FOLIATIONS SINGULAR ALONG A CURVE

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It has been studied in

X. Gómez-Mont and I. Luengo, *The Bott polynomial of a holomorphic foliation by curves*, Ecuaciones diferenciales y singularidades (Medina, 1995), Universidad de Valladolid, 1997.

I.V., *Sur le nombre de singularités dégénérées d'une famille de feuilletages*, Math. Res. Letters, 2004.

Viviana Ferrer, I.V., *Degenerate singularities of one dimensional foliations*, Comment. Math. Helv., 2013

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positive dimensional component

in the singular locus of a foliation

and study the geometry of the subvariety

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formed by the transgressors:

(1) $\dim\Sigma$? $\deg \Sigma$?

(2) ¿If $\operatorname{Sing}\mathscr{F} \supseteq C =$ curve,

How many isolated singularities do persist?

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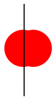

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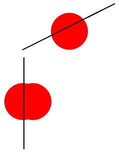

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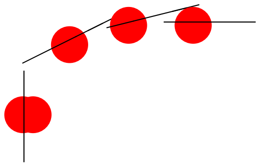

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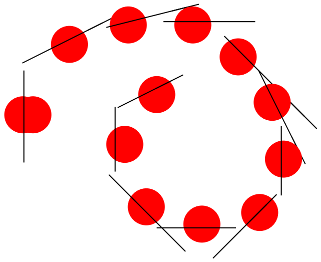

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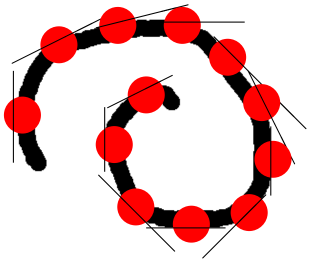

The *leaves* are analytic curves with tangent line specified by the direction attached to the point.

Here we look at *algebraic foliations*
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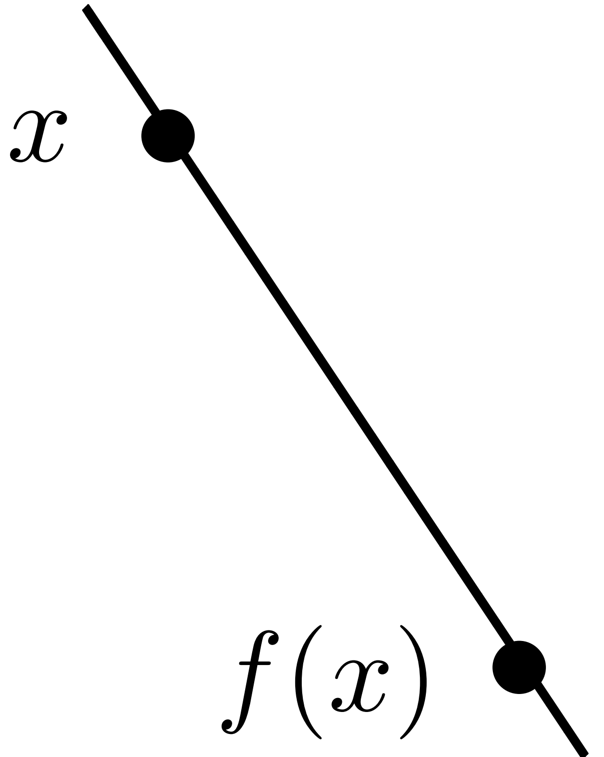
and then take the join

$$\langle x, f(x) \rangle \subset \mathbb{P}^n.$$



The map f is in turn given by $(f_0, \ldots, f_n)$, homogeneous polynomials of the same degree,

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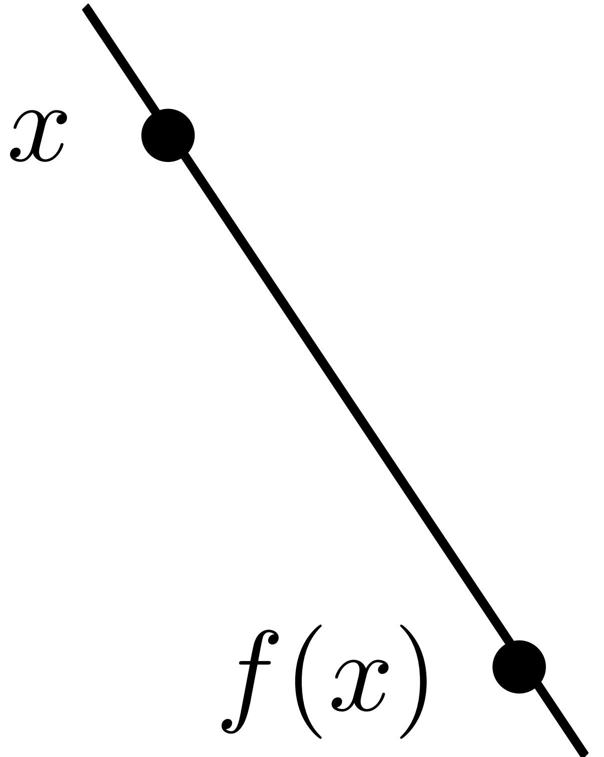

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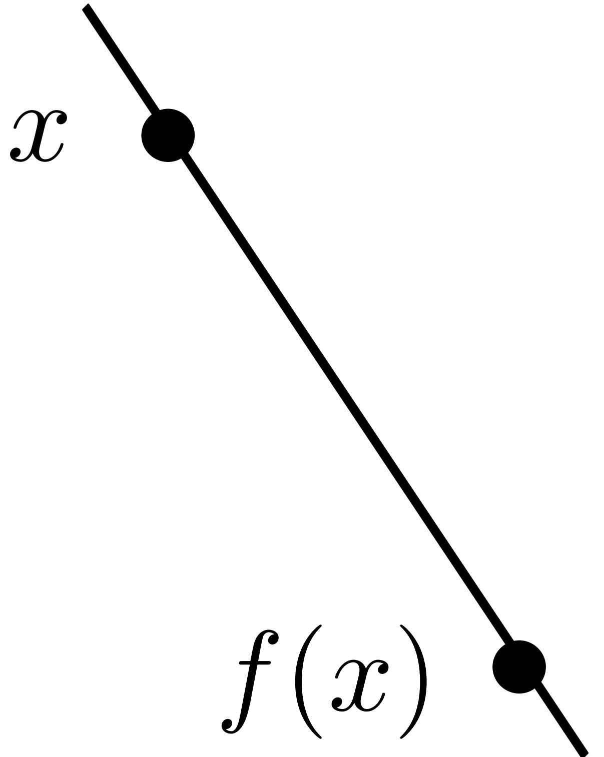
The lines $\langle x, f(x) \rangle$ and $\langle x, g(x) \cdot x + f(x) \rangle$ coincide for any homogeneous polynomial $g(x)$ of degree $d - 1$.



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It can be shown that this *singular locus* is nonempty and, *if finite*, consists of

$$\tfrac{d^{n+1}-1}{d-1} = 1 + d + \cdots + d^n$$

points counted with natural multiplicities.

space of foliations

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$$f = (f_0, \ldots, f_n) \in \mathcal{P}_d^{\oplus(n+1)}$$

$$x = (x_0, \ldots, x_n)$$

$$f \sim g \cdot x + f, \ g \in \mathcal{P}_{d-1}$$

(Euler) exact seq.:

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\mathcal{P}_{d-1} \arrow[r, tail] & \mathcal{P}_d^{\oplus(n+1)} \arrow[r, two heads] & \mathcal{P}_d^{\oplus(n+1)} \Big/ \Big(\mathcal{P}_{d-1} \cdot x\Big) \\
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$$\dim \mathbb{F}(n,d) = (n+1)\binom{n+d}{n} - \binom{n+d-1}{n} - 1.$$

$$\left(\mathcal{P}_{d-1} \rightarrowtail \mathcal{P}_d^{\oplus(n+1)} \twoheadrightarrow H^0(\mathbb{P}^n, T\mathbb{P}^n(d-1)) \right)$$

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Goal: force $\# \operatorname{Sing} \mathscr{F} = \infty$.

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That's our object of study.

Inspired by recent work of

Maurício Barros Correa Jr.,

G. N. Costa, Arturo Ulises Fernandez Perez,

and Renato Vidal da Silva Martins,

Foliations by curves with curves as singularities,

arxiv:1209.5618

foliations singular along a curve

Previous work by G.N.Costa *et al.* assumed the foliation to be *special* along the curve, *i.e.*, blowing up the curve produces a foliation leaving invariant the exceptional divisor.

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Instead of Baum-Bott, we use Fulton's residual intersection toolbox.

isolated residual singularities

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Theorem. *Let* $\mathscr{F}$ *be a foliation of degree* d *in* $\mathbb{P}^3$, *general among those which are singular along a smooth curve* C *of degree* m *and genus* g.

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Theorem. *Let $\mathscr{F}$ be a foliation of degree d in $\mathbb{P}^3$, general among those which are singular along a smooth curve C of degree m and genus g. For all $d >> 0$, the scheme of singularities of $\mathscr{F}$ is a disjoint union of C and a finite set, F, with*

$$\#F = \int c_3\big(T\mathbb{P}^3(d-1) \otimes \mathcal{O}_X(-E)\big)$$

$$= d^3 + d^2 - (3m-1)d + 3m - 1 + 2g.$$

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as well as replacing C by any closed subscheme of $\mathbb{P}^n$ with “known” Segre class.

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If $d >> 0$, *the scheme of singularities of* $\mathscr{F}$ *is a disjoint union,*

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with $F =$ *finite set of isolated singularities*

(to be eventually accounted for.)

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Lemma. *Let* C *be a closed subscheme of an integral projective scheme* X *of dimension* n*, with ideal sheaf* $\mathcal{J}$*. Let* $\mathcal{E}$ *be a locally free sheaf over* X *of rank* n*. Assume* $\mathcal{J} \cdot \mathcal{E}$ *globally generated and* $H^1(X, \mathcal{J} \cdot \mathcal{E}) = 0.$ *Then a general section in* $H^0(X, \mathcal{J} \cdot \mathcal{E})$ *has scheme of zeros supported on* C *union finitely many points away from* C*.*

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Proof. Previous lemma $\Rightarrow$ for general

$$\sigma \in \mathcal{U} = H^0(X, \mathcal{J} \cdot \mathcal{E}),$$

its set of zeros is a disjoint union of C and some finite set.

We must show that the scheme of zeros of σ coincides with C at all points on C.

We have the surjections of sheaves

$$\mathcal{U} \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{J} \cdot \mathcal{E}, \qquad (\mathcal{U} = H^0(X, \mathcal{J} \cdot \mathcal{E}))$$

$$\mathcal{U} \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{J} \cdot \mathcal{E}_{|C} = \mathcal{J}/\mathcal{J}^2 \otimes \mathcal{E}_{|C} = \mathrm{Hom}(\mathcal{E}^\vee_{|C}, \mathcal{J}/\mathcal{J}^2) =: \mathrm{H}.$$

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Now ρ_y sends σ to the element σ' of $\mathbf{H}_y$ such that,
for each coordinate functional $e_i^\vee$, we get

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$$\mathrm{H} = \mathrm{Hom}(\mathcal{E}_{|C}^\vee, \mathcal{J}/\mathcal{J}^2),$$

we get $\mathcal{I}_y + \mathcal{J}_y^2 = \mathcal{J}_y \subseteq \mathcal{O}_{X,y}$,

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$$\mathrm{H} = \mathrm{Hom}(\mathcal{E}_{|C}^\vee, \mathcal{J}/\mathcal{J}^2),$$

we get $\mathcal{I}_y + \mathcal{J}_y^2 = \mathcal{J}_y \subseteq \mathcal{O}_{X,y}$,

whence Nakayama obtains $\mathcal{J}_y = \mathcal{I}_y$ for $y \in C$.

$$\sigma'(e_i^\vee) = f_i + \mathcal{J}_y^2 \quad \text{in} \quad \mathcal{J}_y/\mathcal{J}_y^2.$$

Since $\mathcal{I}_y = \langle f_1, \ldots, f_n \rangle \subseteq \mathcal{O}_{X,y}$,

the image of the map σ' is $\mathcal{I}_y + \mathcal{J}_y^2 \bmod \mathcal{J}_y^2$.

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This shows . . .

Prop. *Let $\mathscr{F}$ be a general foliation of degree d singular along an integral l.c.i. subscheme C.*

If $d >> 0$, the scheme of singularities of $\mathscr{F}$ is a disjoint union,

$$\operatorname{Sing}\mathscr{F} = F \cup C,$$

with F = finite set of isolated singularities

(to be eventually accounted for.)

Henceforth, just assume

$$\mathrm{Sing}(\mathscr{F}) = F \cup C$$

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Let the section $\xi : \mathcal{O} \to T\mathbb{P}^3(d-1)$ define $\mathscr{F}$.

$$\xi : \mathcal{O} \to T\mathbb{P}^3(d-1)$$

$$\begin{tikzcd}
\xi : \mathcal{O} \to T\mathbb{P}^3(d-1) \rightsquigarrow \xi^\vee : & \mathcal{O} & (T\mathbb{P}^3(d-1))^\vee \arrow[l] \arrow[dl, two heads] \\
& \mathcal{I}(C) \cdot \mathcal{I}(F) \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\xi : \mathcal{O} \arrow[r] & T\mathbb{P}^3(d-1) \arrow[r, squiggly] & \xi^\vee : \mathcal{O} & (T\mathbb{P}^3(d-1))^\vee \arrow[l, long] \arrow[ddl, two heads] \\
 & & \cup & \\
 & & \mathcal{I}(C)\cdot\mathcal{I}(F) &
\end{tikzcd}$$

Let $p : \mathbb{X} \to \mathbb{P}^3$ be the blowup along C.

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Let $p : \mathbb{X} \to \mathbb{P}^3$ be the blowup along C.

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& \cup \\
& \mathcal{I}(F)\mathcal{O}_{\mathbb{X}} \subset \mathcal{O}_{\mathbb{X}}
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and therefore get section of $p^\star T_{\mathbb{P}^3}(d-1) \otimes \mathcal{O}_{\mathbb{X}}(-\mathbb{E})$

which vanishes exactly on F.

We find

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In the case of a line,

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$$m = 1, g = 0, \rightsquigarrow d^3 + (d-1)^2 + 1.$$

For instance, a foliation of degree $d = 2$,

general among those which are singular along a line

admits **10** additional isolated singularities.

Heuristically, the line absorbs 5 of the 15 $(= 2^4 - 1)$ expected singularities for a general foliation of degree 2 in $\mathbb{P}^3$.

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For example, the foliation defined by

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presents 10 distinct isolated singularities besides the line $x_0 = x_1 = 0$.

For a general foliation $\mathscr{F}$ in $\mathbb{P}^n$ of degree $>> 0$ singular along a subvariety $C \subset \mathbb{P}^n$ of dimension $0 \leq m < n-1$, we'll have a finite residual scheme $F \subset \operatorname{Sing} \mathscr{F}$,

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$$\#F = \int_{\mathbb{X}} c_n(T\mathbb{P}^n(d-1)(-\mathbb{E})) = \sum_0^n (-1)^i c_{n-i}(T\mathbb{P}^n(d-1))\mathbb{E}^i =$$

$$\textstyle\int_{\mathbb{P}^n} c_n(T\mathbb{P}^n(d-1)) + \sum (-1)^{i-1} \mathrm{seg}_{n-m-i}(C, \mathbb{P}^n) c_i T\mathbb{P}^n(d-1))$$

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at least under suitable hypotheses, *e.g.*, C integral, l.c.i.

a couple of hidim examples

number of residual isolated singularities in presence of a

Veronese

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Next,

enumerate the few

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Noether-Lefschetz theory tells us that

any curve in a very general surface F

of degree $\deg F \geq 4$ in $\mathbb{P}^3$ is of the form

$F \cap G$ for some surface $G \subset \mathbb{P}^3$.

enumerate the few

Those not so general F which contain,

say some (variable) line,

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There are *polynomial* formulas for their degrees.

recent work

D. Maulik & R. Pandharipande:

Gromov-Witten theory and Noether-Lefschetz theory

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J.A.D.Maia, A.Rodrigues Silva, I.V. & F.Xavier,
Enumeration of surfaces . . . curve of low degree, J. P. A. Algebra, 2013

foliations singular along a curve

Similarly, a very general 1-dim foliation in $\mathbb{P}^3$ admits only finitely many singularities.

foliations singular along a curve

Similarly, a very general 1-dim foliation in $\mathbb{P}^3$ admits only finitely many singularities.

Given a family W of curves in $\mathbb{P}^3$, consider the subvarieties

$$\Sigma(W, d) \ \subset \ \mathbb{F}(3, d)$$

in the projective space of foliations of degree d, defined by the condition that the singular locus contain some member of W.

foliations singular along a curve

We show that *the degree of* $\Sigma(W,d) \subset \mathbb{F}(3,d)$ *is given by a polynomial* $q^W(d)$ *for all* $d >> 0$.

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We make it explicit in a few examples.

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We show that *the degree of* $\Sigma(W, d) \subset \mathbb{F}(3, d)$ *is given by a polynomial* $q^W(d)$ *for all* $d >> 0.$ We make it explicit in a few examples.

These examples seem to suggest that the degree of the polynomial $q^W(d)$ is equal to twice the dimension of W, though we only manage to bound it by thrice that dim.

foliations singular along a curve

This is reminiscent of the recent work around Göttsche's conjecture on the polynomial behaviour of the numbers of singular curves on surfaces.

S. Kleiman, R. Piene, *Enumerating singular curves on surfaces*, Cont. Math. 241, 209-238, 1999. math.AG/9903192.

———–, *Node polynomials for families: methods and applications*, Math. Nachr. 271, 69-90, 2004. math.AG/0111299.

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Y. Tzeng, *A Proof of the Göttsche–Yau–Zaslow Formula*, J. Differential Geom. 90 (2012), No. 3, 439-472.

M. Kool, V. Shende and R. Thomas, *A short proof of the Göttsche conjecture*, Geometry & Topology 15 (2011) 397-406

Jun Li, Yu-jong Tzeng, *Universal polynomials for singular curves on surfaces*, arXiv:1203.3180v1

down to earth example

Foliations of degree 1: $\mathbb{F}(n,1) = \mathbb{P}^{(n+1)^2-1-1}$,

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Impose one singular line $\rightsquigarrow$

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W & & \mathbb{P}^{14} \supset \Sigma(W, 1) \arrow[d, equal] \\
& & \{A \mid A_{|V} = c \cdot I_V \text{ for some } V \in W\}
\end{tikzcd}$$

(for some c)

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find $\deg \Sigma(W,1)$ via Schubert Calculus

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$$\{\underbrace{(V, A)}_{\dim V = 2} \mid AV \subseteq V\}$$

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cycle of codimension 3 and degree **20** *in* $\mathbb{P}^{14}$
supported on the set of foliations of degree 1
singular along some variable line.

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and degree given by a polynomial $q^W(d)$ *of degree* $\leq n \dim W$.

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We mimic arguments from S.C.Coutinho-J.V.Pereira (Crelle v.594, 2006) and Cukierman-Lopez-V (Proc.AMS).

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Consider the correspondence

$$\begin{array}{c} \widetilde{\Sigma}(W,d) \subset W \times \mathbb{P}^N \\ || \\ \{(C,\mathscr{F}) \in W \times \mathbb{P}^N \,|\, C \subseteq \operatorname{Sing}\mathscr{F}\} \end{array}$$

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$\widetilde{\Sigma}(W,d)$ is a projective subbundle with total dimension equal to the expected one $\forall\, d >> 0$.

Grosso modo, the condition that
a given C lie in $\operatorname{Sing}\mathscr{F}$ is *linear* on $\mathscr{F}$.

Now the main point is that the number of linearly independent conditions is dictated by the Hilbert polynomial $P_W(d)$ for $d >> 0$.

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projectivization of an explicit vector bundle

$\mathcal{U}_d \to W$, which fits into an exact sequence,

$$\mathcal{U}_d \rightarrowtail H^0(T\mathbb{P}^n(d-1)) \times W \twoheadrightarrow \mathcal{V}_d.$$

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The degree of the image of $\widetilde{\Sigma}(W,d) = \mathbb{P}(\mathcal{U}_d)$,

$$\Sigma(W,d) \subset \mathbb{P}^N = \mathbb{P}\big(H^0(T\mathbb{P}^n(d-1))\big),$$

can be calculated as a

top-dimensional Segre class of the

vector bundle $\mathcal{U}_d$.

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By hypothesis, C is integral and $\dim C > 0$.

We must show that C is the sole member of W which appears as a subscheme of $\mathrm{Sing}(\mathscr{F})$.

Arguing as in Bertini, it can be shown that

$\mathrm{Sing}(\mathscr{F}) \setminus C$ is finite.

Thus, if C' is a member of W appearing in $\mathrm{Sing}\mathscr{F}$,

then $C'_{red} = C \cup F$, disjoint union with F finite.

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Since C is integral and the Hilbert polynomials

are the same, it follows that $F = \emptyset$ and $C' = C$ *as schemes*, hence $\delta = 1$. $\square$

Show next how to compute the vector bundles $\mathcal{U}_d$, $\mathcal{V}_d$.

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The Euler sequence reads

$$(\bigstar) \qquad \begin{tikzcd} \mathcal{O}_{\mathbb{P}^n} \arrow[r, tail] & \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathbb{C}^{n+1} \arrow[r, two heads] & T\mathbb{P}^n \\ 1 \arrow[r, mapsto] & \sum x_i \otimes e_i. & \end{tikzcd}$$

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Take the tensor product of $(\bigstar)$ by $\mathcal{O}_{\mathbb{P}^n}(d-1)$; pullback to $\mathbb{P}^n \times W$ and restrict over Γ :

Get the diagram of sheaves over $\mathbb{P}^n \times W \supset \Gamma$,

$$\begin{tikzcd}\mathcal{O}_{\mathbb{P}^n}(d-1) \arrow[r, tail] & \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_W^{\oplus(n+1)} \arrow[r, two heads] & T\mathbb{P}^n(d-1)\end{tikzcd}$$

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$$\Gamma \subset \mathbb{P}^n \times W \xrightarrow{q} W$$

Take direct image to W; we find the diagram of vector bundles $/W$,

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)) \times W \longrightarrow\!\!\!\!\rightarrow q_\star(\mathcal{O}_{\mathbb{P}^n}(d-1)_{|\Gamma})$$

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exact, as soon as $R^1 q_\star(O_\Gamma(d-1)) = 0$;

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(The previous exact sequences will be used to calculate the equivariant Chern classes of $\mathcal{V}_d$ in the examples below.)

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Recall

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is a direct image, we are in a situation similar to the one in Cukierman-Lopez-V.

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Hence, $c_w \mathcal{V}_d$ is a polynomial in d of degree $\leq wn$.

□

examples: impose one line

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We also remark the degree of $q^W(d)$,

$$\mathbf{3}+\mathbf{5}=\mathbf{2}\dim W.$$

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Similarly, for the case $W = \{$ **plane cubics** $\}$:

$$\begin{gathered}\deg\Sigma(W,d) = \binom{d-1}{2} \prod_{4}^{9} (3d^2 - 7d + 2i)\Big(18225d^{10} - 70227d^9 - 4698d^8 \\ +641154d^7 - 1960839d^6 + 2883213d^5 - 1781136d^4 - 722428d^3 \\ +1736640d^2 - 356160d - 506880\Big)/(2^{19} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11).\end{gathered}$$

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(Compare deg's and dim's.)

$W :=$ {**twisted cubics**}

We use the fixed points as detected in Norway; (*seemywww*)

$$
\begin{gathered}
(d-1)\Big(443753235d^{\mathbf{23}} - 1211154039d^{22} - 3260272437d^{21} + 34746612039d^{20} \\
-98418674160d^{19} + 22215446370d^{18} + 674998119198d^{17} \\
-1675105129746d^{16} - 2797122199293d^{15} + 32236140084573d^{14} - 122958699518769d^{13} \\
+310213229195931d^{12} - 588389379312598d^{11} + 886855102748712d^{10} - \\
1108254006266728d^{9} + 1205114928345920d^{8} - 1193890014288160d^{7} \\
-1074684757123712d^{6} - 794759284927104d^{5} + 364343257677056d^{4} + 40601134718976d^{3} \\
-214458571542528d^{2} + 155901732618240d - 44144787456000\Big)/(2^{17} \cdot 3^{4} \cdot 5^{2} \cdot 7 \cdot 11).
\end{gathered}
$$

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\begin{gathered}
\Big(18952050714507d^{\mathbf{32}} - 402650304098544d^{31} + 4068442396247976d^{30} - 24192653251673664d^{29} + \\
76641743386171764d^{28} + 54079300221748704d^{27} - \ 2162076040303606728d^{26} + \\
13911138159415859232d^{25} - 55198672640285530590d^{24} + 145172858185740767616d^{23} - \\
164119957419188973384d^{22} - 898572428296095163104d^{21} + 865602089052216639598 8d^{20} - \\
4960669410288110782704 0d^{19} + 22931884508782577088064 8d^{18} - 89207213554370867226460 8d^{17} \\
+293504628198249442501604 3d^{16} - 816944543343257290517160 0d^{15} + \\
192338722484533193806357 44d^{14} - 382564613824387936790956 48d^{13} + \\
640579065511309420457615 68d^{12} - 896454186993663869215074 56d^{11} + \\
103477420624760229650806 528d^{10} - 961829304772949694215347 20d^{9} + \\
686087218399496104281484 80d^{8} - 332392410093469283986903 04d^{7} + \\
589254759584514762858905 6d^{6} + 538490147093784302703411 2d^{5} - \\
407935012760544377634816 0d^{4} - 686377277683583753650176d^{3} + 267675945654464000557056 0d^{2} \\
-171986645588284570337280 0d + 418630239595794530304000\Big) / (2^{28} \cdot 3^{5} \cdot 5^{2} \cdot 7^{2} \cdot 11 \cdot 13).
\end{gathered}
$$

higher dimension

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higher dimension

Some experiments seem to indicate that, *for a family W of dimension $\mathbf{w}$ of closed subschemes of dimension $\mathbf{k}$ in $\mathbb{P}^n$, the subvariety of $\mathbb{F}(n,d)$ of one-dim foliations singular along a member of W, has degree given by a polynomial $q^W(d)$ of degree*

$$\boxed{(\mathbf{k}+\mathbf{1})\mathbf{w}.}$$

quadric surfaces in $\mathbb{P}^4$:

$$\begin{gathered}
\binom{d}{\mathbf{2}}\binom{d+3}{\mathbf{2}}\prod_{1}^{5}\left(4d^{\mathbf{3}}+3d^2-7d+6(i+3)\right) \\
\Big(12517376d^{\mathbf{20}}+139198464d^{19}+481345536d^{18} \\
+257327104d^{17}-742813184d^{16}+2949311488d^{15}+ \\
6409660608d^{14}-11158877696d^{13} \\
-1826835065d^{12}+44161688960d^{11}-33681870799d^{10}- \\
34375182142d^{9}+108574356973d^{8}-607306366684d^{7} \\
+33199026071d^{6}+39432298290d^{5}-106330304700d^{4}+ \\
284976484632d^{3}-227173139136d^{2}+155118789504d- \\
49884595200\Big)/(2^{23}\cdot 3^{17}\cdot 5\cdot 7\cdot 11\cdot 13).
\end{gathered}$$

(:- *THANKS FOR LISTENING* :-)