcal{P}_2$ to $\mathbf{P}^4$, determined by the linear system $|C_2|$ without base points, giving a surface $S \subset \mathbf{P}^4$. If the three base points P_3, P_4 and P_5 are in general position then F_1 is the only (-2)-curve.

We get, therefore, a surface $S \subset \mathbf{P}^4$ with one singularity, containing the following 12 lines:

- 6 lines corresponding to the lines $m_{ij}, i, j = 1, 3, 4, 5$.
- 3 lines corresponding to the exceptional divisors $E_i, i = 3, 4, 5$.
- 1 line corresponding to the exceptional divisor E_{11}.
- 1 line corresponding to the curve t_{11}.
- 1 line corresponding to the curve G.

Moreover it is easy to see that these are all the lines on the surface. It follows that the Segre symbol for this Del Pezzo surface is $[2, 1, 1, 1]$.

Now we assume that the three base points P_3, P_4, P_5 are not in general position and we study the possibilities for the number of singularities and the number of lines contained in the Del Pezzo surface. The possible configurations are:

1. The three points P_3, P_4 and P_5 are on a line l, not going through $P_1 = P_2$.

2. The three points P_3, P_4 and P_5 are on a line l and the line $t \neq l$ goes through either P_3, P_4 or P_5.

3. The line t goes through one of the points P_3, P_4 or P_5, and the other two points are in general position.

4. The line l through two of the points P_3, P_4, P_5, goes through $P_1 = P_2$, with $l \neq t$ and the fifth point in general position.

5. The line l through two of the points P_3, P_4, P_5, goes through $P_1 = P_2$, with $l \neq t$ and the fifth point lies on t.

Let us see what are the singularities and the number of lines of each Del Pezzo surface corresponding to each situation above:

1. In $\mathcal{P}_2$ we get two (-2)-curves one being the proper transform F_1 of E_1, as before, the other corresponding to the line l. The Del Pezzo surface S has therefore two singularities. We get eight lines lying on S, four through each singularity. The corresponding Segre symbol is $[(1, 1), 1, 1, 1]$.

2. In $\mathcal{P}_2$ we get three (-2)-curves, two as before and a third one corresponding to the line t. They are mapped to three singularities. There are six lines on the surface. The corresponding Segre symbol is $[2, (1, 1), 1]$.

3. In this configuration we get two (-2)-curves in $\mathcal{P}_2$, one being the curve F_1, the other corresponding to the line t. They are mapped to two singularities of the Del Pezzo surface S. There are nine lines on the surface. The corresponding Segre symbol is $[2, 2, 1]$.

4. We get two (-2)-curves in $\mathcal{P}_2$, one corresponding to l, the other being the curve F_1. These two curves intersect and are therefore mapped to the only singularity of this Del Pezzo surface. We get eight lines on the surface, four through the singularity. The corresponding Segre symbol is $[3, 1, 1]$.

5. In this last case we get three (-2)-curves in $\mathcal{P}_2$, a cycle of two (-2)-curves as before corresponding to the line l and the curve F_1 and a (-2)-curve corresponding to the line t. This last curve is mapped to a second singular point of S. We get six lines on the surface and Segre symbol $[3, 2]$.

Q.E.D.

If the original linear system $|C|$ in $\mathbf{P}^2$ has two base points of length two, we have:

Lemma 3 *Suppose that the complete linear system $|C|$, has two base points of length two, so that the complete linear system $|C_2|$ in $\mathcal{P}_2$ has no base points. Then the corresponding Del Pezzo surface has Segre symbol of one of the following types:*

$$[2,2,1],[3,2],[(1,1),(1,1),1],[2,(1,1),1],[3,(1,1)] \textit{ or } [4,1]$$

Proof: This proof, very similar to the last Lemma, will be ommitted. For the sake of completeness let us list the base point configurations that appear in this case:

1. Two points P_1, P_2 infinitely near, two other points P_3, P_4 infinitely near and a fifth point in general position.($[2,2,1]$)
2. Two points P_1, P_2 infinitely near, with tangent direction t, two other points P_3, P_4 infinitely near, with t going through P_3, and a fifth point in general position.($[3,2]$)
3. Three points P_1, P_2, P_3 lying on a line t, two of them infinitely near. Two other points P_4, P_5 infinitely near, with tangent direction s, with $t \cap s = P \neq P_i, i = 1, \ldots, 5$. ($[2,(1,1),1]$)
4. Two points P_1, P_2 infinitely near, two other points P_3, P_4, infinitely near, so that the line through P_1, P_3, goes through P_5. ($[4,1]$)
5. Two points P_1, P_2 infinitely near with corresponding tangent direction t, two other points P_3, P_4 infinitely near with corresponding tangent direction s and a fifth point P_5 in the intersection of t and s. $[(1,1),(1,1)]$
6. Three points P_1, P_2, P_3 on a line t, two of them (P_1, P_2, say) infinitely near, P_3, P_4, P_5 on a line s, with P_3 and P_4 infinitely near.($[3,(1,1)]$)

Q.E.D.

If the linear system $|C_2|$ in $\mathcal{P}_2$, has one base point (let us call it P_{iii}) we blow it up again to get a surface $\mathcal{P}_3$. Let E_{iii} be the exceptional divisor corresponding to the base point P_{iii}. Let $p_3 : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$, be the third blow up map and $|C_3|$ be the complete linear system induced by p_3, as before. The base point P_{iii} corresponds to a base point P_i in the original complete linear system $|C|$, such that:

$$dim_{\mathbf{C}}(\mathcal{O}_{P_i}/(c_0, c_1, c_2, c_3, c_4)) \geq 3,$$

where c_0, c_1, c_2, c_3, c_4 denote a basis for $|C|$.

Consider a linear system of cubics with a base point P_1, of length three and common tangent t at P_1. The fact that the base point P_1 has length three means that after the first blow up all curves in the linear system $|C_1|$ still have a common tangent t_1' at a point P_{11} in E_1. We have to distinguish two different cases here:

1. The tangent t_1' is different from the proper transform t_1 of t.
2. The proper transform of t is tangent to all cubics in the linear system C_1 at the base point P_{11}, i.e. $t_1 = t_1'$.

In both cases we have that $p_1^*(t) = t_1 + E_1$, so that $t_1^2 = 1 - 1 = 0$. When we perform the second blow up we have: $p_2^*(t_1) = t_{11} + E_{11}$, where t_{11} is the proper transform of t_1, so that $t_{11}^2 = -1$. When we do the third blow up, however, the two cases are different. In the first case, since the tangent to all curves of $|C_1|$ at P_{11} is not t_1, it follows that the third blow up will not be done at the intersection of t_{11} with E_{11}, but at some other point of E_{11}. As a result, when we consider the map determined by the linear system $|C_3|$, to get a Del Pezzo surface S the line t_{11} is mapped to a line of S. On the other hand, in the second case the third blow up will be done at the intersection of t_{11} and E_{11}, so we get: $p_3^*(t_{11}) = t_{111} + E_{111}$, where t_{111} is the proper transform of t_{11}. We have that: $t_{111}^2 = -1 - 1 = -2$, so that the line t_{111} is contracted to a singularity of the corresponding Del Pezzo surface. Keeping in mind this observation, it is straightforward to figure out what are the configurations when there is a base point of length three and also two other possible cases when we have base points of length four and five. Since we are interested in configurations of base points that correspond to strictly semi-stable Del Pezzo surfaces, and considering the fact that if the base point has length greater than or equal to three, all configurations correspond to unstable pencils, we do not describe here these configurations explicitly.

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