Characteristic Classes in Algebraic Geometry

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to my father, in memoriam

Preface

The aim of this book is to present a few basic results of intersection theory *à la* Fulton-MacPherson. It grew out of class notes, elaborating on material covered by the Portuguese version, "Classes características em Geometria Algébrica", course notes for the XV Colóquio Brasileiro de Matemática, 1985. Our exposition is meant to be utilitarian, in the sense that we avoid excessive generality and do not worry much about self-sufficiency.

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product of the degrees of the curves. A proof will be given later on, cf. 3.16.
(2) Let t be a constant and let Z_t denote the subscheme of $\mathbb{A}^4$ defined by the ideal

$$I_t = \langle x, y\rangle \cap \langle z, w\rangle + \langle x - z - t, y - w\rangle.$$

This is the intersection of the plane

$$\pi_t : \ x = z + t, y = w$$

with the union, call it U, of two other planes passing through the origin in $\mathbb{A}^4$. We may compute the cycle

$$[Z_t] = (0, 0, -t, 0) + (t, 0, 0, 0) \quad \text{for} \quad t \neq 0$$

whereas

$$[Z_0] = 3(0, 0, 0, 0).$$

This example illustrates why the notion of intersection multiplicity should not always be taken naively as the length of the local ring of the scheme intersection. While the length of the scheme Z_0 is 3, the (not yet defined!) intersection of the two cycles $[U]$ and $[\pi_0]$ should capture somehow the number 2 suggested by the moving cycle $[Z_t]$.

Perhaps one of the most subtle points in intersection theory is the "correct" attribution of multiplicities in such a way that certain properties supported by intuition and interesting examples be always true. One such cherished property is the so-called "principle of continuity", clearly violated in the above example were we stubbornly sticking to the naïve geometric multiplicity. We shall see later that the correct intersection multiplicity is in fact expressible as a geometric multiplicity of a different scheme, namely, the normal cone (cf. §13, especially (13.7)). As a sort of warm up, the reader should study carefully the exercises 3, 4 below.

Exercises

1. Let $X = \mathbb{A}^n$, the affine space over a field K. The subvarieties of X of codimension 1 correspond to the nonzero principal prime ideals $\langle p(X_1, \ldots, X_n)\rangle \subseteq K[X_1, \ldots, X_n]$. Show that $\mathcal{C}_{n-1}\mathbb{A}^n$ is naturally isomorphic to $R(X)^\star/K^\star$, the multiplicative group of nonzero rational functions, modulo constants.

2. Let $F \in K[X_0, \ldots, X_n]$ be a homogeneous polynomial and let $Z_1, \ldots, Z_m$ denote the irreducible components of the hypersurface $\mathcal{Z}(F) \subset \mathbb{P}^n$, so that

Introduction

Algebraic geometry is often described in simple terms as the study of systems of polynomial equations. We could say likewise that intersection theory is mainly aimed at the calculation of the number of solutions, counted with appropriate multiplicities. A typical statement is Bézout's theorem: two plane curves of degrees m, n and without common components meet in mn points, counted with multiplicities. Intuitively, a curve defined by a polynomial equation of degree m can be deformed, under a continuous variation of its coefficients, into the union of m lines in general position. Thus, some sort of principle of conservation of numbers should convince us that, as each curve is deformed into a union of lines, the solutions should deform continuously to the mn intersections of the lines. Intersection theory provides us with a rigorous formalization of the process of deformation as well as with a justification for the invariance of the desired result under such permissible deformations. Moreover, it gives us quite a machinery for the actual computation of the number of solutions.

Blanket assumptions. Schemes will be assumed of finite type over a field, often $\mathbb{C}$, unless stated otherwise.

Variety means reduced and irreducible scheme. A subvariety of a scheme is assumed to be closed, reduced and irreducible. The function field of a variety V will be written $R(V)$.

Chapter 1

Cycles

The notion of a cycle appears as a natural extension of the idea of divisor of zeros and poles of rational functions.

1.1. Definition. Let X be a scheme. The **group of cycles of dimension** k is the free abelian group generated by the closed, irreducible subvarieties of dimension k of X. It will be denoted by $\mathcal{C}_k X$. The **group of cycles** is the graded group

$$\mathcal{C}_\star = \bigoplus \mathcal{C}_k X.$$

Notice that $\mathcal{C}_k X$ is nonzero only for $0 \leq k \leq \dim X$. We also observe that, since the closed, irreducible subvarieties of X and its associated reduced subscheme X_{red} are the same, we have

$$\mathcal{C}_k X = \mathcal{C}_k X_{red}, \, \forall k.$$

By definition, each k-cycle $c \in \mathcal{C}_k X$ can be written in a unique way as a linear combination with coefficients in $\mathbb{Z}$,

$$c = \sum_V n_V V,$$

where V runs in the collection of closed, irreducible subvarieties of X of dimension k. Here, the integral coefficient n_V is zero except for finitely many V's.

The **support** of c is defined as

$$|c| = \bigcup_{n_V \neq 0} V. \tag{1.1}$$

1.2. Example. (1) If X is an irreducible variety of dimension n, then $\mathcal{C}_n X = \mathbb{Z} \cdot X$.

(2) Suppose $X = \mathbb{A}^1$, the affine line over a field K. The subvarieties of X of dimension 0 correspond to the maximal ideals $\langle p(t) \rangle \subseteq K[t]$. Hence, $\mathcal{C}_0 \mathbb{A}^1$ is naturally isomorphic to $K(t)^\star / K^\star$, the multiplicative group of nonzero rational functions, modulo constants. If $f \in K(t)^\star$, we may write $f = a \prod p^{e_p}$, with $p \in K[t]$ monic and irreducible, $a \in K^\star$ and the exponents $e_p \in \mathbb{Z}$. We associate to the class $\overline{f} \in K(t)^\star / K^\star$ the cycle $\sum e_p \mathcal{Z}(p)$, where $\mathcal{Z}(p)$ stands for the subvariety defined by p.

1.3. Definition. Let $X_1, \ldots, X_m$ be the irreducible components (with reduced structure) of a scheme X. The **fundamental cycle** of X is

$$[X] = \sum m_i X_i \tag{1.2}$$

where

$$m_i = \ell(\mathcal{O}_{X,X_i})$$

denotes the length of the local ring of X at the generic point of X_i.

We recall that $\mathcal{O}_{X,X_i}$ is an artinian local ring; thus, its length m_i is a positive integer, called the geometric multiplicity of X along X_i.

Similarly, we define the fundamental cycle associated to a closed subscheme $Z \subseteq X$, either as an element of $\mathcal{C}_\star Z$ or $\mathcal{C}_\star X$.

1.4. Examples. (1) Let F, G be plane curves with no components in common. Set $Z = F \cap G$, scheme intersection. For each point $P \in Z$, the local ring $\mathcal{O}_{Z,P}$ is of finite length, denoted $i(P, F \cdot G)$. This is also called the intersection index or intersection multiplicity of F, G at P (cf. (11.4), p. 89 for a generalization). We have, in the present case, the formula

$$i(P, F \cdot G) = \dim_K \mathcal{O}_{Z,P},$$

vector–space dimension over the ground field K, assumed algebraically closed. Indeed, if an artinian local ring $\mathcal{O}$ contains a field K such that the residue field R is a finite algebraic extension, then we have $\ell(\mathcal{O}) = \dim_K \mathcal{O} / \dim_K R$; see (3.5), p. 11. Since K is algebraically closed, we have $R = K$, hence $\ell(\mathcal{O}) = \dim_K \mathcal{O}$.

The classical theorem of Bézout asserts that

$$\sum_P i(P, F \cdot G) = \deg(F) \deg(G).$$

2.7. Important remark. If X is of dimension n, then $\mathcal{R}_n X = 0$ by the very reason that there is no subvariety of X of dimension $n+1$. Hence $\mathcal{A}_n(X) = \mathcal{C}_n X$ holds.

Of course we also have $\mathcal{A}_i(X) = 0$ if $i < 0$ and for $i > n$. If X is a variety of dimension n, we have $\mathcal{A}_n(X) = \mathcal{C}_n X = \mathbb{Z}$.

2.8. Examples. (1) $\mathcal{A}_{n-1}(\mathbb{A}^n) = 0$.

(2) $\mathcal{A}_{n-1}(\mathbb{P}^n) = \mathbb{Z} \cdot h$, the free abelian group generated by h, the class of a hyperplane. Indeed, let F_d be a homogeneous polynomial of degree d and let $\mathcal{Z}(F_d) \subset \mathbb{P}^n$ be the corresponding hypersurface. Consider the rational function $r = F_d/F_1^d \in R(\mathbb{P}^n)$. We may write

$$[r] = [\mathcal{Z}(F_d)] - d[\mathcal{Z}(F_1)].$$

Since $h = [\mathcal{Z}(F_1)]$ clearly is not torsion, it follows that $\mathcal{A}_{n-1}(\mathbb{P}^n)$ is in fact free, generated by the hyperplane class. We shall see later on that the other $\mathcal{A}_k(\mathbb{P}^n)$ are also isomorphic to $\mathbb{Z}$ for k between 0 and n (cf. (5.3), p. 30), generated by the class of a subspace of dimension k.

(3) Let X be a smooth curve. Each non constant rational function $r \in R(X)$ induces a dominant morphism $\widetilde{r} : X \to \mathbb{P}^1$. The non-empty fibers of $\widetilde{r}$ give rise to 0-cycles rationally equivalent to each other. Indeed, for each $P \in X$ we have $\mathrm{ord}_P(r) > 0$ if r is regular at P and $r(P) = 0$. Assume $0, \infty$ are in the image of $\widetilde{r}$. It follows that

$$[\widetilde{r}^{-1}(0)] - [\widetilde{r}^{-1}(\infty)] = [r].$$

Now for any pair of distinct points $Q, Q' \in \mathbb{P}^1$ lying in the image of $\widetilde{r}$, we may find an automorphism α of $\mathbb{P}^1$ sending the pair to $0, \infty$.

Exercises

8. Show that $\mathcal{A}_0(\mathbb{A}^n) = 0$ for $n > 0$.

9. *Some properties of the length.* Let M be a finitely generated $A-$module. Suppose $M \neq 0$. Let P be a maximal element in the collection of annihilators, $\{an(m) \mid m \in M,\, m \neq 0\}$. Show that P is a prime ideal. Deduce by noetherian induction that M has a filtration, $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ consisting of submodules with quotients M_i/M_{i+1} isomorphic to A/P_i where each P_i is a prime ideal. Show that M is of finite length if and only if each P_i is a maximal ideal.

each $Z_i = \mathcal{Z}(F_i)$ for some irreducible polynomial F_i. Show the equality of cycles,

$$[\mathcal{Z}(F)] = \sum_i e_i Z_i$$

where $\prod_i F^{e_i} = F$ is the decomposition of F into irreducible factors.

3. Let X be an irreducible (not necessarily reduced) scheme and let $X_0 = X_{red}$ be the supporting subvariety. Show that if $[X] = m[X_0]$ for some $m \in \mathbb{Z}$ then likewise $[X \times \mathbb{A}^n] = m[X_0 \times \mathbb{A}^n]$.

4. Set $A = k[x,y]/\langle x^2, xy, y^2\rangle$ and let $X = \mathrm{Spec}(A)$. Put $N = X \times \mathbb{A}^2 = \mathrm{Spec}(A[u,v])$. Let $C \subset N$ be the subscheme defined by the ideal $J = \langle xv - yu\rangle$. Compute the cycle $[C]$.

5. Set $B = k[x,y,z,w]/\langle xz, xw, yz, yw\rangle$ and let $I = \langle x-z, y-w\rangle B$. Put $R = \bigoplus I^n/I^{n+1} = A \bigoplus I/I^2 \bigoplus \cdots$ with A as in the previous exercise. Show that there is a surjection $A[u,v] \twoheadrightarrow R$ and determine the kernel.

6. Verify the assertions in example *1.4(2), p. 3* .

7. Let Z_t denote the subscheme of $\mathbb{A}^3$ defined by the ideal $\langle y,z\rangle \cap \langle x, z-t\rangle$ (the union of two skew lines). Let π be the plane $y = x$. Calculate the cycle of $\pi \cap Z_t$ for each $t \in \mathbb{A}^1$.

Chapter 2

Rational equivalence

The group of cycles is much too big to be interesting. We introduce now an adequate subgroup of relations. The quotient, *grosso modo*, identifies cycles that arise as fibers of a morphism $X \to \mathbb{P}^1$.

First we discuss the notion of order of a rational function.

2.1. Definition. Let V be a variety and let $r \in R(V)$ be a non-zero rational function. The **order** of r along a subvariety $W \subset V$ of codimension one is defined by the formula

$$\mathrm{ord}_W^V(r) = \mathrm{ord}_W(r) = \ell(A/\langle a\rangle) - \ell(A/\langle b\rangle),$$

where $A = \mathcal{O}_{V,W}$, $r = a/b$, with $a, b \in A$.

Of course we are required to show the following.

2.2. Lemma. *Let A be a noetherian ring of Krull dimension one. For each non zero divisors $a, b \in A$, we have*

$$\ell(A/\langle ab\rangle) = \ell(A/\langle a\rangle) + \ell(A/\langle b\rangle).$$

Proof. For any A-module M and any ideal $I \subseteq A$ such that $IM = 0$, it is clear that $\ell_A(M) = \ell_{A/I}(M)$. Since the length is additive, the result follows from the natural exact sequence of A-modules,

$$R/\langle b\rangle \simeq \langle a\rangle/\langle ab\rangle \hookrightarrow A/\langle ab\rangle \twoheadrightarrow A/\langle a\rangle.$$

□

The lemma ensures that $\mathrm{ord}_W(\cdot)$ is well defined *i.e.*, it does not depend on the representation $r = a/b$ and it is furthermore additive:

$$\mathrm{ord}_W(rr') = \mathrm{ord}_W(r) + \mathrm{ord}_W(r') \quad \forall\, r, r' \in R(V)^\star. \tag{2.1}$$

2.3. Example. If A is a discrete valuation ring with uniformizer t, then $\ell(A/\langle a\rangle) = e$ whenever $a \in A$, $a \neq 0$ is written in the form $a = ut^e$ with u invertible in A.

2.4. Definition. The **cycle of a rational function** $r \in R(V)^\star$ is defined by

$$[r] = \sum_W \mathrm{ord}_W(r) \cdot W$$

where the sum extends over the collection of subvarieties of codimension one of V.

We must verify that $\mathrm{ord}_W(r) = 0$ except for finitely many W's.

Let $U \subseteq V$ be an open affine subset. Write $r = a/b$ with a, b regular functions on U. It follows that the set

$$\{W \mid W \cap U \neq \emptyset \text{ and } \mathrm{ord}_W(r) \neq 0\}$$

is contained in

$$\{W \mid W \cap U \text{ is an irreducible component of } \mathcal{Z}(ab)\}.$$

Since the number of irreducible components is finite and recalling that V can be covered by finitely many affine open subsets, the verification is complete.

2.5. Example. If $V = \mathbb{A}^n$ and $r \in R(V)^\star \setminus K^\star$, we may write $r = \prod f^{e_f}$ with f an irreducible polynomial. We have $[r] = \sum_f e_f \mathcal{Z}(f)$.

2.6. Definition. Let X be a scheme. The **group of $k-$cycles rationally equivalent to zero** is the subgroup $\mathcal{R}_k X \subset \mathcal{C}_k X$ generated by the cycles of rational functions of subvarieties of X of dimension $k+1$.

The quotient graded group,

$$\mathcal{A}_\star(X) = \bigoplus \mathcal{A}_k(X) = \bigoplus \mathcal{C}_k X \big/ \mathcal{R}_k X \tag{2.2}$$

is called the **Chow group** of X.

Two cycles are said to be **rationally equivalent** to each other when they represent the same class modulo $\mathcal{R}_\star X$.

Our next goal is to show that rational equivalence is preserved by arbitrary proper pushforward. The following lemmata will be needed.

3.4. Lemma. *Let B be a ring*[1] *and let M be a nonzero module of finite type. Then M admits a composition series,*

$$M = M_0 \supset M_1 \supset \cdots \supset M_\ell = 0$$

with quotients $M_i/M_{i+1} = B/P_i$ where P_i is a prime ideal in B. Moreover, M is of finite length if an only iff each P_i is maximal.

Proof. Consider the collection $\{\mathrm{ann}(m) \mid m \in M \setminus 0\}$. Let $P = \mathrm{ann}(v)$ be a maximal member. Then P is a prime ideal. Indeed, let $a, b \in B, ab \in \mathrm{ann}(v), a \notin P$. Then $av \neq 0, \mathrm{ann}(av) \supseteq \mathrm{ann}(v)$, hence $b \in \mathrm{ann}(av) = \mathrm{ann}(v)$ by maximality. Thus B/P sits in M as the submodule spanned by v. By noetherian induction, $\overline{M} = M/\langle v\rangle$ admits a composition series as stated, and so does M. Now if P is a prime ideal, then $ovB = B/P$ is a domain. If the length of $\overline{B}$ is finite, then it must be a field. Indeed, pick $b \in \overline{B}, b \neq 0$; we have $\langle b^n\rangle = \langle b^{n+1}\rangle$ for some $n \geq 1$. It follows $b^n = xb^{n+1}$ for some x, hence $xb = 1$. □

3.5. Lemma. (1) *Let B be a ring and let M be a module of finite length. Then we have*

$$\ell_B(M) = \sum_{P \in \mathrm{Spec}(B)} \ell_{B_P}(M_P).$$

(2) *If $\alpha : A \to B$ is a local homomorphism then*

$$\ell_A(M) = \ell_B(M)[R(B) : R(A)] \quad (R(A) = \text{residual field.})$$

(3) *Let $A \hookrightarrow B$ be a finite extension of one-dimensional rings. Assume there is some $c \in A$ which is a nonzero divisor in B such that $cB \subseteq A$. Then $\ell_A(B/A) < \infty$ and for any $a \in A$ we have $\ell_A(A/aA) = \ell_A(B/aB)$.*

3.6. Remark. The existence of the conductor c in (3) holds if B is the normalization of a domain (essentially) of finite type over a field.

Proof. Since length is additive, we may assume $M = B/P$. Now the first formula is trivial. For the second formula, first note that $\ell_A(M) = \ell_{A/Q}(M)$ for any ideal $Q \subset A$ such that $QM = 0$. Apply this to $M = B/P, P$ prime

[1] noetherian!

10. Let F, G be plane curves without common component. Show that

1. $i(P, F\cdot G)$, as defined in (1.4), p. 2, is equal to $\ell_{\mathcal{O}_{F,P}}(\mathcal{O}_{F\cap G,P})$.
2. Show that $i(P, F\cdot(GH)) = i(P, F\cdot G) + i(P, F\cdot H)$.

11. Let Z be an irreducible component of a scheme W. Let $Z_1, \ldots, Z_n$ be subvarieties of W distinct from Z. Show that any relation $mZ + \sum m_i Z_i = 0$ in $\mathcal{A}_\star(W)$ implies $m = 0$.

12. Let K be a field and write, as customary, $\mathbb{P}^1 = \mathrm{Spec}\,(K[t]) \cup \mathrm{Spec}\,(K[u])$ with $u = t^{-1}$. Let $\infty \in \mathrm{Spec}\,(K[u])$ correspond to the prime ideal $\langle u\rangle$. Show that a cycle $z \in \mathcal{C}_0\mathbb{P}^1$ is in $\mathcal{R}_0\mathbb{P}^1$ if and only if there are irreducible polynomials $p_i(t) \in K[t]$ and integers d, m_i such that $z = d\infty + \sum m_i \mathcal{Z}(p_i)$ with $d + \sum m_i \deg(p_i) = 0$.

Chapter 3

Direct Image

If X denotes a closed subscheme of a scheme Y, we have a natural inclusion of free abelian groups $\mathcal{C}_\star X \subseteq \mathcal{C}_\star Y$. We will show that it preserves rational equivalence. More generally, given a morphism $f : X \to Y$ of schemes, we shall define below a natural homomorphism $f_\star : \mathcal{C}_\star X \to \mathcal{C}_\star Y$. Whenever the morphism f is *proper*, we will show that $f_\star \mathcal{R}_\star X \subseteq \mathcal{R}_\star Y$, thereby inducing a homomorphism $\mathcal{A}_\star(X) \to \mathcal{A}_\star(Y)$.

3.1. Definition. Let $f : V \to W$ be a dominant morphism of irreducible varieties. We define the **degree** of f as

$$\deg(f) = \begin{cases} 0 & \text{if} \quad \dim V > \dim W; \\ [R(V) : R(W)] & \text{if} \quad \dim V = \dim W. \end{cases}$$

In the latter case, $R(V)$ and $R(W)$ are finitely generated field extensions with the same transcendence degree over the base field. Hence $\deg(f)$ is finite. The geometric content is the following: there exists an open dense subset $U \subseteq W$ such that for any $Q \in U$, the fiber $f^{-1}Q$ consists of $\deg(f)/i$ distinct points, where i is the inseparable degree of the field extension.

Let $p : X \to Y$ be a proper map of schemes. Recall that properness implies that $p(V)$ is closed in Y for each closed subset $V \subseteq X$. Let $V \subseteq X$ be a (closed irreducible) subvariety and let $W = p(V)$. Let $f : V \to W$ be induced by p. We put

$$p_\star V = \deg(f)\, W \quad \text{ in } \mathcal{C}_\star W. \tag{3.1}$$

We extend it by linearity to a homomorphism $p_\star : \mathcal{C}_\star X \to \mathcal{C}_\star Y$, called **direct image** or **pushforward**.

3.2. Remarks. (1) $p_\star$ preserves the grading by dimension:

$$p_\star \mathcal{C}_k X \subseteq \mathcal{C}_k Y \quad \forall k.$$

(2) *Functoriality*: if $q : Y \to Z$ is another proper map, we have

$$(qp)_\star = q_\star p_\star.$$

This follows at once from the multiplicativity of the degree of finite extension fields.

3.3. Examples. (1) If $Y = \operatorname{Spec}(K)$ with K a field, not assumed algebraically closed, then we have $\mathcal{C}_\star Y = \mathcal{C}_0 Y \simeq \mathbb{Z}$. If X is a complete scheme, *i.e.*, proper over K, for each closed point P in X we have $p_\star P = [R(P) : K]$, the degree of the residual field extension of P over K. Of course we have $p_\star \mathcal{C}_k X = 0$ for all $k > 0$ by trivial dimension reason!

(2) Let $X = \mathbb{P}^1_K$, $Y = \operatorname{Spec}(K)$ and let $p : X \to Y$ be the structure morphism. Let $r \in R(X)$ be a rational function ($\neq 0$). Then $p_\star[r] = 0$ in $\mathbb{Z} = \mathcal{C}_0 Y$. To see this, write $X = \mathbb{A}^1_K \cup \{\infty\}$, with $\mathbb{A}^1_K = \operatorname{Spec}(K[t])$. We have $r = f/g$ for some co-prime polynomials $p, q \in K[t]$. By additivity, it suffices to verify that $p_\star[f] = 0$. By the same reason, we may assume f irreducible in $K[t]$. The scheme of zeros $\mathcal{Z}(f)$ is therefore an irreducible subvariety of $\mathbb{A}^1_K \subset X$, consisting of a single point P, possibly non-rational over K. We have $R(P) = K[t]/\langle f \rangle$, a field extension of degree equal to the degree $d = \deg(f)$ of the polynomial f. We need the values of $\operatorname{ord}_Q(f)$ at all points of X. We may write

$$\operatorname{ord}_Q(f) = \begin{cases} 1 & \text{for} \quad Q = P; \\ 0 & \text{for} \quad Q \in \mathbb{A}^1 \setminus \{P\}. \end{cases}$$

At infinity, we have that $u = 1/t$ is a local uniformizer. From the identity $f(t) = a_d t^d + \cdots + a_1 t + a_0 = t^d(a_d + a_{d-1}u + \cdots + a_0 u^d)$ we deduce

$$\operatorname{ord}_\infty(f) = \operatorname{ord}_\infty(t^d f'(u)) = -d$$

where f' is a polynomial in u with nonzero constant term a_d. We get at once

$$[f] = P - d\infty$$

whence

$$p_\star[f] = p_\star P - d p_\star \infty = dY - dY = 0.$$

(3) The calculation above also shows that the structure map $\mathbb{A}^1_K \to \operatorname{Spec}(K)$ does *not* preserve rational equivalence.

3.10. Lemma. *Let A be a principal ideal domain. Let B denote a finitely generated free A–module. Let $r : B \to B$ be an injective A–linear map. Then B/rB is of finite length and we have*

$$\ell_A\left(B/rB\right) = \ell_A\left(A/\det(r)A\right).$$

Proof. According to the structure theorem for modules over PID's, there exists a basis $b_1, \ldots, b_n$ of B over A and there are $a_1, \ldots, a_n \in A$ such that $a_1 b_1, \ldots, a_n b_n$ is a basis of rB. Put $b_i' = r^{-1}(a_i b_i)$. It follows that the matrix of b with respect to the pair of basis $\{b_i'\}$, $\{b_i\}$ is the diagonal matrix $\operatorname{diag}(a_1, \ldots, a_n)$. Hence we get

$$B/rB \simeq \prod A/a_iA.$$

Therefore we may compute

$$\ell_A\left(B/rB\right) = \sum \ell_A\left(A/a_iA\right) = \ell_A(A/(\textstyle\prod a_i)A).$$

Finally, $\det(r) = c \prod a_i$, where c is the determinant of an invertible matrix over A, whence $\det(r)A = (\prod a_i)A$ as desired. □

3.11. Proposition. *Let $p : V \to W$ be a proper, surjective map of varieties. Suppose $\dim V > \dim W$. Then we have*

$$p_\star[r] = 0 \quad \forall r \in R(V)^{\cdot}.$$

Proof. The assertion is trivial for $\dim V > \dim W + 1$. So assume $\dim V = \dim W + 1$. Write $K = R(W)$ and look at the generic fiber, V_K, of p,

$$\begin{array}{ccccc} V \times_W \operatorname{Spec}(K) & =: & V_K & \longrightarrow & V \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec}(K) & \longrightarrow & W. \end{array}$$

Note that V_K is a complete curve over K and $R(V_K) = R(V)$. We have a natural bijection $T \leftrightarrow T_K$ between the collection of subvarieties of V of codimension one such that $p(T) = W$ and the set of closed points in the curve V_K. In this bijection, the local rings $\mathcal{O}_{V,T}$ and $\mathcal{O}_{V_K,T_K}$ are canonically isomorphic as well as the residue fields $R(T)$ and $R(T_K)$. This is more or less straightforward from the construction of the fiber product. One takes compatible

in B, $Q = \alpha^{-1}P$ the contraction to A. If $\ell_B(M) = \infty$ clearly $\ell_A(M) = \infty$ and the formula is trivial. In the finite case, P and Q are maximal ideals and $\ell_A(M) = \dim_{A/Q}(B/P) = [R(B) : R(A)]$. Hence assertion (2) follows.

To prove (3), first note that $c \cdot (B/A) = 0$ implies that B/A is a finitely generated modulo over the Artin ring $A/\langle c\rangle$. Thus

$$\ell_A(B/A) = \ell_{A/\langle c\rangle}(B/A) < \infty.$$

Now consider the snake diagram below.

$$\begin{tikzcd}
\mathcal{K} \arrow[r, tail] & B/A \arrow[r] & B/A \arrow[r, two heads] & \mathcal{C} \\
& B \arrow[u] \arrow[r, hook, "a"] & B \arrow[u] \arrow[r, two heads] & B/aB \arrow[u] \\
& A \arrow[u, hook] \arrow[r, hook, "a"] & A \arrow[u, hook] \arrow[r, two heads] & A/aA \arrow[u] \\
& & & \mathcal{R} \arrow[u, hook] \arrow[uuulll, dashed]
\end{tikzcd}$$

In order to prove the main result asserting invariance of rational equivalence under proper pushforward, we begin with a "reduction to the normal case".

3.7. Lemma. *Let W be a variety and let $\nu : V \to W$ be the normalization map. Then we have $R(V) = R(W)$ and for each $r \in R(V)^{\cdot}$*

$$\nu_\star([r]_V) = [r]_W \quad \text{in } \mathcal{C}_\star W. \tag{3.2}$$

Proof. First recall that ν is a finite map. Hence it is proper. Let $T \subset W$ be a subvariety of codimension one. We must show that the coefficient of T is the same in either member of (3.2). On the right hand side, we get $\operatorname{ord}_T^W(r)$. On the left hand side, we get the sum of the contributions over all subvarieties $T' \subset V$ such that $\nu(T') = T$. We are thus reduced to show the equality

$$\sum_{T'} \operatorname{ord}_{T'}^V(r)[R(T') : R(T)] = \operatorname{ord}_T^W(r). \tag{3.3}$$

Put $A = \mathcal{O}_{W,T}$ and let B denote its normalization in $R(W)$. Since B is finite over A, we have a natural bijection between the set of irreducible

components T' of $\nu^{-1}T$ and the set of maximal ideals of B, cf. 3.8 below. The equality stated is a consequence of the previous lemma. Indeed, we may assume, without loss of generality, $r \in A$ and invoke the fact that there is a conductor $c \in A$ such that $cB \subseteq A$. Now take $M = B/rB$. □

3.8. Lemma. *Let $A \subseteq B$ be a finite ring extension. Let P be a prime ideal of A. Then the set of prime ideals Q of B which contract to P is in natural bijection with the set of maximal ideals of B_P.*

Proof. Let Q be a prime ideal of B contracting to P. Clearly Q_P contracts to the maximal ideal P_P of A_P. Form the diagram

$$\begin{array}{ccc} A_P & \subseteq & B_P \\ \downarrow & & \downarrow \\ A_P/P_P & \subseteq & B_P/Q_P. \end{array}$$

Since the bottom extension is finite, we see that Q_P is maximal. Conversely, if Q_P is maximal, its contraction to A_P is of the form P'_P for some prime $P' \subseteq P$. Again since the extension $A_P/P'_P \subseteq B_P/Q_P$ is finite and the latter is a field, it follows that P'_P is maximal, hence $P' = P$. □

3.9. Proposition. *Let $p : V \to W$ be a proper, surjective map of varieties of the same dimension. Given $r \in R(V)^\bullet$, we have*

$$p_\star[r] = [N(r)] \quad \textit{in } \mathcal{C}_\star W, \tag{3.4}$$

where $N(r)$ stands for the norm, i.e.*, the determinant of the $R(W)-$linear map $R(V) \to R(V)$ defined by multiplication by r.*

Proof. Let $T \subset W$ be a subvariety of codimension one. The calculation of the contribution of T in both sides of (3.4) is local. Indeed, it depends only on the degrees of residual field extensions $[R(T') : R(T)]$ for each component T' of $p^{-1}T$. Therefore, we are allowed to replace W by any open subset W_0 that meet T. We shall use this freedom to reduce the proof to the case when p is a finite map (not just *generically* finite). Let

$$Z = \{w \in W \mid \dim p^{-1}(w) \geq 1\}.$$

We know that Z is a closed subset of W. Furthermore, T is not contained in Z, otherwise $p^{-1}T = V$ and p is not surjective. Put $W_0 = W \smallsetminus Z$. It follows that the induced map $p^{-1}W_0 \to W_0$ is finite, since the fibers are finite and the map is proper (cf. [?, EGA-III 14.4.2]). Henceforth we shall assume p finite. Let

$$\nu_V : V' \to V, \quad \nu_W : W' \to W$$

be the normalizations of V, W. There is a canonical morphism p' induced by p fitting into the diagram

$$\begin{array}{ccc} V' & \xrightarrow{p'} & W' \\ \nu_V \downarrow & & \downarrow \nu_W \\ V & \xrightarrow{p} & W \end{array}$$

In view of 3.2 and recalling that $p_\star \nu_{V\star} = \nu_{W\star} p'_\star$, we may as well assume V, W normal. (In fact only the normality of W is required in the sequel).

Put $A = \mathcal{O}_{W,T}$. Since W is normal, we have that A is a discrete valuation ring. Let W^0 be an affine open subset that meets T and put $V^0 = p^{-1}W^0$. Since p is finite, we have that V^0 is also affine. Let C be the coordinate ring of V^0. Each component of $p^{-1}T$ corresponds to a prime ideal of height one in C. Put $B = C \otimes_{\mathcal{O}(W^0)} A$, the ring of fractions of C with respect to the multiplicative system $\mathcal{O}(W^0) \setminus P$, where P denotes the prime ideal corresponding to T. Note that the maximal ideals of B correspond bijectively to the primes of C lying over the maximal ideal of A.

Back to the main argument, we may also assume $r \in B$ because both sides in the formula are additive. Now the coefficient of T in $p_\star[r]$ is equal to

$$m := \sum_Q \ell_{B_Q}(B_Q/rB_Q)\,[R(Q) : R(T)],$$

where the sum ranges over the maximal ideals Q of B (each of which corresponds to a component of $p^{-1}T$). Using lemma 3.5, item (2), with $M = B_Q/rB_Q$, we find

$$\begin{aligned} m &= \textstyle\sum_Q \ell_A(B_Q/rB_Q) \\ &= \ell_A(B/rB). \end{aligned}$$

The last equality follows from the decomposition

$$B/rB = \prod B_Q/rB_Q$$

of an Artin ring as the product of its localizations. It remains to show the following lemma. □

Chapter 4

Inverse Image

Let $f : X \to Y$ be a morphism. Under appropriate conditions on f, we shall define a homomorphism $f^\star : \mathcal{C}_\star Y \to \mathcal{C}_\star X$ induced by the inverse image of subvarieties, compatible with rational equivalence. We also examine its relation to the proper pushforward.

4.1. Definition. Let $V \subset Y$ be a subvariety. The **inverse image cycle** of V under f is given by

$$f^\star V = [f^{-1}V]$$

The cycle on the right hand side is the cycle (cf. 1.3) of the scheme theoretic inverse image. We extend by linearity to the homomorphism

$$f^\star : \mathcal{C}_\star Y \to \mathcal{C}_\star X.$$

4.2. Examples. (1) Let $i : U \to X$ be the inclusion map of an open subscheme. For each subvariety V of X, we have $i^\star V = U \cap V$. We also note that for any $r \in R(V)^\bullet$, we may write $i^\star([r]_V) = [r]_V$. This is immediate from the fact that $R(V) = R(V \cap U)$ for $U \cap V \neq \emptyset$. Thus $i^\star(\mathcal{R}_\star X)$ is contained in $\mathcal{R}_\star U$. We obtain therefore an induced homomorphism

$$i^\star : \mathcal{A}_\star(X) \longrightarrow \mathcal{A}_\star(U).$$

Note that $i^\star$ is homogeneous of degree zero. See 4.8 below for a more general case.

(2) Let X, Y be varieties over an algebraically closed field. Then we know that $X \times Y$ is also a variety. Let $p : X \times Y \to Y$ denote the projection. For each subvariety $V \subseteq Y$ we have $p^\star[V] = [X \times V]$. Let $r \in R(V)^\bullet$ be a rational

affine open subsets Spec $(B) \subseteq V$, Spec $(A) \subseteq W$. Then Spec $(B \otimes_A K)$ is a typical affine open subset of V_K. Here is the corresponding diagrams for the affine situation:

$$\begin{array}{ccccc} R(W) & \supseteq & B \otimes_A K & \hookleftarrow & B \\ & & \uparrow & & \uparrow \\ & & K & \hookleftarrow & A. \end{array}$$

Recall that here $B \otimes_A K$ is nothing but the ring of fractions of B with respect to the multiplicative system $S = A \setminus \{0\}$. Thus, the field of fractions of $B \otimes_A K$ is $R(W) = R(V_K)$. The transcendence degree of of $R(V_K)$ over K is $1 = \dim V - \dim W$. Also recall that the prime ideals of $B \otimes_A K$ are all of the form $S^{-1}P$ where P is a prime sucht that $P \cap A = 0$. It can easily be seen that B_P is canonically isomorphic to $(S^{-1}B)_{S^{-1}P}$. This implies the identifications $\mathcal{O}_{V,T} \simeq \mathcal{O}_{V_K,T_K}$, $R(T) \simeq R(T_K)$.

Now, only the subvarieties $T \subset V$ of codimension one such that $p(T) = W$ may occur in the cycle $p_\star[r]$. According to the previous discussion, we may as well replace $V \to W$ by $V_K \to$ Spec (K). In other words, we may just assume V is a complete curve over Spec (K). Passing to the normalization if necessary, we may also assume V is normal. In this case, we may pick any finite morphism $f : V \to \mathbb{P}^1_K$ (simply choose any non-constant rational function $f \in R(V)$). We have a commutative diagram,

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{P}^1_K \arrow[dl, "q"] \\ & W & \end{tikzcd}$$

This allows us to write

$$\begin{array}{rcll} p_\star[r] & = & q_\star f_\star[r] & \\ & = & q_\star[N(r)] & \text{(by 3.4)} \\ & = & 0 & \text{(by 3.3 (2))}. \end{array}$$

□

3.12. Remark. The passage to the normalization of the curve V used at the end of the proof above may be replaced by taking f as the map induced by a projection. Of course one has to use then the fact that V is projective.

3.13. Theorem. *Let $p : X \to Y$ be a proper map. Then the proper pushforward map $p_\star : \mathcal{C}_\star X \to \mathcal{C}_\star Y$ preserves rational equivalence,* i.e., *we have*

$$p_\star \mathcal{R}_\star X \subseteq \mathcal{R}_\star Y.$$

Proof. Since $\mathcal{R}_\star X$ is generated by the cycles of rational functions on subvarieties of X, the result follows from (3.9) and (3.11). $\square$

3.14. Definition. Let $p : X \to Y$ be a proper map. The **pushforward homomorphism** is the induced homomorphism of quotient groups, denoted by the same symbol,

$$p_\star : \mathcal{A}_k(X) \longrightarrow \mathcal{A}_k(Y), \quad \forall k = 0, 1, \ldots$$

If $Y = \operatorname{Spec}(K)$, we may identify

$$\mathcal{A}_\star(Y) = \mathbb{Z}.$$

In this case, the homomorphism is also denoted by $\int_X$ or simply $\int$. Explictly, for each cycle class z in $\mathcal{A}_\star(X)$ we have

$$\int z = \sum m_i [R(P_i) : K] \tag{3.5}$$

where $\sum m_i P_i$ is the homogeneous part of dimension zero of z, *i.e.*, the P_i denote closed points of X.

The integer $\int Z$ is called the **degree** of the cycle z. By definition, the degree of a cycle is the same as the degree of its zero dimensional part.

3.15. Important remark. Let $p : X \to Y$ be a proper map of complete schemes over K. Then we have

$$\int_X z = \int_Y p_\star z$$

for each $z \in \mathcal{A}_\star(X)$. In fact, this is but a special case of 3.2.

We are ready to use the machinery developed thus far to recover the classical theorem of Bézout for plane curves. See ?? for a generalization.

3.16. Theorem. *Let F, G denote plane projective curves with no common component. Then* (cf. 1.4) *we have,*

$$\sum_P i(P, F \cdot G) = \deg(F) \cdot \deg(G).$$

Proof. Say $m = \deg(F)$. We may assume G is irreducible in view of the additivity of the index (cf. ??). Let L be a linear form that is not a multiple of G. Consider the rational function $r = F/L^m \in R(G)$. Let us calculate the cycle

$$[r] = \sum \operatorname{ord}_P^G(r) P.$$

For each P we choose a linear form H_P nonzero at P and write

$$r = \frac{(F/H^m)}{(L/H)^m}.$$

By additivity, we have

$$\operatorname{ord}_P^G(r) = \operatorname{ord}_P^G(F/H^m) - m \cdot \operatorname{ord}_P^G(L/H).$$

Now it is clear by definition of the index (cf. 1.4) that

$$\operatorname{ord}_P^G(F/H^m) = \ell(\mathcal{O}_{G,P}/\langle F/H^m \rangle) = \ell(\mathcal{O}_{\mathbb{P}^2,P}/\langle F/H^m, G/H^n \rangle) = i(P, F \cdot G).$$

Hence we get

$$0 = \int_G [r] = \sum_P \left(i(P, F \cdot G) - m \cdot i(P, L \cdot G) \right). \tag{3.6}$$

The first equality is obvious since $[r] = 0$ in $\mathcal{A}_\star(G)$. (In practice, we have replaced F by m copies of L.) We get

$$\sum_P i(P, F \cdot G) = m \cdot i(P, L \cdot G).$$

By the same token, we find

$$\sum_P i(P, L \cdot G) = n \cdot i(P, L \cdot L') = n,$$

where $n = \deg(G)$ and L' stands for a linear form co-prime with L. $\square$

3.17. Remark. The proof given above captures the spirit of the classical argument that consists in deforming each curve into a union of lines in general position.

Exercises

13. Let X be a complete scheme of positive dimension. Show that $\mathcal{A}_\star(X)$ contains a direct summand isomorphic to $\mathbb{Z}$.

14. Let X be a complete rational curve. Check whether $\mathcal{A}_0(X)$=$\mathbb{Z}$ holds.

15. Prove a version of Bézout's theorem for curves on $\mathbb{P}^1 \times \mathbb{P}^1$.

Thus we are reduced to show that

$$g'_\star[X'] = d[X] \tag{4.3}$$

where $d = \deg(g)$. Let V be a component of X and let $V'_1, \dots, V'_m$ be the components of X' dominating V. We set for short

$$C = \mathcal{O}_{X,V}, \ C'_i = \mathcal{O}_{X',V_i}.$$

Note that C, C'_i are artinian local rings. We must now show that

$$\sum \ell_{C'_i}(C'_i)[R(C'_i) : R(C)] = d\ell_C(C)$$

given that the left hand side above is precisely the coefficient of V in the left hand side of (4.3), and likewise for the right hand sides.

For this task, we first replace $g : Y' \to Y$ by a restriction to compatible affine open subsets corresponding to a ring homomorphism $g^\star : A \to A'$; this is an injective map of domains of finite type over the ground field and $\dim A = \dim A'$. Since g is proper and generically finite, we may assume g finite and $\mathrm{Spec}\,(A') = g^{-1}\,\mathrm{Spec}\,(A)$. Flatness of X over Y implies that the image of any component $V \subset X$ is dense in Y. Hence we may choose an affine open subset $\mathrm{Spec}\,(B) \subseteq X$ that meets V and maps to $\mathrm{Spec}\,(A)$. Put $B' = A' \otimes_A B$. Since g is finite (hence affine), it follows that $\mathrm{Spec}\,(B') = (g')^{-1}\,\mathrm{Spec}\,(B)$. The diagram in the statement gives rise to the tensor product diagram of coordinate rings,

$$\begin{array}{ccccc} B' & := & B \otimes_A A' & \longleftarrow & A' \\ & & \uparrow & & \uparrow \\ & & B & \underset{f^\star}{\longleftarrow} & A \end{array}$$

Let $\mathfrak{p} \subset B$ be the minimal prime corresponding to V. We have $C = B_{\mathfrak{p}}$. Put $K = R(Y)$, $K' = R(Y')$, the fraction fields of A, A'. Flatness implies $(f^\star)^{-1}\mathfrak{p} = 0$. Hence $A \to B \to C$ factors through $A \hookrightarrow K \hookrightarrow B \otimes_A K \to C$. Let $\mathfrak{p}' \subset B'$ be any prime that survives in $A' \otimes_A C = A' \otimes_A B_{\mathfrak{p}}$. Then $\mathfrak{p}'$ is a minimal prime of B'. Indeed, we have $\overline{B} := B/\mathfrak{p} \hookrightarrow \overline{B}' := B'/\mathfrak{p}'$. Since

$$\dim B = \dim \overline{B} \le \dim \overline{B}' \le \dim B' = \dim B$$

function and consider its image $p^\star r$ in $R(X \times V)$. One checks easily that the cycles $p^\star[r]$ and $[p^\star r]$ are one and the same. It follows that $p^\star$ induces a homomorphism

$$p^\star : \mathcal{A}_k(Y) \longrightarrow \mathcal{A}_{k+n}((X \times Y))$$

where $n = \dim X$.

(3) Let X be a variety and let $f : X \to \mathbb{P}^1$ be a dominant morphism. We may also consider f as an element r of the field $R(X)$ and compute the corresponding cycle. Indeed, let $U_0 \subset \mathbb{P}^1$ be the standard affine neighborhood of $0 = [0,1] \in \mathbb{P}^1$ and let t be the coordinate function there. Put $r = f^\star t$. We then have

$$[r] = [f^{-1}(0)] - [f^{-1}(\infty)],$$

where $\infty = [1,0]$. To see this, let $V \subset X$ be a subvariety of codimension one and put $A = \mathcal{O}_{X,V}$. Note that $f^{-1}(0)$ and $f^{-1}(\infty)$ are disjoint and that r (*resp.* $1/r$) is regular in the open subset $X_0 = X \smallsetminus f^{-1}(\infty)$ (*resp.* $X_\infty = X \smallsetminus f^{-1}(0)$). Hence, if $V \not\subset f^{-1}(\infty)$, we have $r \in A$. Put $B = A/rA$. Then we have $B = \mathcal{O}_{f^{-1}(0),V}$. We may write

$$\ell_B(B) = \ell_A(B) = \mathrm{ord}_V(r).$$

Arguing the same way when $V \not\subset f^{-1}(0)$, we deduce that the coefficient of V in the cycle of the rational function $[r]$ is the same as in $[f^{-1}(0)] - [f^{-1}(\infty)]$.

(4) Let Y be the curve $y^2 = x^3$ and let $f : \mathbb{A}^1 \to Y$ be the usual parametrization $f(t) = (t^2, t^3)$. Let $\overline{x}$, $\overline{y}$ denote the restrictions of the coordinate functions to Y. We have $f^\star\overline{x} = t^2$, $f^\star\overline{y} = t^3$. Note that the cycle of the function $\overline{y}$ is $3Q$, with $Q = (0,0)$. On the other hand we have $f^\star Q = 2P$, where $P = 0 \in \mathbb{A}^1$. Thus, in the present case, we have

$$f^\star[\overline{y}] = 3f^\star Q = 6P \neq 3P = [t^3] = [f^\star\overline{y}].$$

This last example also shows that $(gf)^\star$ and $f^\star g^\star$ may not coincide for arbitrary morphisms $f : X \to Y$, $g : Y \to Z$. Nevertheless we argue below that the validity of pleasant properties such as functoriality or compatibility with rational equivalence holds under the assumption of *flatness*. Let us recall the following.

4.3. Definition. A map $f : X \to Y$ is **flat** if for all pairs of affine open subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that $f(X_0) \subseteq Y_0$, the ring homomorphism of coordinate rings

$$f^\star : \mathcal{O}(Y_0) \longrightarrow \mathcal{O}(X_0)$$

is flat. Equivalently, for each subvariety V of X, if $W = \overline{f(V)} \subseteq Y$, then $\mathcal{O}_{X,V}$ is flat over $\mathcal{O}_{Y,W}$.

We say $f : X \to Y$ is of **relative dimension** n if for each subvariety W of Y, any component V of $f^{-1}(W)$ is of dimension

$$\dim V = n + \dim W.$$

4.4. Proposition. *Let $f : X \to Y$ be a flat morphism of relative dimension n. Then for each closed subscheme $Z \subseteq Y$ of pure dimension k, we have*

$$f^\star[Z] = [f^{-1}Z] \quad \text{in } \mathcal{C}_{k+n}X.$$

Proof. We may replace X by $X \times_Y Z$ with the benefit of assuming $Z = Y$, $X = f^{-1}(Z)$. Let V be a component of X and set $W = \overline{f(V)}$. Put $A = \mathcal{O}_{Y,W}$, $B = \mathcal{O}_{X,V}$. We have that B is artinian and is flat over A. We claim that A is also artinian (and consequently W is a component of Y too). Indeed, since A is local, B is in fact faithfully flat over A. Hence, given a chain of ideals,

$$A = A_0 \supset \cdots \supset A_\ell, \tag{4.1}$$

it follows that the chain

$$B = A_0 \otimes_A B \supset \cdots \supset A_\ell \otimes_A B, \tag{4.2}$$

has length ℓ. Hence ℓ is bounded by $\ell_B(B)$. Assuming in (4.1) $\ell = \ell_A(A)$, we must have $A_i/A_{i+1} \simeq R(A)$, the residual field of A. It follows from 4.2 that

$$\ell_B(B) = \sum_0^{\ell-1} \ell_B\left((A_i/A_{i+1}) \otimes_A B\right) = \ell_A(A) \cdot \ell_B\left(R(A) \otimes_A B\right).$$

This last formula shows that the coefficient $\ell_B(B)$ of V in the cycle $[X]$ is equal to $\ell_A(A)$ times the coefficient $\ell_B(R(A) \otimes_A B)$ of V in $[f^{-1}W]$, as desired. ☐

4.5. Corollary. *Let $f : X \to Y$, $g{:}Y{\to}Z$ be flat maps of relative dimensions m,n. Then we have*

$$(gf)^\star = f^\star g^\star : \mathcal{C}_k Z \longrightarrow \mathcal{C}_{k+m+n}X \quad \forall k.$$

Proof. Pick a subvariety V of Z. We have

$$\begin{aligned}(gf)^\star V &= [(gf)^{-1}(V)] && \text{(by definition)}\\ &= [f^{-1}(g^{-1}(V))] &&\\ &= f^\star[g^{-1}(V)] && \text{(by (4.4))}\\ &= f^\star g^\star V.\end{aligned}$$

4.6. Proposition. (Compatibility with proper pushforward). *Let be given a cartesian diagram,*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & Y \end{array}$$

where we assume f is flat of relative dimension n and g is proper. Then f' (resp. g') *is flat of relative dimension n* (resp. proper) *and we have*

$$g'_\star f'^\star = f^\star g_\star : \mathcal{C}_k Y' \longrightarrow \mathcal{C}_{k+n}X.$$

Proof. It suffices to verify the formula when applying to subvarieties of Y'. Thus, we may suppose Y' is a variety. Indeed, say $V' \subseteq Y'$ is one such; then $g_{|V'} : V' \to Y$ is proper. Let $\iota : V' \hookrightarrow Y'$, $f'_{V'} : X'_{V'} = X \times_Y V' \to V'$, $\iota' : X'_{V'} \hookrightarrow X'$ and $g'_{V'} : X'_{V'} \to X$ be the induced maps. We have

$$\begin{aligned}&g'_\star f'^\star[V'] = g'_\star[(f')^{-1}V'] = g'_\star \iota'_\star[(f'_{V'})^{-1}V'] = (g'_{V'})_\star[(f'_{V'})^{-1}V']\\ &= (g'_{V'})_\star(f'_{V'})^\star V' = f^\star(g_{|V'})_\star[V'] \text{ (assuming true for } V')\\ &= f^\star g_\star[V'].\end{aligned}$$

We may also suppose $Y = g(Y')$ with the same dimension as Y'. Indeed, put $Z = g(Y') \subseteq Y$. Let

$$\iota : Z \hookrightarrow Y,\ f_Z : X_Z = f^{-1}Z \to Z \text{ and } g'_Z : X'_Z = (g')^{-1}f^{-1}Z \to X_Z$$

be the natural maps. We have $X' = X'_Z$ because

$$(g')^{-1}f^{-1}Z = (f')^{-1}g^{-1}Z = (f')^{-1}g^{-1}Y$$

Also, g' factors through $X_Z \hookrightarrow X$. Setting $d = \deg(g)_Z$, we may write,

$$\begin{aligned}&f^\star g_\star[Y'] = df^\star[Z] = d[f^{-1}Z] \text{ (by 4.4)}\\ &= d[(f_Z)^{-1}Z] = d(f_Z)^\star Z = (f_Z)^\star[dZ]\\ &= (f_Z)^\star(g_Z)_\star[Y'] = (g'_Z)_\star(f'_Z)^\star[Y'] \text{ (assuming true)}\\ &= (g'_Z)_\star[X'] = g'_\star(f')^\star[Y'].\end{aligned}$$

18. Where does the hypothesis of pure dimension enter the proof of *4.8.1*?

19. Let X, Y be complete schemes of pure dimensions. Let p, q denote the projections from $X \times Y$ to x and Y. Let $x \in \mathcal{A}_0(X)$ be a zero cycle. Show that $q_\star p^\star x = m[Y]$, where $m = \int x$.

20. Let X, Y be schemes, and let $V \subseteq X$, $W \subseteq Y$ be subvarieties. Define the **exterior product**,

$$\begin{array}{ccc} \mathcal{C}_r X \otimes \mathcal{C}_s Y & \longrightarrow & \mathcal{C}_{r+s}(X \times Y) \\ x \otimes y & \longmapsto & x \times y \end{array}$$

sending generators $[V] \otimes [W]$ to $[V \times W]$.

(i) Show that for $y \in \mathcal{R}_r X$, we have $x \times y \in \mathcal{R}_{r+s}(X \times Y)\, \forall\, y \in \mathcal{C}_s Y$.

(ii) Let $f : X \to X'$, $g : Y \to Y'$ be morphisms and let $f \times g : X \times Y \to X' \times Y'$ be the induced map. If f, g are proper (*resp.* flat of relative dimensions m, n) then $f \times g$ is proper (*resp.* flat of relative dimension $m + n$) and the following hold.

(1) $(f \times g)_\star(x \times y) = f_\star(x) \times g_\star(y)$ for all cycles $x \in \mathcal{C}_\star X, y \in \mathcal{C}_\star Y$.

(2) (resp. $(f \times g)^\star(x' \times y') = f^\star x' \times g^\star y',\ \forall\, x' \in \mathcal{C}_\star X', y' \in \mathcal{C}_\star Y'$).

(iii) Show that the exterior product induces a homomorphism (called by the same name),

$$\mathcal{A}_\star(X) \otimes \mathcal{A}_\star(Y) \longrightarrow \mathcal{A}_\star(X \times Y)$$

satisfying formulas as in (ii).

(iv) Show that the exterior product satisfies associativity,

$$x \times (y \times z) = (x \times y) \times z.$$

(v) Suppose Y complete and let $p : X \times Y \to X$ be the projection. Show that $p_\star(x \times y) = (\int y)x$ for all $x \in \mathcal{A}_\star(X)$, $y \in \mathcal{A}_\star(Y)$.

holds, it follows that $\mathfrak{p}'$ is minimal. Hence the primes of $A' \otimes_A C = B' \otimes_B B_\mathfrak{p}$ correspond precisely to the minimal primes of B' that contract to $\mathfrak{p}$, *i.e.*, the components $V' \subset X'$ that dominate V. Now observe that since $A \hookrightarrow A'$ is finite, it follows that the ring of fractions $A' \otimes_A K$ is a domain which is finite over the field K, hence we have $A' \otimes_A K = K'$. We may write

$$K' \otimes_K C = A' \otimes_A K \otimes_K C = A' \otimes_A C$$

Since $[K' : K] = d$, it follows that $C' := K' \otimes_K C = A' \otimes_A C$ is a free C–module of the same rank d. The net effect was replacing Y by $\mathrm{Spec}\,(K)$, and also Y' by $\mathrm{Spec}\,(K')$. Furthermore, the prime ideals $P'_1, \ldots, P'_m$ of C' correspond to the components V'_i of X' and $C'_i = C'_{P'_i}$. With this at hand, we may now write

$$\begin{array}{rcll} d\,\ell_C(C) & = & \ell_C(C') & (\text{since } C' \simeq C^d) \\ & = & \sum \ell_C(C'_i) & (C' \simeq \prod C'_i) \\ & = & \sum \ell_{C'_i}(C'_i)[R(C'_i) : R(C)] & (\text{by } 3.5(2)). \end{array}$$

□

4.7. Remark. Keep the notation and hypothesis of 4.6. In view of 4.8 below, we also get a formula expressing the compatibility between flat pullback and proper pushforward for the homomorphisms at the Chow groups level,

$$g'_\star f'^\star = f^\star g_\star : \mathcal{A}_k(Y)' \longrightarrow \mathcal{A}_{k+n}(X).$$

4.8. Proposition. (Compatibility with rational equivalence). *Let*

$$f : X \to Y$$

be a flat morphism of relative dimension n. Then we have

$$f^\star \mathcal{R}_\star Y \subseteq \mathcal{R}_\star X,$$

thereby inducing homomorphisms

$$f^\star \mathcal{A}_k(Y) \longrightarrow \mathcal{A}_{k+n}(X), \quad \forall k.$$

Proof. Pick a subvariety $V \subseteq Y$ and $r \in R(V)^\bullet$. We must show that $f^\star[r]$ lies in $\mathcal{R}_\star X$. Using the special case of 4.6 with $g =$ inclusion map $V \subset Y$, we

may assume $Y = V$. In particular, f is dominant. Let $Y' \subset Y \times \mathbb{P}^1$ be the closure of the graph of the rational function $r : Y \dashrightarrow \mathbb{P}^1$. We work around the diagram with cartesian square

$$\begin{array}{ccccccc} X \times_Y Y' & =: & Z & \xrightarrow{f'} & Y' & \subset & Y \times \mathbb{P}^1 \\ & & g' \downarrow & & \downarrow g & \searrow p & \\ & & X & \xrightarrow{f} & Y & & \mathbb{P}^1 \end{array}$$

where p, g are induced by the projection maps. We note that g is birational and proper while p is flat. Also note that f' and pf' are dominant. We may write

$$\begin{aligned} f^\star[r] &= f^\star g_\star[r] && \text{(by 3.9)} \\ &= f^\star g_\star(p^\star 0 - p^\star\infty) && \text{(4.2(3))} \\ &= g'_\star f'^\star(p^\star 0 - p^\star\infty) && \text{(4.6)} \\ &= g'_\star (pf')^\star(0 - \infty) && \text{(4.5).} \end{aligned}$$

Now, if Z also were a variety, a second application of 4.2(3) would allow us to write the last expression above as $g'_\star[pf']$, thinking of pf' as a rational function on Z, and then using the fact that proper pushforward preserves rational equivalence to finish the argument. However, since the fiber product often yields non-integral schemes, all we know about Z is that it is pure-dimensional. We will be done invoking the following.

4.8.1. Claim. *Let $[Z] = \sum m_i Z_i$, where the Z_i denote the irreducible components of a pure dimensional scheme Z. Let $h : Z \to \mathbb{P}^1$ be a dominant morphism and write $h_i := h_{|Z_i}$. Then we have*

$$(\star) \qquad [h^{-1}(P)] = \sum m_i [h_i^{-1}(P)], \ \forall P \in \mathbb{P}^1.$$

The formula clearly implies that $g'_\star h^\star(0 - \infty)$ belongs to $\mathcal{R}_\star X$, thereby completing the proof.

To prove the claim, we must compare the coefficients of each subvariety V of Z of codimension one occurring in each side of the formula.

Put $A = \mathcal{O}_{Z,V}$. Each component Z_i containing V corresponds to a minimal prime $P_i \subset A$. Let t be a uniformizing parameter of P in $\mathbb{P}^1$. Thus $a := h^\star t$ is a local equation for $h^{-1}(P)$. We may assume $V \subset h^{-1}(P)$. The coefficient of V in the left hand side is $\ell_A(A/\langle a\rangle)$. For the right hand side we find $\sum m_i \ell_{A/P_i}(A/(P_i + \langle a\rangle))$. The equality stated in the claim is now a consequence of the following lemma (taking $M = A$ and noting that presently we have ${}_aM = 0$). $\square$

4.9. Lemma. *Let A be a one dimensional ring. Let $P_1, \dots, P_s$ be the minimal primes of A. Let $a \in A$ be a nonzero divisor. For each finitely generated A–module M put*

$${}_aM = \{x \in M \mid ax = 0\} \quad \text{and} \quad M_a = M\big/(aM)\,.$$

Then we have the following.

(1) *${}_aM$ and M_a have finite length;*

(2) *$e(a, M) := \ell_A(M_a) - \ell_A({}_aM)$ is an additive function over exact sequences of M's;*

(3) *$e(a, M) = \sum \ell_{A_{P_i}}(M_{P_i}) \ell_{A/P_i}(A/(P_i + \langle a\rangle))$.*

Proof. (1) Note that both modules are killed by a, hence they are in fact modules over the artinian ring $A/\langle a\rangle$. This implies the first assertion.

(2) The additivity follows from the snake diagram,

$$\begin{array}{ccccccc} & & & & & & \star \\ & & & & & & \downarrow \\ {}_aM' & \hookrightarrow & M' & \xrightarrow{a} & M' & \twoheadrightarrow & M'_a \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}_aM & \hookrightarrow & M & \xrightarrow{a} & M & \twoheadrightarrow & M_a \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}_aM'' & \hookrightarrow & M'' & \xrightarrow{a} & M'' & \twoheadrightarrow & M''_a \\ \downarrow & & & & & & \\ \star & & & & & & \end{array}$$

(3) Since both members are additive, we may invoke a composition series $M = M_0 \supset \cdots \supset M_n = 0$ and replace M by A/P where either $P = P_i$ or P is maximal. In the first case ($P = P_i$ minimal), with $M = A/P_i$, we have ${}_aM = 0$ and $M_a = A/(P + \langle a\rangle)$. In the second case, $M = A/P$ with P maximal, we have $M_{P_i} = 0$ and $e(a, M) = 0$. $\square$

Exercises

16. Find examples of morphisms $f : X \to Y$, $g : Y \to Z$ for which

$$(gf)^\star \neq f^\star g^\star : \mathcal{C}_\star Z \to \mathcal{C}_\star X.$$

17. Examine *4.8.1* letting $Z = \operatorname{Spec}(k[x, y, z]/\langle zx, zy\rangle)$ and $h = x - z$, $P = 0 \in \mathbb{A}^1$.

where $Z = \cup V_i$. Thus we get $m[\mathbb{P}^i] = 0$ in $\mathcal{A}_i(Z)$. There exists a finite map $p : Z \to \mathbb{P}^{i+1}$ (*e.g.*, induced by a linear projection if the base field is infinite or by a Noether normalization argument). We find

$$mp_\star[\mathbb{P}^i] = 0 \quad \text{in} \quad \mathcal{A}_i\big(\mathbb{P}^{i+1}\big),$$

which is torsion free. Hence $m = 0$ as desired. □

5.4. Definition. Let $z = \sum_0^k m_i[\mathbb{P}^i]$ be a cycle in $\mathbb{P}^n$. If $m_k \neq 0$ we define the **degree** of z by the formula

$$\deg(z) = m_k.$$

If X is a subscheme of $\mathbb{P}^n$ of dimension k and $[X] = z$ as above, we define the **degree** of X as

$$\deg(X) = \deg(z) = m_k.$$

5.5. Remark. We will see in 8.8 that $\deg(X)$ is the degree of the zero cycle of intersection of X with a subspace of codimension k.

Exercises

21. Let $X \subset \mathbb{P}^n$ be the hypersurface defined by some homogeneous polynomial of degree d. Show that $\deg(X) = d$.

22. Find $\mathcal{A}_\star(\mathbb{P}^m \times \mathbb{A}^n)$ and $\mathcal{A}_\star(\mathbb{P}^m \times \mathbb{P}^n)$.

23. Determine $\mathcal{A}_\star(U)$, $U=$ complement of a point in $\mathbb{P}^2$.

24. Let $p : X \to \mathbb{P}^2$ be the blowup of a point O in $\mathbb{P}^2$. Show that $p_\star : \mathcal{A}_\star(X) \to \mathcal{A}_\star(\mathbb{P}^2)$ is surjective, with kernel the subgroup of $\mathcal{A}_1(X)$ generated by the class of the exceptional divisor $E = p^{-1}\mathrm{O}$.

25. Let $Z \subseteq \mathbb{P}^n$ be a nonempty subvariety. Show that $[Z] \neq 0$ in $\mathcal{A}_\star(\mathbb{P}^n)$.

26. *(The Chow group of $G(2,4)$)* Let X be the grassmannian of lines in $\mathbb{P}^3$. We may identify X to a quadric hypersurface $X \subset \mathbb{P}^5$ given by Plücker's relation,

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$$

(cf. [?, p. 62]). Fix $\ell_0 \in X$, say $\ell_0 = \mathcal{Z}(x_1, x_2)$, and let

$$Z = \{\ell \in X \mid \ell \cap \ell_0 \neq \emptyset\}.$$

Chapter 5

The excision sequence

5.1. Proposition. *Let $i : Z \hookrightarrow X$, $j : U \hookrightarrow X$ be the inclusion maps of a closed subscheme Z and its complement U. Then we have the following commutative diagram with horizontal exact sequences,*

$$\begin{tikzcd}
\mathcal{C}_\star Z \arrow[r, "i_\star"] \arrow[d] & \mathcal{C}_\star X \arrow[r, two heads, "j^\star"] \arrow[d] & \mathcal{C}_\star U \arrow[d] \\
\mathcal{A}_\star(Z) \arrow[r, "i_\star"] & \mathcal{A}_\star(X) \arrow[r, two heads, "j^\star"] & \mathcal{A}_\star(U).
\end{tikzcd}$$

Proof. It is clear that the top sequence is exact. Now pick a cycle $c \in \mathcal{C}_\star X$ such that $j^\star c$ lies in $\mathcal{R}_\star U$. Hence we may write

$$j^\star c = \sum m_i [r_i]_{V_i},$$

for some rational functions $r_i \in R(V_i)$ in subvarieties $V_i \subset U_i$. Let W_i be the closure of V_i in X. Recalling that $R(V_i) = R(W_i)$, it is easy to see that we must have

$$c = \sum m_i [r_i]_{W_i} + i_\star z$$

for some $z \in \mathcal{C}_\star Z$. It follows that $c = i_\star z$ holds in $\mathcal{A}_\star(X)$. □

We have already enough tools to calculate a few Chow groups.

5.2. Lemma. *Let X be a scheme and let $p : X \times \mathbb{A}^n \to X$ be the projection. Then*

$$p^\star : \mathcal{A}_\star(X) \twoheadrightarrow \mathcal{A}_\star(X \times \mathbb{A}^n)$$

is surjective.
(We will see later on that it is in fact an isomorphism.)

Proof. We may obviously assume $n = 1$. Let $V \subset X \times \mathbb{A}^1$ be a subvariety. We wish to exhibit some $c \in \mathcal{A}_\star(X)$ such that $p^\star c = [V]$ holds. Replacing X by the closure of $p(V)$ we may as well assume X irreducible and $p_{|V}$ dominant (use 4.6).In this situation, let U be an affine open subset of X with coordinate ring A. Now we claim that, replacing U by a smaller neighborhood if needed, the ideal $P \subset A[t]$ of V in U is principal. Indeed, let K be the fraction field of A and pick $f, f_1, \ldots, f_m \in P$ such that $P = \langle f_1, \ldots, f_m \rangle$ and $P\,K[t] = \langle f \rangle$. We may write each $f_i = g_i f$ for some $g_i \in K[t]$. Let $a \in A$ be a common denominator for the coefficients of the g_i. It is clear that, upon replacing A by the ring of fractions $A[1/a]$, we have got the desired affine open subset as claimed. Put $Z = X \setminus U$. Let $i : Z \hookrightarrow X$ and $i' : Z \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$ be the inclusion maps. By construction we have $[V \cap (U \times \mathbb{A}^1] = [f] = 0$ in $\mathcal{A}_\star(U \times \mathbb{A}^1)$. By excision (5.1), it follows that $V = i'_\star z$ for some $z \in \mathcal{A}_\star(Z \times \mathbb{A}^1)$. By induction on $\dim X$, we get $z = q^\star x$ for some $x \in \mathcal{A}_\star(Z)$ where $q = p_{|Z} : Z \times \mathbb{A}^1 \to Z$. At last, we may invoke (4.6) and write $[V] = i'_\star q^\star x = p^\star i_\star x$. □

5.3. Proposition. *We have the following*

(1) $\mathcal{A}_i(\mathbb{A}^n) = 0$ *for all* $i \neq n$ *and* $\mathcal{A}_n(\mathbb{A}^n) = \mathbb{Z}$.

(2) $\mathcal{A}_i(\mathbb{P}^n) = \mathbb{Z} \cdot [\mathbb{P}^i]$, *the free group generated by the class of a subspace* $\mathbb{P}^i \subset \mathbb{P}^n$ *for all* $0 \leq i \leq n$.

Proof. (1) We already know that $\mathcal{A}_n(\mathbb{A}^n) = \mathbb{Z}$ and that $\mathcal{A}_{n-1}(\mathbb{A}^n) = 0$ (cf. 2.8). By the previous lemma, the map $p^\star : \mathcal{A}_{i-n+1}(\mathbb{A}^1) \to \mathcal{A}_i(\mathbb{A}^1 \times \mathbb{A}^{n-1})$ is surjective. This proves (1).

(2) Let us consider the excision exact sequence,

$$\mathcal{A}_i(\mathbb{P}^{n-1}) \longrightarrow \mathcal{A}_i(\mathbb{P}^n) \twoheadrightarrow \mathcal{A}_i(\mathbb{A}^n).$$

Using (1) and induction, it follows that $\mathcal{A}_i(\mathbb{P}^n) = \mathbb{Z} \cdot [\mathbb{P}^i]$. It remains to be seen that the latter group is free, a fact already proven for $i = n, n-1$ (2.8). Now assume $m[\mathbb{P}^i] = 0$. Let $V_1, \ldots, V_s$ be subvarieties of $\mathbb{P}^n$ of dimension $i+1$ and let $r_j \in R(V_j)$ be rational functions such that

$$m[\mathbb{P}^i] = \sum m_i [r_i] \quad \text{in} \quad \mathcal{C}_i Z$$

6.3. Examples. (1) Let $X = \mathbb{P}^n$ and let $F(x_0, \dots, x_n)$ be a nonzero homogeneous polynomial of degree m. Let U_i be the standard affine open subset complementary of the hyperplane $x_i = 0$. The coordinate ring of U_i is the polynomial ring in the indeterminates $x_0/x_i, \dots, x_n/x_i$. Put

$$f_i = x_i^{-m} F = F(x_0/x_i, \dots, x_n/x_i) \in \mathcal{O}_X(U_i).$$

Then (U_i, f_i), $i = 0, \dots, n$ is an effective Cartier divisor. It is equal to the hypersurface defined by F.

(2) Let X be a curve and let $P_1, \dots, P_n$ be nonsingular points of X. Let t_i be a local uniformazing parameter at P_i and let U_i be an open subset of X such that t_i is regular on U_i and vanishes only at P_i on U_i and such that $P_i \notin U_j \, \forall i \neq j$. Take $U_0 = X \setminus \{P_1, \dots, P_n\}$ and set $t_0 = 1$. Pick integers m_i. Then $(U_i, t_i^{m_i})$ defines a Cartier divisor on X, naturally associated to the zero cycle $m_1 P_1 + \cdots + m_n P_n$.

The last example suggests the following generalization.

6.4. Definition. Let $D = (\{U_i\}, f_i)$ be a Cartier divisor on X. The **cycle associated** to D is

$$[D] = \sum \mathrm{ord}_V(D) V,$$

where the sum is taken over the subvarieties of codimension one and the coefficient is defined by

$$\mathrm{ord}_V(D) = \mathrm{ord}_{V_i}(f_i) \tag{6.2}$$

with $V_i = U_i \cap V \neq \emptyset$.

A word of caution about the meaning of subvariety of codimension one is in order whenever X is not equidimensional. We say a subvariety $V \subset X$ is of codimension one if the dimension of the local ring $\mathcal{O}_{X,V}$ of X at the generic point of V is equal to one.

The reader may object that the right hand side in (6.2) has been defined only when X (and hence the U_i) is a variety. It can be easily checked however that, in view of the lemma 2.2, the same formula given in (2.1) applies. Furthermore, $\mathrm{ord}_V(D)$ is well defined and is additive and nonzero only for finitely many V's.

6.5. Remark. It follows immediately that the map $D \mapsto [D]$ is a group homomorphism from $\mathrm{CDiv}\,(X)$ into $\mathcal{C}_\star X$.

(vi) Show that Z is the intersection of X with the tangent hyperplane to X at ℓ_0. Show that $X \setminus Z \simeq \mathbb{A}^4$. Deduce that $\mathcal{A}_3(X)$ is free, generated by $[Z]$.

(vii) Verify that Z is a cone over a smooth quadric in $\mathbb{P}^3$, with vertex the point ℓ_0. Prove that $Z \setminus \{\ell_0\}$ is isomorphic to a line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$.

(viii) Show that $Z \setminus \{\ell_0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^1$. Conclude that $\mathcal{A}_2(X)$ is free of rank 2, generated by the classes of

$$\{\ell \in X \mid \ell \text{ is contained in a fixed plane }\}$$

and of

$$\{\ell \in X \mid \ell \text{ contains a given point }\}.$$

(ix) Show that $\mathcal{A}_0(X)$ is infinite cyclic generated by the class of a point $\ell_0 \in X$.

(x) Fix a point P in a plane π in $\mathbb{P}^3$. Show that $\mathcal{A}_1(X)$ is infinite cyclic generated by the class of $\{\ell \in X \mid P \in \ell \subset \pi\}$.

Chapter 6

The first Chern class

We start now the program to construct a product of cycles. In this first step, one of the cycles will be a hypersurface. More precisely, we shall define the first Chern class of a line bundle as an operator mapping cycles to cycle classes in the Chow group. Geometrically, this can be interpreted as taking the intersection with a hypersurface. We begin recalling some basic definitions. The reader acquainted with the notions of vector bundles and Cartier divisor may jump directly to §6.20.

6.1. Definition. Let X be a scheme. A **Cartier divisor** D on X is given by an affine open cover $\{U_i\}$ of X together with a choice of an invertible element f_i in the total ring of fractions $R(U_i)$ of the coordinate ring $\mathcal{O}_X(U_i)$ such that $f_i f_j^{-1}$ is invertible in $\mathcal{O}_X(U_{ij})$, with $U_{ij} = U_i \cap U_j$, $\forall i, j$. Each f_i is said to be a **local equation of** D **in** U_i.

The data $(\{U_i\}, f_i)$ and $(\{V_\alpha\}, g_\alpha)$ determine the same Cartier divisor if there exists a refinement $\{W_\lambda\}$ such that

$$(f_{i\,|W_\lambda})(g_{\alpha\,|W_\lambda})^{-1} \quad \text{is invertible in } \mathcal{O}_X(W_\lambda) \tag{6.1}$$

for all $i = i(\lambda)$, $\alpha = \alpha(\lambda), W_\lambda \subseteq U_i \cap V_\alpha$.

We are implicitly using the following elementary facts about rings of fractions.

6.2. Lemma. *Let A be a ring, $a \in A$ and $A \to A_a$ the natural map. If b is a nonzero divisor in A then $b/1$ is a nonzero divisor in A_a. If $P \subset A$ is a prime ideal and $b/1$ is a nonzero divisor in A_P then there exists $a \in A \setminus P$ such that $b/1$ is a nonzero divisor in A_a. If $R(A)$ is the total ring of fractions of A then the natural map $A \to R(A)$ is injective.*

6.2.1. The natural maps $R(A) \to R(A_P)$ for $P \in \operatorname{Spec}(A)$ show that for each Cartier divisor on a scheme X, we are given a collection of local equations $f_x \in R(\mathcal{O}_{X,x})$, $\forall x \in X$ with the following property. For each $x \in X$, there exists an affine neighborhood U_x together with some $\widetilde{f} \in R(U_x)$ such that f_y is the image of $\widetilde{f}$ in $R(U_y)$ for all $y \in U_x$. Two such collections $\{f_x\}, \{g_x\}$ define the same Cartier divisor if and only if for all x we have that $f_x {g_x}^{-1}$ lies in $\mathcal{O}^\star_{X,x}$, the subgroup of invertible elements. Put in other words, a Cartier divisor is an element of $H^0(X, R^\star/\mathcal{O}^\star)$.

6.2.2. The **support** $|D|$ of D is the subset of all $x \in X$ such that $f_x \in R(\mathcal{O}_{X,x})^\star$ *is not* in $\mathcal{O}^\star_{X,x}$. This condition on x is equivalent to requiring that for any representation $f_x = a_x/b_x$ with $a_x, b_x \in \mathcal{O}_{X,x}$ non zero divisors, we have $a_x(x)b_x(x) = 0$.

Indeed, suppose $f_x \in R(\mathcal{O}_{X,x})^\star \setminus \mathcal{O}^\star_{X,x}$ is written as $f_x = a_x/b_x$ with $a_x, b_x \in \mathcal{O}_{X,x}$ non zero divisors. If $a_x(x)b_x(x) \neq 0$ then both a_x, b_x are invertible in $\mathcal{O}_{X,x}$ and so f_x lies in $\mathcal{O}^\star_{X,x}$. Conversely, it is clear that if f_x lies in $\mathcal{O}^\star_{X,x}$ we have a representation with $a_x = f_x, b_x = 1$ and so $a_x(x)b_x(x) \neq 0$.

6.2.3. We define the **sum of Cartier divisors** by multiplying their local equations. Thus we have defined a structure of abelian group on the set $\operatorname{CDiv}(X)$ of Cartier divisors, as the reader may easily check.

A Cartier divisor is said to be **effective** if it admits a representation by local equations $(\{U_i\}, f_i)$ such that f_i is a regular function, *i.e.*, f_i lies in $\mathcal{O}_X(U_i)$ for all i. This is the same as a closed subscheme locally defined by a nonzero divisor.

$D = (\{U_i\}, f_i)$ is said to be **principal** if $f_i = f_j$ in U_{ij}, $\forall i, j$. In other words, the given local equations are compatible along the intersection, thereby yielding a global section f of the subsheaf $R^\star_X$ of invertible elements of the sheaf of total ring of fractions, so that we may also write $D = (\{X\}, f)$, a single equation.

6.2.4. The **inverse image** $f^\star D$ of a Cartier divisor D on X by a morphism $f : X' \to X$ is defined in a natural way whenever X, X' are varieties and f is dominant: just apply the injective homomorphism of fields of functions, $f^\star : R(X) \to R(X')$ to the local equations. Two other instances where $f^\star D$ is well defined are as follows: (1) whenever f is flat or (2) f is the inclusion of a subvariety not contained in the support of D. See exercise 29, p.43.

6.13. Definition. Let $s : X \to \mathcal{E}$ be a section of a vector bundle of rank n. The **scheme of zeros** of s is the subscheme $\mathcal{Z}(s)$ of X with ideal locally generated by the local expressions of s. We say that the section s is **regular** if it admits, locally around each $x \in X$, a local expression whose n coordinates form a regular sequence in $\mathcal{O}_{X,x}$.

6.14. Remark. If X is a locally Cohen-Macaulay scheme then the regularity of a section s is equivalent to the requirement that each component of $\mathcal{Z}(s)$ be of codimension n in X.

6.15. Proposition. *Let $\mathcal{L} \to X$ be a line bundle and let $s : X \to \mathcal{L}$ be a regular section. Then the scheme of zeros $\mathcal{Z}(s)$ is an effective Cartier divisor whose associated line bundle is isomorphic to $\mathcal{L}$. Conversely, if D is an effective Cartier divisor such that $\mathcal{O}(D)$ is isomorphic to $\mathcal{L}$, then there exists a regular section whose scheme of zeros is D.*

Proof. Let $\{U_i\}$ be an affine open cover of X trivializing $\mathcal{L}$ and let $s_i \in \mathcal{O}_X(U_i)$ be a local expression of s. By hypothesis, s_i is a non zero divisor, thereby defining an effective Cartier divisor $D = (\{U_i\}, s_i)$. The transition functions for $\mathcal{O}(D)$ are $s_i\, s_j^{-1}$. Now, if (f_{ij}) are transition functions for $\mathcal{L}$, we have from (6.3)

$$s_i = f_{ij} s_j \quad \text{in} \quad \mathcal{O}_X(U_{ij}).$$

Thus we may write $f_{ij} = s_i\, s_j^{-1}$ in $\mathcal{O}_X(U_{ij})$. We leave the converse for the reader. □

6.16. Proposition. *Let D be a Cartier divisor on X. Then $\mathcal{O}(D)$ is isomorphic to the trivial line bundle $X \times \mathbb{A}^1$ if and only if D is principal.*

Proof. Assume $\mathcal{O}(D)$ trivial. Say $D = (\{U_i\}, f_i)$. The transition functions for $\mathcal{O}(D)$ are therefore the $f_i f_j^{-1}$. The constant function 1 yields a section of $\mathcal{O}(D)$ with local expressions (s_i) with each s_i invertible in $\mathcal{O}_X(U_i)$. In view of (6.3) we deduce

$$f_i f_j^{-1} s_j = s_i \quad \text{in} \quad \mathcal{O}_X(U_{ij})^\star.$$

Hence we have

$$f_i s_i^{-1} = f_j s_j^{-1} \quad \text{in} \quad R(U_{ij}).$$

This last relation means that there exists an $f \in R(X)$ such that

$$f_i f^{-1} = s_i \quad \text{in} \quad \mathcal{O}_X(U_{ij})^\star \subseteq R(U_i)\ \forall i.$$

6.6. Example. Any Cartier divisor on $\mathbb{A}^n$ is principal. In fact, this is so for any integral affine scheme X such that the coordinate ring $\mathcal{O}(X)$ is a UFD. Let $V \subset X$ be an irreducible hypersurface. It corresponds to a height-one prime $\mathfrak{p} \subset \mathcal{O}(X)$. Since $\mathcal{O}(X)$ is a UFD, we have that $\mathfrak{p} = \langle p \rangle$ is a principal ideal, generated by an irreducible element. One sees at once that the principal Cartier divisor defined by p maps to the generator V in $\mathcal{C}_{n-1}X$. To prove injectivity, let $D = (f_i, U_i)_{i \in I}$ map to zero in $\mathcal{C}_{n-1}X$. We may assume U_i affine and $A = \mathcal{O}(U_i)$ is a UFD, because principal open subsets of X are UFD. Write $f_i = a/b$ with $a, b \in A$ and $\gcd(a, b) = 1$. Let $\mathfrak{p} = \langle p \rangle \subset A$ be a prime of height one. For any nonzero $a \in A$, we have that the length of $A_\mathfrak{p}/\langle a \rangle$ is the largest integer α such that p^α divides a. Let the codimension one $V \subset X$ correspond to $\mathfrak{p}$. Since $\mathrm{ord}_V(D) = \ell(A_\mathfrak{p}/\langle a \rangle) - \ell(A_\mathfrak{p}/\langle b \rangle)$, it follows that p does not divide a nor b. Since p is an arbitrary irreducible element of A, it follows that a, b are invertible in A. Hence $D = 0$.

6.7. Lemma. *If X is an integral, locally factorial scheme of dimension n, then the natural homomorphism* $\mathrm{CDiv}\,(X) \to \mathcal{C}_{n-1}X$ *is an isomorphism.*

Proof. Indeed, let $V \subset X$ be an irreducible hypersurface. For any $x \in V$, we have that the stalk of the sheaf $I(V)$ is a height-one prime $\mathfrak{p}_x \subset \mathcal{O}_x$. Since $\mathcal{O}_x$ is a UFD, we have again $\mathfrak{p}_x = \langle f_x \rangle$. For $x \in X \setminus V$, put $f_x = 1$. Then the f_x define a Cartier divisor which maps to V. For the injectivity, suppose $D = (f_x)_{x \in X}$ is a Cartier divisor such that $[D] = 0$. We must show that each local equation f_x is in fact a unit in the local ring $\mathcal{O}_x$. Let $\mathfrak{p} = \langle p \rangle \subset \mathcal{O}_x$ be a height one prime. It corresponds to an irreducible hypersurface $V \subset X$. Write $f_x = a/b$ with $a, b \in \mathcal{O}_x$ relatively prime. Arguing as before, one checks that the condition $\mathrm{ord}_V(D) = 0$ (for all V) implies that a, b are units in $\mathcal{O}_x$. See [**?**, II-6.11, p. 141]. □

6.8. Definition. Let $D = (\{U_i\}, f_i)$ be a Cartier divisor on X. We write $\mathcal{O}_X(D)$ for the *line bundle associated* to D, defined by the transition functions $f_{ij} = f_i f_j^{-1}$ on U_{ij} (cf. Shafarevich, p.270).

Explicitly, $\mathcal{O}_X(D)$ is the scheme over X obtained by glueing. One takes the disjoint union

$$\coprod U_i \times \mathbb{A}^1$$

and identify pairs

$$(x, v) \in U_i \times \mathbb{A}^1,\ (y, w) \in U_j \times \mathbb{A}^1$$

if and only if

$$x = y \in U_{ij} \quad \text{and} \quad v = f_{ij}(x)w.$$

In other words, we glue the open affine subsets $U_i \times \mathbb{A}^1, U_j \times \mathbb{A}^1$ identifying the open subsets $U_{ij} \times \mathbb{A}^1 \subseteq U_i \times \mathbb{A}^1$, $U_{ij} \times \mathbb{A}^1 \subseteq U_j \times \mathbb{A}^1$ via the isomorphism $A[T] \simeq A[T]$ defined by $T \mapsto f_{ij} \cdot T$, where A denotes the coordinate ring of U_{ij}.

6.9. Example. Take $X = \mathbb{P}^n$ and U_i as in the example 6.3. Let $\mathcal{L} \subset \mathbb{P}^n \times \mathbb{A}^{n+1}$ be the tautological line bundle over $\mathbb{P}^n$, *i.e.*, the fiber of $\mathcal{L}$ over each point $P \in \mathbb{P}^n$ is the line that P represents. Thus, we have

$$\mathcal{L} = \{(P, v) \in \mathbb{P}^n \times \mathbb{A}^{n+1} \,|\, v \in P\}.$$

Let us consider the local trivializations

$$\begin{array}{cccc} \varphi_i : & \mathcal{L}_{|U_i} & \longrightarrow & U_i \times \mathbb{A}^1 \\ & (P, v) & \longmapsto & (P, v_i) \\ & || & & \end{array}$$

$$\overbrace{([x_0, \ldots, 1, \ldots, x_n]\,,\,(x_0 v_i, \ldots, v_i, \ldots, x_n v_i))}$$

We find

$$\begin{array}{cccc} \varphi_i {\varphi_j}^{-1} : & U_{ij} \times \mathbb{A}^1 & \longrightarrow & U_{ij} \times \mathbb{A}^1 \\ & (P, c) & \longmapsto & (P, (x_i/x_j) \cdot c). \end{array}$$

Therefore, the transition functions of the line bundle $\mathcal{L}$ are x_i/x_j on U_{ij}. The tensor powers $\mathcal{L}^{\otimes m}, m \in \mathbb{Z}$ are given by $(U_i, (x_i/x_j)^{\otimes m})$. Comparing with 6.3, we see that if F is a hypersurface of degree m, the associated line bundle $\mathcal{O}(F)$ has transition functions $(x_i^{-m}F)/(x_j^{-m}F) = (x_i/x_j)^{-m}$.

The tautological line bundle $\mathcal{L} \to \mathbb{P}^n$ is usually denoted by $\mathcal{O}_{\mathbb{P}^n}(-1)$. More generally, we write

$$\mathcal{O}_{\mathbb{P}^n}(m) = \mathcal{L}^{\otimes(-m)},\, m \in \mathbb{Z}.$$

One knows that every line bundle over $\mathbb{P}^n$ is isomorphic to one of the $\mathcal{O}_{\mathbb{P}^n}(m)$. The transition functions computed above show that $\mathcal{O}(F) = \mathcal{O}(m)$.

6.10. Proposition. *Let $\mathcal{L} \to X$ be a line bundle over a variety. Then there exists a Cartier divisor D on X such that $\mathcal{O}(D)$ is isomorphic to $\mathcal{L}$.*

Proof. Let $\{U_i\}$ be an affine open cover of X and let $f_{ij} \in \mathcal{O}_X(U_{ij})^\star$ be transition functions for $\mathcal{L}$. Since X is a variety, each coordinate ring $\mathcal{O}_X(U_{ij})$ is a domain, contained in the function field $R(X) = R(U)$ for any open subset $U \neq \emptyset$. Fix an index i_0, and write it 0 for short. Set $f_i = f_{i0}$. It is clear that $(\{U_i\}), f_i)$ defines a Cartier divisor D. Furthermore, the associated line bundle $\mathcal{O}(D)$ is given by the transition functions $f_i {f_j}^{-1} = f_{i0} {f_{j0}}^{-1} = f_{ij}$, whence $\mathcal{O}(D)$ is isomorphic to $\mathcal{L}$. □

6.11. Remark. The result above does not hold for arbitrary schemes, cf. Hartshorne, "Ample subvarieties of algebraic varieties", LNM 156 Springer-Verlag (1970).

6.12. Let us recall briefly what are the local expressions of a section of a vector bundle $\pi : \mathcal{E} \to X$ of rank e. Let $\{U_i\}$ be an affine open cover of X endowed with trivializations, $\varphi_i : \mathcal{E}_{U_i} := \pi^{-1}(U_i) \simeq U_i \times \mathbb{A}^n$. We have the isomorphisms of trivial vector bundles,

$$\begin{array}{cccc} \varphi_i {\varphi_j}^{-1} : & U_{ij} \times \mathbb{A}^n & \longrightarrow & U_{ij} \times \mathbb{A}^n \\ & (x, v) & \mapsto & (x, f_{ij}(x) v) \end{array}$$

where f_{ij} is a morphism from U_{ij} into the general linear group GL_n. There are cocycle relations,

$$f_{ij} f_{jk} = f_{ik}.$$

Let $s : X \to \mathcal{E}$ be a section. The restriction s_{U_j} composed with φ_j can be written in the form

$$\varphi_j \, s_{U_j}(x) = (x, s_j(x)), \text{ for } x \in U_j$$

where the row vector $s_j \in \mathcal{O}_X(U_j)^n$ is the local expression of s with respect to the trivialization φ_j. Therefore, we may write for each $x \in U_{ij}$,

$$\begin{aligned} \varphi_i {\varphi_j}^{-1}(x, s_j(x)) &= \varphi_i \, s_{U_j}(x) \\ &= \varphi_i \, s_{U_i}(x). \end{aligned}$$

Hence, we get

$$f_{ij} s_j = s_i \text{ in } \mathcal{O}_X(U_{ij})^n, \quad \forall\, i, j. \tag{6.3}$$

Conversely, given a collection of vectors $s_i \in \mathcal{O}_X(U_i)^n$ satisfying the above compatibility relations, there exists a unique section $s : X \to \mathcal{E}$ associated to the local data s_i.

29. Let $A \to A'$ be a flat homomorphism of rings.

(i) Show that if $a \in A$ is a nonzero divisor (nzd) so remains its image in A'. Deduce the existence of a natural induced homomorphism of total rings of fractions $R(A) \to R(A')$.

(ii) Show that if $f : X' \to X$ is a flat morphism of schemes and D is a Cartier divisor on X, then there exists a well defined Cartier divisor D' on X' satisfying the following requirement. If $U = \operatorname{Spec}(A)$, $U' = \operatorname{Spec}(A')$ are affine open subsets such that $f(U') \subseteq U$ and $r \in R = R(A)$ is a local equation of D on U then $f^\star r \in R'$ is a local equation of D' on U', where $f^\star : R \to R'$ is as in (i).

(iii) Show that $f^\star \mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_{X'}(D')$.

30. Let D, D' be Cartier divisors on X. Show that $\mathcal{O}(D) \otimes \mathcal{O}(D')$ is isomorphic to $\mathcal{O}(D + D')$. Deduce that $D \longmapsto \mathcal{O}(D)$ gives a homomorphism from $\operatorname{CDiv}(X) \big/ \mathrm{P}(X)$ in $\operatorname{Pic}(X)$, the Picard group of isomorphism classes of line bundles on X.

According to (6.1), we have $D = (\{U_i\}, s_i) = (\{X\}, f)$. The converse is easy. $\square$

6.17. Corollary. *Let D, D' be Cartier divisors. If $\mathcal{O}(D)$ and $\mathcal{O}(D')$ are isomorphic line bundles then the associated cycles $[D], [D']$ are rationally equivalent.*

Proof. If $\mathcal{O}(D)$ is isomorphic to $\mathcal{O}(D')$ then $\mathcal{O}(D - D')$ is the trivial line bundle. It follows that $[D] - [D'] = [D - D'] = [r]$ for some $r \in R(X)$. $\square$

6.18. Remark. Two Cartier divisors are said to be linearly equivalent if they differ by a principal Cartier divisor. The previous result can now be restated thus: linear equivalence of divisors implies rational equivalence of the corresponding cycles.

6.19. Example. Let X be the projective plane curve $zy^2 - x^3$. For any point P in X there exists a function $r \in R(X)$ such that $[r] = \mathrm{P} - \mathrm{O}$, with $\mathrm{O} = [0, 0, 1]$. Indeed, let F be a homogeneous polynomial defining the line OP and let $r = \frac{F}{y}$. It can be checked that $[r] = \mathrm{P} + 2\mathrm{O} - 3\mathrm{O}$. Let D be the Cartier divisor with local equations $f_1 = \frac{x}{x-y}$ on the open subset $U_1 = X \setminus \{\mathrm{O}\}$ and $f_2 = 1$ on $U_2 = X \setminus \{[1, 1, 1], [0, 1, 0]\}$. It can be checked that D is not principal. Nevertheless, we do have $[D] = [0, 1, 0] - [1, 1, 1]$, which lies in $\mathcal{R}_0 X$ (cf. Hartshorne p. 142).

Let $\mathrm{P}(X)$ denote the subgroup of $\operatorname{CDiv}(X)$ formed by the principal Cartier divisors. We have a natural homomorphism,

$$\operatorname{CDiv}(X) \big/ \mathrm{P}(X) \longrightarrow \mathcal{A}_{n-1}(X).$$

It can be shown to be injective (*resp.* surjective) if X is a normal (*resp.* locally factorial) variety.

6.20. Definition. (**1st Chern class**) Let $\mathcal{L} \to X$ be a line bundle and let V be a subvariety of X. Let C be a Cartier divisor in V such that $\mathcal{O}(C)$ is isomorphic to the restriction $\mathcal{L}_{|V}$ (cf. 6.10). We define

$$c_1(\mathcal{L}) \cap V = [C] \quad \text{in } \mathcal{A}_\star(V).$$

The ambiguity for the choice of C in fact vanishes modulo $\mathcal{R}_\star V$ (by 6.17). Recalling the natural homomorphism $\mathcal{A}_\star(V) \to \mathcal{A}_\star(X)$, we may define the

1st Chern class operator,

$$\begin{array}{rccc} c_1(\mathcal{L}): & \mathcal{C}_\star X & \longrightarrow & \mathcal{A}_\star(X) \\ & z := \sum m_i V_i & \longmapsto & c_1(\mathcal{L}) \cap z := \sum m_i c_1(\mathcal{L}) \cap V_i. \end{array}$$

($c_1(\mathcal{L}) \cap z$ is read $c_1(\mathcal{L})$ "cap" z, from topology's cap product.)

It follows from 6.17 that if $\mathcal{L}, \mathcal{L}'$ are isomorphic line bundles then we have $c_1(\mathcal{L}) = c_1(\mathcal{L}')$.

6.21. Definition. Let D be a Cartier divisor on a scheme X and let V be a subvariety of dimension k. We define the **intersection class** of V by D as

$$D \cdot V = c_1(\mathcal{O}(D)) \cap V \quad \text{in } \mathcal{A}_{k-1}(V \cap |D|). \tag{6.4}$$

6.22. Remark. If $V \subset |D|$, the class defined just above is computed as in the above definition 6.20. Now, if $V \not\subset |D|$, then local equations of D restrict to local equations of the Cartier divisor $i^\star D$, where $i: V \hookrightarrow X$ is the inclusion map. In this case, the class $D \cdot V$ is represented by a well defined cycle with support contained in $|D| \cap V$.

For an arbitrary $k-$cycle $z = \sum n_i V_i$, we extend (6.4) by linearity, setting

$$D \cdot z = \sum n_i D \cdot V_i \quad \text{in } \mathcal{A}_{k-1}(|z| \cap |D|). \tag{6.5}$$

Thus we have defined a **homomorphism of intersection** by D,

$$\begin{array}{rccc} D\cdot: & \mathcal{C}_k X & \longrightarrow & \mathcal{A}_{k-1}(|D|) \\ & z & \mapsto & D \cdot z \end{array} \tag{6.6}$$

One might interpret the presence of the line bundle $\mathcal{O}(D)$ in the definition of $D \cdot V$ as a a way of placing D in general position with respect to V. This is indeed the case whenever the space of sections of $\mathcal{O}(D)$ is sufficiently ample, so that, given finitely many subvarieties $V_1, \ldots, V_t$, there exists a section $s \in H^0(\mathcal{O}(D))$ such that the divisor $\mathcal{Z}(s)$ does not contain any V_i, $1 \le i \le t$.

6.23. Examples.(1) Let $H \subset \mathbb{P}^n$ a hyperplane. Let $V \subset \mathbb{P}^n$ a subvariety. Then we have

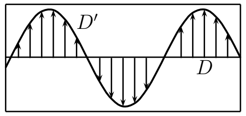
$$H \cdot V = [H' \cap V] \quad \text{in } \mathcal{A}_{k-1}(V \cap H) \quad (k = \dim V),$$

whre H' denotes a hyperplane that does not contain V.

(2) Let D be an effective Cartier divisor on X and let $i: D \hookrightarrow X$ be the inclusion map. Then $i^\star\mathcal{O}(D)$ is the normal bundle of D in X. If D is a variety (see **??** for a generalization), we have the self intersection formula,

$$D \cdot D = c_1(i^\star\mathcal{O}(D)) \cap [D] \quad \text{in } \mathcal{A}_\star(D).$$

If the normal bundle admits a nonzero global section, *i.e.*, a normal field, we may think of an infinitesimal motion of D along the normal flow to the position D'. The formula above can be interpreted as a relation between $D \cap D'$ and the zeros of the section of the normal bundle.



$N_D X$

(2) Let C be a smooth curve. Put $X = C \times C$ and let $i: C \hookrightarrow X$ be the diagonal embedding. The normal bundle of C in X can be identified to the tangent bundle TC. For projective C we may calculate

$$\int_C c_1(TC) \cap [C] = 2 - 2g,$$

where g denotes the genus of C. Indeed, if D is a canonical divisor on C, the corresponding line bundle $\mathcal{O}_C(D)$ is equal to the cotangent bundle $(TC)^\star$. Recalling that the degree of the canonical class is $2g-2$, the formula above follows. In particular, if $C = \mathbb{P}^1$, we have $T\mathbb{P}^1 = \mathcal{O}_{\mathbb{P}^1}(2)$. The formula $\int c_1(T\mathbb{P}^1) \cap [\mathbb{P}^1] = 2$ means that every tangent field ($\neq 0$) vanishes at two points (counted with multiplicity). This is in agreement with the well known topological assertion for the sphere $S^2 = \mathbb{P}^1_{\mathbb{C}}$.

Exercises

27. Show that the support of a Cartier divisor is closed.

28. Verify the assertions of the examples.

$\dim X' > \dim X$. Hence all fibers of p are positive dimensional. Therefore p shrinks the dimension of any component of $p^{-1}V$ for any subvariety $V \subseteq X$, so that we get trivially zero on both sides of (7.4). Assuming $d > 0$ we may write

$$\begin{aligned} p_\star[p^\star r] &= [r^d] \quad \text{(by (3.4))} \\ &= d[r], \end{aligned}$$

as desired. □

7.2. Remark. The hypothesis of pure dimension is essential in (7.1), cf. exercise 36.

Exercises

31. Let D, E be Cartier divisors on a scheme X. Show that for any $z \in \mathcal{C}_k X$, we have

$$(D+E)\cdot z = D\cdot z + E\cdot z \ \text{ in } \ \mathcal{A}_{k-1}((|D|\cup|E|)\cap|z|).$$

32. Let $f : X' \to X$ be a flat morphism of relative dimension n and let $D \in \mathrm{CDiv}\,(X)$. Given $z \in \mathcal{C}_k X$, set $Z = |D| \cap |z|$, $Z' = f^{-1}Z$ and let $g : Z' \to Z$ be the induced map. Show that

$$g^\star(D\cdot z) = f^\star D \cdot f^\star z \ \text{ in } \ \mathcal{A}_{k+n-1}(Z').$$

33. Let $p : X' \to X$ be a proper map and let D be a Cartier divisor on X such that $p^\star D$ is defined *(cf. 6.2.4)*. Pick $z' \in \mathcal{C}_k X'$ and put $Z' = p^{-1}|D| \cap |z'|$, $Z = |D| \cap |p_\star z|$. Let $p' : Z' \to Z$ be the induced map. Show that

$$p'_\star(p^\star D\cdot z') = D\cdot p'_\star z' \ \text{ in } \ \mathcal{A}_{k-1}(Z).$$

34. Let D be a Cartier divisor on a scheme X. Suppose there exists an open subset U of X such that $|D| \subset U$ and the restriction D_U is principal. Let V be a subvariety of dimension k. Define the cycle in $\mathcal{C}_{k-1}|D|$

$$D\cdot V = \begin{cases} 0 & \text{if} \quad V \subset |D| \\ [i^\star D] & \text{if} \quad V \not\subset |D| \end{cases}, i : V \hookrightarrow X.$$

Show that the class of $D\cdot V$ in $\mathcal{A}_{k-1}(|D|)$ is the same as the intersection class defined in (6.4).

35. Let $X \subset \mathbb{A}^3$ be the surface $y = xz$ and let $D = \mathcal{Z}(y)$, $D' = \mathcal{Z}(y-x)$. Compute both $D\cdot[D']$ and $D'\cdot[D]$ in $\mathcal{C}_0(|D|\cap|D'|)$ using the recipe of the previous exercise.

Chapter 7

Elementary properties of the first Chern class

7.1. Proposition. *The operator first Chern class satisfies the following.*

(1) (**Normalization**) *Let X be a scheme of pure dimension n and let D be a Cartier divisor on X. Then we have*

$$c_1(\mathcal{O}_X(D)) \cap [X] = [D] \quad \text{in} \quad \mathcal{A}_{n-1}(X). \tag{7.1}$$

(2) (**Additivity**) *Let $\mathcal{L}, \mathcal{M}$ be line bundles on X. Then we have*

$$c_1(\mathcal{L} \otimes \mathcal{M}) = c_1(\mathcal{L}) + c_1(\mathcal{M}).$$

(3) (**Naturality**) *Let $f : X' \to X$ be a flat morphism. Then, for each cycle $z \in \mathcal{A}_\star(X)$ and for any line bundle $\mathcal{L}$ on X, we have*

$$f^\star\left(c_1(\mathcal{L}) \cap z\right) = c_1(f^\star\mathcal{L}) \cap f^\star z \quad \text{in} \quad \mathcal{A}_\star(X'). \tag{7.2}$$

(4) (**Projection formula**) *Let $p : X' \to X$ be a proper morphism. Then, for each cycle $z' \in \mathcal{A}_\star(X')$ and for any line bundle $\mathcal{L}$ on X, we have*

$$p_\star\left(c_1(p^\star\mathcal{L}) \cap z'\right) = c_1(\mathcal{L}) \cap p_\star(z') \quad \text{in} \quad \mathcal{A}_\star(X). \tag{7.3}$$

Proof. (1) Note that if X is a variety, the formula is but the very definition of the first Chern class 6.20. For the general case, let $X_1, \ldots, X_t$ be the

irreducible components of X. Write $[X] = \sum m_i[X_i]$ with $m_i =$ multiplicity of X along X_i (cf. 1.3). Let D_i be the restriction of D to X_i as in 6.22. Pick a subvariety V of codimension one and set for short $A = \mathcal{O}_{X,V}$. Let $a, b \in A$ be nonzero divisors such that $r = a/b$ is a local equation of D in A, *i.e.*, at the generic point of V (see 6.2.1). The coefficient of V in the cycle $[D]$ is $\ell_A(A/\langle a\rangle) - \ell_A(A/\langle b\rangle)$. Let P_i be the minimal prime of A corresponding to the component X_i. Write $A_i = A/P_i$ and $a_i = a + P_i$ for the residue class and likewise for b. Thus, a_i/b_i is a local equation for D_i in A_i. It follows that the coefficient of V in $[D_i]$ is equal to $\ell_{A_i}(A_i/\langle a_i\rangle) - \ell_{A_i}(A_i/\langle b_i\rangle)$. Recalling that the left hand side in (7.1) is, by definition, $\sum m_i[D_i]$, the asserted formula follows from the lemma 4.9 (3).

(2) The assertion is clear from the definition in view of 6.2.3, p. 34.

(3) As usual, we may assume $z = [V]$ for some variety V, and in fact, we may as well suppose $X = V$. Thus, we have $\mathcal{L} = \mathcal{O}_X(D)$ for a Cartier divisor D. Therefore, we have $f^\star\mathcal{L} = \mathcal{O}_{X'}(f^\star D)$. Now the assertion follows from the following equality of cycles on X',

$$f^\star[D] = [f^\star D]. \tag{$\star$}$$

It has already been verified for D effective (cf. 4.4). For the general case, fix a subvariety $W' \subseteq X'$ of codimension one occurring in either member of ($\star$). By definition, W' is a component of $f^{-1}W$ for some subvariety W of codimension one in X. If we replace X by an affine neighborhood of the generic point of W we loose no information regarding data contributing to the coefficients. Thus we may assume D is principal and write it as a difference of two effective Cartier divisors, in which case ($\star$) is clear.

(4) Note first that the formula is an immediate consequence of the definitions in the case when p is the inclusion of a subvariety $X' \subseteq X$. For the general case, we may assume $z = [X']$, with X' a variety and replace X by $p(X')$, also a variety. Now we have $\mathcal{L} = \mathcal{O}_X(D)$ for some Cartier divisor D on X. The projection formula follows from the following version, valid at the cycle level,

$$p_\star[p^\star D] = \deg(p)[D] \quad \text{in} \quad \mathcal{C}_\star X. \tag{7.4}$$

As in the proof of (3), the verification is local on X, around the generic point of subvarieties of codimension one. Thus, we may assume D principal, say with equation $r \in R(X)$. Write $d = \deg(p)$. If $d = 0$, we have that

(iv) $p^\star D = E + C$, $p^\star D' = E + C'$, *with* C, C' *denoting disjoint effective Cartier divisors such that the restriction of* p *to* C *(resp.* C'*) is an isomorphism onto a closed subscheme of* D *(resp.* D'*).*

(v) *If* $e(D, D') = \varepsilon > 0$ *then* $e(C, E)$ *and* $e(C', E)$ *are both strictly smaller than* ε.

Proof. (i) Recall at first some facts about the blowup. Let $U = \operatorname{Spec}(A)$ be an open affine subset of X and let $a, b \in A$ denote local equations for D, D' so that the local ideal of the scheme intersection Z is $I = \langle a, b\rangle$. For each $c \in I$ put

$$(I_c)_0 = \{d/c^n \mid d \in I^n,\ n \geq 0\},$$

subring of the ring of fractions $(\bigoplus_n I^n)_c$. Then the affine schemes

$$U'_c = \operatorname{Spec}((I_c)_0)$$

provides us with an affine open cover for $U' = p^{-1}U$ as c varies through a set of generators of I (*e.g.*, $\{a, b\}$). We may write

$$\frac{a}{1} = \frac{a}{c} \cdot \frac{c}{1} \quad \text{in} \quad (I_c)_0.$$

In the above expression, $\frac{a}{1}$ (resp. $\frac{c}{1}$) is a local equation for $p^\star D$ (resp. E) in $(I_c)_0$. It follows that $\frac{a}{c}$ gives a local equation for an effective Cartier divisor C such that $p^\star D = E + C$. Similarly, $\frac{b}{c}$ works for C' so that. $p^\star D' = E + C'$. Furthermore, a relation of the form $c = ax + by$ in A implies

$$1 = \frac{a}{c} \cdot \frac{x}{1} + \frac{b}{c} \cdot \frac{y}{1} \quad \text{in} \quad (I_c)_0.$$

This shows that C and C' are disjoint. Setting $c = a$, we see that $C \cap U'_a$ is empty, so that $C \cap U' = C \cap U'_b$. We have the following commutative diagram of coordinate rings induced by p,

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & (I_b)_0 \arrow[d] \\ A/aA \arrow[r, two heads, "\pi"] & (I_b)_0/\frac{a}{b}(I_b)_0. \end{tikzcd}$$

36. Let $X = V(I) \subset \mathbb{P}^3$ with $I = \langle zx, zy\rangle$. Let D be defined by $x = z$. Show that D is a Cartier divisor on X. Compute $[D]$. Compare with (7.1). Repeat for $I = \langle z^2, zx, zy, xy\rangle$, $D := y = x$.

Chapter 8

Commutativity

A nagging aspect of the construction of the intersection cycle $D \cdot z$ of a cycle by a Cartier divisor is the lack of symmetry between the roles of D and z (cf. exercise 35). The cornerstone towards the definition of an intersection class by cycles of codimension ≥ 1, and in fact, for the construction of the Chow ring, is the following important result.

8.1. Theorem. *Let X be a variety and let D, D' be Cartier divisors on X. Then we have*

$$(\star) \qquad D \cdot [D'] = D' \cdot [D] \quad \text{in} \quad \mathcal{A}_{n-2}(|D| \cap |D'|), \ (n = \dim X).$$

Proof. The argument will be split into several steps, successively weakening appropriate additional hypotheses.

(1) *Assume X is normal and D, D' are effective and intersect properly,* i.e.*, without common components.*

Recall that presently, $D \cdot [D']$, $D' \cdot [D]$ are defined as cycles, not just classes (6.22). Let W be a subvariety of codimension two in X and set $B = \mathcal{O}_{X,W}$. Thus B is a normal domain of dimension two, hence Cohen-Macaulay. Let $a, b \in B$ be local equations of D, D' respectively. The ideal $I = \langle a, b \rangle$ defines the intersection of D and D' in a neighborhood of the generic point of W. If it is equal to B then W is not in the support of $|D| \cap |D'|$ and the coefficient of W in both sides of $(\star)$ is zero. If $I \neq B$, then B/I is artinian because I is not contained in any prime of height one. Since B is Cohen-Macaulay, it follows that a is a nonzero divisor modulo b (cp. exc (37), p. 56). The subvarieties V of codimension one in X which contain W correspond to the prime ideals, P, of height one in B. The coefficient of any such V in the cycle $[D']$ is

$\ell_{B_P}(B_P/bB_P)$. The coefficient of W in $D \cdot V$ is $\ell_{B/P}(B/(P + aB))$. At last, the coefficient of W in $D \cdot [D']$ is equal to

$$\sum_P \ell_{B_P}(B_P/bB_P)\, \ell_{B/P}(B/(P + aB)) \tag{8.1}$$

Employing (4.9) with $M = B/bB =: A$, we recognize (8.1) as the quantitiy

$$e(a, A) = \ell_A(A/aA)$$

since ${}_aM = 0$ because a is a nonzero divisor modulo b. Recalling that

$$\ell_A(A/aA) = \ell_B(B/\langle a, b\rangle)$$

it follows at once that (8.1) also matches the coefficient of W in $D' \cdot [D]$.

(2) *Assume D, D' are effective and meet properly.*

Let $f : X' \to X$ be the normalization of X. By the first step, we have

$$f^\star D \cdot [f^\star D'] = f^\star D' \cdot [f^\star D].$$

Applying $f_\star$, we get from the projection formula,

$$\begin{array}{ccc} D \cdot f_\star[f^\star D'] & = & D' \cdot f_\star[f^\star D] \\ || & & || \\ D \cdot [D'] & & D' \cdot [D]. \end{array}$$

(3) *Assume D, D' are effective.*

The plan now is to blowup the excess part of $D \cap D'$ and reduce to the previous case. We define the *excess* of the intersection of D, D' as the integer

$$e(D, D') = \max \{\mathrm{ord}_V(D)\mathrm{ord}_V(D') \,|\, V \subset X\}$$

for all subvarieties V of codimension one in X.

We clearly have $e(D, D') \geq 0$, with equality holding only if D, D' meet properly. We shall proceed by induction on the excess. In order to be able to cut it down we will need the following.

8.2. Lemma. *Let X be a variety and let D, D' be effective Cartier divisors Let $Z = D \cap D'$ denote their scheme intersection. Let $p : X' \to X$ denote the blowup along Z and set $E = p^{-1}Z$. Then the following assertions hold.*

Proof. The assertion follows from the formulas

$$\begin{array}{rcll}[X] &=& dh^{n-k}\cap[\mathbb{P}^n] & \text{in } \mathcal{A}_k(\mathbb{P}^n);\\ h\cap[\mathbb{P}^i] &=& [\mathbb{P}^{i-1}] & \text{in } \mathcal{A}_{i-1}(\mathbb{P}^n).\end{array}$$

□

We shall end this section giving a proof for the agreement of the degree à la Hilbert-Samuel with the one introduced in (5.4).

8.9. Proposition. *Let X be a subscheme of $\mathbb{P}^n_K$ of pure dimension $m \geq 0$. Let*

$$S(X)=\bigoplus_t S(X)_t$$

be the homogeneous coordinate ring of X. Let

$$\dim_K S(X)_t = d\,\frac{t^m}{m!}+\cdots, t\gg 0,$$

be the Hilbert-Samuel function of $S(X)$. Put $h=c_1\mathcal{O}(1)$. Then we have

$$[X]=dh^{n-m}\cap[\mathbb{P}^n]\quad \textit{in } \mathcal{A}_m(\mathbb{P}^n),$$

i.e., *d times the class of a subspace of dimension m.*

Proof. We argue by induction on $m=\dim X$. We should recall that the coefficient $d=d_m(X)$, which is the degree à la Hilbert-Samuel, satisfies the following properties (cf. [**?**] p.52-53):

(1) If $X_1,\dots,X_u$ denote the irreducible components of X with respective multiplicities $\mu_1,\dots,\mu_u$, then

$$d_m(X)=\sum_1^u \mu_i d_m(X_i).$$

(2) If $m>0$ and H denotes a hyperplane that does not contain any irreducible component of X, then

$$d_m(X)=d_{m-1}(X\cap H.)$$

For the reader's convenience we sketch a proof. Noetherian induction as in (3.4), p. 11 shows that there exists a filtration

$$S(X)=M^r\supset\cdots\supset M^1\supset M^0=0$$

The bottom map denoted by π in the above diagram corresponds to the restricition of p to $C\cap U'_b$. Let us check that π is surjective. Pick $x\in I^n$. We may write $x=F(a,b)$ for some homogeneous polynomial $F(x,y)=a_0x^n+a_1x^{n-1}y\cdots+a_ny^n$ with coefficients $a_i\in A$. We then have $x/b^n=F(a/b,1)$ in $(I_b)_0$. Since $F(a/b,1)\equiv a_n \mod \frac{a}{b}(I_b)_0$, it follows that π is surjective. This proves the final assertion in (i).

(ii) Say $\varepsilon' := e(C,E)\geq e(D,D')=\varepsilon>0$. Let V' be a subvariety of codimension one of X' for which

$$\varepsilon'=\mathrm{ord}_{V'}(C)\mathrm{ord}_{V'}(E)$$

holds. Since we have $V'\subseteq C$, its image $V=p(V')$ is a subvariety of D of codimension 1 in X because $p_{|C}: C \xrightarrow{\sim} p(C)$. On the other hand, we have (by (7.4))

$$D=p_\star[p^\star D]=p_\star[C+E].$$

This implies

$$\mathrm{ord}_V(D)\geq \mathrm{ord}_{V'}(C+E)$$

and likewise for D'. Now we may compute,

$$\begin{array}{rcl}\varepsilon &\geq& \mathrm{ord}_V(D)\mathrm{ord}_V(D')\geq \mathrm{ord}_{V'}(C+E)\mathrm{ord}_{V'}(C'+E)\\ &\geq& e(C,E)+(\mathrm{ord}_{V'}(E))^2>\varepsilon'.\end{array}$$

The last inequality is strict because $\mathrm{ord}_{V'}(E)>0$. This contradicts the assumption $\varepsilon'\geq\varepsilon$.

Let us proceed with the proof of the theorem 8.1 with the assumptions as in step 3 on p. 50. We are given effective Cartier divisors D, D' with positive excess intersection. Write

$$p': p^{-1}(|D|\cap|D'|)\longrightarrow |D|\cap|D'|$$

for the map induced by p. Using the exercise 33 we may write

$$\begin{array}{rcll}D\cdot[D'] &=& p'_\star(p^\star D\cdot[p^\star D']) & \text{in } \mathcal{A}_{n-2}(|D|\cap|D'|)\\ &=& p'_\star((C+E)\cdot[C'+E]) &\\ &=& p'_\star(C\cdot[E]+E\cdot[C']+E\cdot[E]) &\end{array}$$

(by induction on the excess)

$$= p'_\star(E\cdot[C]+C'\cdot[E]+E\cdot[E])$$

(reverting the previous calculation)

$$= \quad D' \cdot [D].$$

(3) *The general case.*

Let $U = \operatorname{Spec}(A)$ be an affine open subset of X where both D, D' have local equations $a, b \in R(X)$. Let

$$\mathcal{I}(U) = \{d \in A \mid da, db \in A\}.$$

It can easily be checked that this recipe defines a sheaf of ideals on X, associated to a closed subscheme $Z \subseteq X$. If $Z = X$ then D, D' are effective. If $Z \neq X$, let $p : X' \to X$ be the blowup along Z. A local argument similar to that at the beginning of the proof of lemma 8.2 shows that the Cartier divisors,

$$C = p^\star D + E \quad \text{and} \quad C' = p^\star D' + E$$

are both effective. The argument may now be completed by reduction to the previous case and performing a calculation similar to the one made there. □

8.3. Corollary. *Let L be a line bundle over a scheme X. If z is a cycle rationally equivalent to zero on X then,*

$$c_1(L) \cap z = 0 \quad \textit{in } \mathcal{A}_\star(X).$$

Hence, the first Chern class induces an operator, still denoted

$$c_1(L) : \mathcal{A}_\star(X) \longrightarrow \mathcal{A}_\star(X)$$

which takes classes of k-cycles to classes of k–1-cycles.

Proof. We may assume $z = [r]$, the divisor of some rational function $r \in R(V)$ for some subvariety $V \subset X$. Let D be a Cartier divisor associated to $\mathrm{L}_{|V}$ (cf. 6.10) and let D' be the principal Cartier divisor defined by r. We then have,

$$\begin{aligned} c_1(\mathrm{L}) \cap z &= D \cdot [D'] && \text{in } \mathcal{A}_\star(|D| \cap |D'|) \\ &= D' \cdot [D] && \text{(by 8.1)} \\ &= c_1(\mathcal{O}_V) \cap [D] = 0 && \text{in } \mathcal{A}_\star(|D|) \end{aligned}$$

since $\mathcal{O}(D') = \mathcal{O}_V$, the trivial line bundle. □

Essentially the same argument also gives the following variant.

8.4. Corollary. *Let D be a Cartier divisor on a scheme X. The homomorphism given by intersecting with D (cf. 6.4) passes to the quotient under rational equivalence thereby inducing,*

$$D : \mathcal{A}_k(X) \longrightarrow \mathcal{A}_{k-1}(|D|).$$

□

8.5. Warning! If D is principal, even though we certainly have that $D \cdot z = 0$ holds in $\mathcal{A}_\star(X)$, it may well happen that $D \cdot z$ be non zero in $\mathcal{A}_\star(|D|)$. Check for instance in the example $z = X = \mathbb{A}^1$, $D = \{0\}$ (a single point).

8.6. Corollary. *Let $i : D \hookrightarrow X$ be the inclusion map of an effective Cartier divisor. Then, for each cycle $z \in \mathcal{A}_\star(D)$ we have*

$$D \cdot z = c_1(i^\star \mathcal{O}(D)) \cap z \quad \textit{in } \mathcal{A}_\star(D).$$

□

Once we know that c_1 defines operators on $\mathcal{A}_\star(X)$, we may compose them. The following is an easy consequence.

8.7. Corollary. *Let $L, \mathcal{M}$ be line bundles over a scheme X. We have, for all $z \in \mathcal{A}_k(X)$,*

$$c_1(L) \cap (c_1(\mathcal{M}) \cap z) = c_1(\mathcal{M}) \cap (c_1(L) \cap z) \quad \textit{in } \mathcal{A}_{k-2}(X).$$

□

This last result enables us to define new operators by taking polynomials on the Chern classes of line bundles $\mathrm{L}_1, \ldots, \mathrm{L}_n$. More precisely, given $P(t_1, \ldots, t_n) \in \mathbb{Z}[t_1, \ldots, t_n]$, we may take $P(c_1(\mathrm{L}_1), \ldots, c_1(\mathrm{L}_n))$ as an operator on $\mathcal{A}_\star(X)$.

8.8. Corollary. *Let $X \subset \mathbb{P}^n$ be a closed subscheme of pure dimension k and degree d (cf. 5.4). Put $h = c_1\mathcal{O}(1)$. Then we have*

$$d = h^k \cap [X].$$

Proof. We write for short

$$\epsilon = e - 1.$$

(1) Let $V \subset X$ be a subvariety of dimension k. By (3) we have that $s_i(E) \cap V$ comes from $\mathcal{A}_{k-i}(V)$. This group is zero for $i < 0$ and is equal to $\mathbb{Z}V$ for $i = 0$. Thus $s_0(E) \cap V = mV$ for some $m \in \mathbb{Z}$. In order to find m, we apply (2) to the inclusion map $U \hookrightarrow V$ of an open subset trivializing E. Now assuming E trivial we may write

$$c_1(\mathcal{O}_E(1)) \cap [\mathbb{P}^\epsilon \times V] = [H \times V]$$

where $H = \mathbb{P}^{\epsilon - 1}$ denotes a hyperplane. Iterating we find

$$s_0(E) \cap V = p_\star \left(c_1(\mathcal{O}_E(1))^\epsilon \cap [\mathbb{P}^\epsilon \times V]\right) = p_\star[\mathbb{P}^0 \times V] = V$$

whence $m = 1$.

(2) We may compute, with the notation as in 9.1, p. 57,

$$\begin{aligned}
s_i(g^\star E) \cap g^\star z &= p'_\star \left(c_1\left(\mathcal{O}_{E'}(1)\right)^{\epsilon+i} \cap p'^\star g^\star z\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (9.2) \\
&= p'_\star \left(c_1\left(g'^\star \mathcal{O}_E(1)\right)^{\epsilon+i} \cap g'^\star p^\star z\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (7.2) \\
&= p'_\star g'^\star \left(c_1\left(\mathcal{O}_E(1)\right)^{\epsilon+i} \cap p^\star z\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{prop. } 4.7) \\
&= g^\star p_\star \left(c_1\left(\mathcal{O}_E(1)\right)^{\epsilon+i} \cap p^\star z\right) \\
&= g^\star \left(s_i(E) \cap z\right).
\end{aligned}$$

The verification of (4) is similar, using now (7.3).

In order to prove (5) we consider the cartesian diagram

$$\begin{array}{ccc}
\mathbb{P}(E) \times_X \mathbb{P}(F) & \xrightarrow{p'} & \mathbb{P}(F) \\
q' \downarrow & & \downarrow q \\
\mathbb{P}(E) & \xrightarrow[p]{} & X.
\end{array}$$

by graded S−submodules, with $S = K[t_0, \ldots, t_n]$, such that

$$M^i / M^{i-1} \simeq (S/P_i)(n_i)$$

for suitable homogeneous primes $P_i \subset S$ and integers n_i. The minimal elements amongst the P_i's are the ideals of corresponding X_j's. Moreover, the number of times that each minimal prime P occurs is equal to the length of $S(X)_P$ as S_P−module. Indeed, it suffices to notice that $(S/P_i)_P = 0$ if and only if $P \neq P_i$. Well, the same thing holds for the localization in degree zero, $(S/P_i)_{(P)}$, formed by all fractions with numerator and denominator of like degree. Thus, that length coincides with the length μ_j of $S(X)_{(P)}$, since the latter ring is the local ring of X along the X_j corresponding to P. Recalling that, for $t \gg 0$, $\dim_K \left(\left(S/P_i\right)(n_i)\right)_t$ is a polinomial in t of degree equal to the dimension of the projective variety defined by P_i, it follows that the leading coefficient $d_m(X)/m!$ is the sum of the $d_m(X_i)/m!$, each one of these occurring μ_i times.

Exercises

37. This exercise shows that the argument at the beginning of the proof of the theorem *8.1* fails in the absence of normality. Take a projection of the quartic normal curve of $\mathbb{P}^4$ to $\mathbb{P}^3$, say given by the parametrization $[u, v] \mapsto [v^4, uv^3, u^3v, u^4], \in \mathbb{P}^3$. Let A be the local ring at the vertex of the corresponding cone in $\mathbb{A}^4$. Set $a = v^4$, $b = u^4 \in A$. Then $A/\langle a, b\rangle$ is artinian but a, b is not a regular sequence in A.

```
ring r=0,(x(0..3)),dp;ring rr=0,(x(0..3),s,t),dp;
def q=ideal(s,t); q=q^4;q; q= ideal(q[1..2],q[4..5]);q;
for(i=1;i<=4;i++){q[i]=q[i]-x(i-1);}q=eliminate(q,st);
setring r;imapall(rr);q=std(q);q;def qx=std(q+x(0));qx;
dim(qx); quotient(qx,x(3));
reduce(_,qx); def qx1=std(qx+x(3));qx1;
dim(_);
```

Chapter 9

The Chern classes

Let $f : E \to X$ be a vector bundle of rank e. We shall define Segre classes $s_i(E)$ and Chern classes $c_i(E)$ $(i = 0, 1, \dots)$ as homogeneous operators of degree $-i$ acting on the Chow group $\mathcal{A}_\star(X)$, *i.e.*, the image of $\mathcal{A}_k(X)$ lies in $\mathcal{A}_{k-i}(X)$.

The construction will strongly rely on the structure of the Chow group of the associated projective bundle $p : \mathbb{P}(E) \to X$.

9.1. We recall briefly some basic properties of the projective bundle associated to a vector bundle. If $U \subseteq X$ is an open subset such that $E_U := f^{-1}(U) \simeq U \times \mathbb{A}^e$, we have

$$\mathbb{P}(E)_U := p^{-1}(U) \simeq U \times \mathbb{P}^\epsilon \quad (\text{with } \epsilon = e - 1).$$

In particular, it follows that p is proper and flat (of relative dimension r). The vector bundle $p^\star E$ admits a tautological subbundle of rank one, written $\mathcal{O}_E(-1)$ (or simply $\mathcal{O}(-1)$ if no confusion is possible). For a point $t \in \mathbb{P}(E)$ put $x = p(t)$. Then t lies in the fiber $\mathbb{P}(E_x) \simeq \mathbb{P}^\epsilon$. The fiber of $\mathcal{O}_E(-1)$ at t is precisely the one dimensional subspace of E_x that t represents. The dual line bundle $\mathcal{O}_E(-1)^\star$ is denoted $\mathcal{O}_E(1)$, and is also called the tautological quotient line bundle of $p^\star E^\star$

Given a map $g : X' \to X$, write $E' := g^\star E$. Form the cartesian diagram,

$$\begin{array}{ccc} \mathbb{P}(E') & \xrightarrow{g'} & \mathbb{P}(E) \\ p' \downarrow & & \downarrow p \\ X' & \xrightarrow{g} & X \end{array} \tag{9.1}$$

We have

$$(g')^\star \mathcal{O}_E(1) = \mathcal{O}_{E'}(1). \tag{9.2}$$

9.2. Definition. For each cycle $z \in \mathcal{A}_k(X)$ we set

$$s_i(E) \cap z = p_\star \left(c_1 \left(\mathcal{O}_E(1) \right)^{\epsilon+i} \cap p^\star z \right) \in \mathcal{A}_{k-i}(X).$$

Note that $p^\star z$ lies in $\mathcal{A}_{k+\epsilon}(\mathbb{P}(E))$. Hence, letting $c_1(\mathcal{O}_E(1))$ act $\epsilon + i$ times on $p^\star z$ lands in $\mathcal{A}_{k-i}(X)$.

The operator $s_i(E)$ thus defined is called the $i-$**th Segre class** of the vector bundle E.

9.3. Remark. For a nonsingular variety X, the group $\mathcal{A}_\star(X)$ is endowed with a product (cf. 13.14). In this case, the operator $s_i(E)$ coincides with the multiplication by the cycle class $s_i(E) \cap [X]$ (cf. 13.15(3)).

9.4. Proposition. *We have the following properties of the Segre class.*

(1) $s_0(E) = 1$ (= identity operator) *and* $s_i(E) = 0$ *for* $i < 0$ *and for* $i > \dim X$.

(2) (naturality) *If* $g : X' \to X$ *is a flat map and* $z \in \mathcal{A}_\star(X)$ *then*

$$s_i(g^\star E) \cap g^\star z = g^\star \left(s_i(E) \cap z \right).$$

(3) (projection formula) *If* $g : X' \to X$ *is proper and* $z' \in \mathcal{A}_\star(X')$ *then*

$$g_\star \left(s_i(g^\star E) \cap z' \right) = s_i(E) \cap g_\star z'.$$

(4) (commutativity) *If* E, F *are vector bundles over* X *and* $z \in \mathcal{A}_\star(X)$ *then*

$$s_i(E) \cap s_j(F) \cap z = s_j(F) \cap s_i(E) \cap z.$$

(5) (normalization) *If* E *is a line bundle then*

$$s_1(E) = -c_1(E).$$

(cf. (9.3), p. 61 for higher rank).

the elementary symmetric functions. Then we have

(1) *If* E *admits a section* s *which is nowhere zero then* $\prod_{1}^{e} \left(c_1(L^{(j)})\right) = 0.$

(2) $c(E) = \prod_{1}^{e} \left(1 + c_1(L^{(j)})\right)$, i.e., $c_i(E) = \begin{cases} \sigma_i & \textit{for each } i = 1, \ldots, e; \\ 0 & \textit{for } i > e. \end{cases}$

Proof. (1) Let $V \subseteq X$ be a subvariety. We must show that $\prod c_1(L^{(j)}) \cap V = 0$. Using the projection formula (7.3) for the inclusion map of V in X, we may as well assume $V = X$. We form the diagram of maps of vector bundles

$$\begin{tikzcd} & \mathcal{O}_X \arrow[dl, dotted, "\widetilde{s}"'] \arrow[d, "s"] \arrow[dr, "s'"] & \\ F = E^{(1)} \arrow[r, tail] & E \arrow[r, two heads] & L^{(1)}. \end{tikzcd} \tag{9.5}$$

Let W be the scheme of zeros of s'. If $W = X$ then the section s factors as indicated by the dotted arrow. By induction on the rank, we find $c_2(L^{(2)}) \cdots c_e(L^{(e)}) = 0$. So we may now assume W is an effective Cartier divisor. Therefore, we have $c_1(L^{(1)}) \cap [X] = [W]$ in $\mathcal{A}_\star(X)$ cf. (6.20), p. 40. If W is empty then $c_1(L^{(1)}) \cap [X] = 0$ and we are done. Else, let $i : W \hookrightarrow X$ denote the inclusion map. Restricting the diagram above to W, we have $i^\star s' = 0$, hence $i^\star s$ induces a section $\widetilde{s}$ of F as indicated by the dotted arrow. By induction on the rank of E we may write

$$\prod_{2}^{e} c_1(i^\star L^{(j)}) = 0.$$

Hence we have

$$\begin{aligned} \prod_{1}^{e} c_1(L^{(j)}) \cap [X] &= \Big(\prod_{2}^{e} c_1(L^{(j)})\Big) \cap \left(c_1(L_1) \cap [X]\right) \\ &= \Big(\prod_{2}^{e} c_1(L^{(j)}) \cap i_\star[W] \\ \text{(by 7.3, p. 45)} \quad &= i_\star\Big(\prod_{2}^{e} c_1(i^\star L^{(j)}) \cap [W]\Big) = 0. \end{aligned}$$

Set $\varphi = \operatorname{rank} F$. Using tools similar to the proof of (2) and (3), we may write

$$\begin{aligned} s_i(E) \cap s_j(F) \cap z &= p_\star\Big(c_1(\mathcal{O}_E(1))^{\epsilon+i} \cap p^\star q_\star\Big(c_1(\mathcal{O}_F(1))^{\varphi+j} \cap q^\star z\Big)\Big) \\ &= p_\star\Big(c_1(\mathcal{O}_E(1))^{\epsilon+i} \cap q'_\star p'^\star\Big(c_1(\mathcal{O}_F(1))^{\varphi+j} \cap q^\star z\Big)\Big) \\ &= p_\star\Big(c_1(\mathcal{O}_E(1))^{\epsilon+i} \cap q'_\star\Big(c_1(p'^\star\mathcal{O}_F(1))^{\varphi+j} \cap p'^\star q^\star z\Big)\Big) \\ &= p_\star q'_\star\Big(c_1(q'^\star\mathcal{O}_E(1))^{\epsilon+i} \cap \Big(c_1(p'^\star\mathcal{O}_F(1))^{\varphi+j} \cap p'^\star q^\star z\Big)\Big) \\ \text{(by 8.7)} \quad & \\ &= q_\star p'_\star\Big(c_1(p'^\star\mathcal{O}_F(1))^{\varphi+j} \cap \Big(c_1(q'^\star\mathcal{O}_E(1))^{\epsilon+i} \cap q'^\star p^\star z\Big)\Big) \\ &= q_\star\Big(c_1(\mathcal{O}_F(1))^{\varphi+j} \cap p'_\star q'^\star\Big(c_1(\mathcal{O}_E(1))^{\epsilon+i} \cap p^\star z\Big)\Big) \\ &= s_j(F) \cap s_i(E) \cap z. \end{aligned}$$

This completes the verification of (4).

(5) The formula results from the fact that

$$p : \mathbb{P}(E) \to X \text{ and } p^\star E^\star \to \mathcal{O}_E(1)$$

are isomorphisms. Indeed, for $z \in \mathcal{A}_\star(X)$ we have

$$\begin{aligned} s_1(E) \cap z &= p_\star\left(c_1(\mathcal{O}_E(1)) \cap p^\star z\right) \\ &= p_\star\left(c_1(p^\star E^\star) \cap p^\star z\right) \\ &= p_\star p^\star\left(c_1(E^\star) \cap z\right) \\ &= -c_1(E) \cap z. \end{aligned}$$

$\square$

9.5. Definition. Let $E \to X$ be a vector bundle. We define the **total Segre class** of E as

$$s(E) = s_0(E) + s_1(E) + \cdots = \sum s_i(E).$$

Knowing already that $s_0(E) = 1$ and that the operators $s_i(E), s_j(E)$ with $i, j > 0$ are nilpotent and commute, it follows that the operator total Segre class $s(E)$ is invertible in the ring of endomorphisms of $\mathcal{A}_\star(X)$. Thus we may define the **total Chern class** of E by

$$c(E) = s(E)^{-1}.$$

We may write

$$c(E) = c_0(E) + c_1(E) + \cdots,$$

decomposition into homogeneous components, so that

$$c_i(E) \cap \mathcal{A}_k(X) \subseteq \mathcal{A}_{k-i}(X).$$

It follows from the definition that we have the identities,

$$\begin{cases} c_0(E) = 1, \\ c_1(E) = -s_1(E), \\ \sum_k s_{i-k}c_k = 0, \quad \text{for} \quad i \geq 1. \end{cases} \tag{9.3}$$

We may compute, for $i \geq 2$, the determinant

$$c_i = (-1)^i \begin{vmatrix} s_1 & s_2 & s_3 & \cdots & s_{i-1} & s_i \\ 1 & s_1 & s_2 & \cdots & s_{i-2} & s_{i-1} \\ 0 & 1 & s_2 & \cdots & s_{i-3} & s_{i-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & s_1 \end{vmatrix}. \tag{9.4}$$

9.6. Remark. It is clear that the Chern classes satisfy properties similar to those listed in prop. 9.4, p. 58.

More of a surprise and quite central are the following next two properties.

9.7. Proposition. (1) *Let $E \to X$ be a vector bundle of rank e. Then*

$$c_i(E) = 0 \;\forall i > e.$$

(2) *Let*

$$E' \rightarrowtail E \twoheadrightarrow E''$$

be an exact sequence of vector bundles. Then we have
(additivity) $\qquad c(E) = c(E')c(E'')$
i.e., *for each m,*

$$c_m(E) = \sum_{i+j=m} c_i(E')c_j(E'').$$

Proof. The argument for both assertions is by reduction to the case of line bundles. The technique employed is known by the name of

9.8. Proposition. (The splitting principle.) *Let (E_i) be a finite family of vector bundles over X. Then there exists a flat map $f : X' \to X$ such that*

(1) *the induced homomorphism $f^\star : \mathcal{A}_\star(X) \to \mathcal{A}_\star(X')$ is injective;*

(2) *each $f^\star E_i$ admits a filtration by vector subbundles*

$$E_i^{e_i} = 0 \subset \cdots \subset E_i^1 \subset E_i^0 = f^\star E_i$$

whose successive quotients are line bundles, $L_i^j = E_i^{j-1}/E_i^j$.

Proof of the splitting principle. Let be given just one vector bundle E. Let $p : \mathbb{P}(E) \to X$ be the associated projective bundle. We have the tautological exact sequence,

$$\mathcal{O}_E(-1) \rightarrowtail p^\star E \twoheadrightarrow F.$$

Furthermore, the map

$$p^\star : \mathcal{A}_\star(X) \longrightarrow \mathcal{A}_\star(\mathbb{P}(E))$$

is injective. Indeed, setting $\epsilon = \operatorname{rank} E - 1$, we have for any $z \in \mathcal{A}_\star(X)$,

$$z = s_0(E) \cap z = p_\star\left(c_1\left(\mathcal{O}_E(1)\right)^\epsilon \cap p^\star z\right).$$

By induction on the rank of E, there exists a flat map $q : X' \to \mathbb{P}(E)$ such that $q^\star F$ admits a filtration as stated and with $q^\star$ injective. Composing with p, we have shown the assertions for the case of just one vector bundle. The general case follows at once. □

As a first application of the splitting principle, we have the following important step towards the proof of 9.7 which is also of independent interest.

9.9. Lemma. *Let E be a vector bundle endowed with a filtration $E = E^{(0)} \supset \cdots \supset E^{(e)} = 0$ with successive line bundle quotients $L^{(j)} = E^{(j-1)}/E^{(j)}$. Put $\lambda_j = c_1(L^{(j)})$ and define*

$$\begin{cases} \sigma_1 &= \sum c_1(L^{(j)}), \\ \sigma_2 &= \sum_{i<j} c_1(L^{(i)})c_1(L^{(j)}), \\ & \vdots \\ \sigma_e &= c_1(L^{(1)}) \cdots c_1(L^{(e)}), \end{cases}$$

Proof. The splitting principle 9.8 together with property (3) above and additivity 9.7 reduce the verification to the case of line bundles. This case is covered by the recipe (1). Since 9.7 depends only on (1),(2) and (3) just above, the uniqueness follows readily. □

Exercises

38. Compute the total Segre class $s(\mathcal{O}_{\mathbb{P}^n}(1))$.

39. With the notation as in 9.9, show that any symmetric polynomial on the Chern classes of $L^{(1)}, \ldots, L^{(e)}$ can be expressed as a polynomial on the Chern classes of E. In particular, find a formula for $\sum c_1(L^{(j)})^2$ in terms of the $c_i(E)$.

40. Let $L, L_1, \ldots, L_e$ be line bundles and put $E = L_1 \oplus \cdots \oplus L_e$. (i) Compute $c(E \otimes L)$ in terms of $c_1(L)$ and $c_\star(E)$. (ii) Show that the second symmetric power S_2E is isomorphic to $\bigoplus_{i \leq j} L_i \otimes L_j$. (iii) Use the previous item to express $c(S_2E)$ in terms of $c_\star(E)$. (iv) Show that $\overset{e}{\wedge} E$ is isomorphic to $L_1 \otimes \cdots L_e$. Deduce that $c_1(\overset{e}{\wedge} E) = c_1(E)$ for *any* vector bundle of rank e. (v) Show that $c_i(E^\star) = (-1)^i c_i(E)$ where$E^\star =$ dual of E.

41. Let E, F be vector bundles of rank 2. Compute $c_\star(E \otimes F)$ in terms of $c_\star(E), c_\star(F)$.

42. Let E be a vector bundle of rank e endowed with a nowhere zero section. Then $c_e(E) = 0$.

43. Let X be a complete scheme. Let C be an operator on $\mathcal{A}_\star(X)$ and pick $z \in \mathcal{A}_\star(X)$. Write $\int_z C := \int Cz$, the degree of the zero cycle part of Cz (cf. 3.5). Now let E, F be vector bundles over complete varieties V, W respectively. Show that

$$\int_{V\times W} c_\star(p^\star E)\, c_\star(q^\star F) = \Big(\int_V c_\star(E)\Big)\Big(\int_W c_\star(F)\Big),$$

where p, q denote the projection maps from $V \times W$.

We may now prove (2). Look at the projective bundle $p : \mathbb{P}(E) \to X$. The tautological line subbundle $\mathcal{O}_E(-1) \rightarrowtail p^\star E$ yields a nowhere zero section of $p^\star E \otimes \mathcal{O}_E(1)$. Thus, writing $h = c_1\mathcal{O}_E(1), \lambda'_j = c_1(p^\star L^{(j)})$ and using (1) we may compute

$$\begin{aligned} 0 &= \prod_1^e c_1(p^\star L^{(j)} \otimes \mathcal{O}_E(1)) \\ &= \prod_1^e (h + \lambda'_j) \qquad (9.6) \\ &= \sum_1^e h^{r-k} \sigma'_k, \end{aligned}$$

where σ'_k denotes the kth elementary symmetric function on the λ'_j. Multiplying by h^{i-1} with $i \geq 1$, $\epsilon = e - 1$ and applying $p_\star$, we find for any $z \in \mathcal{A}_\star(X)$,

$$0 = \sum p_\star(h^{\epsilon+i-k}\sigma'_k \cap p^\star z) = \sum p_\star(h^{\epsilon+i-k} p^\star(\sigma_k \cap z)) = \sum s_{i-k}(E)\sigma_k \cap z.$$

Comparing with (9.3), p. 61, the result follows easily. □

We may put the pieces together and finish the

Proof of 9.7. By the splitting principle, in order to prove the statement (1) in 9.7 we may assume E admits a filtration as in 9.9. Therefore, we find $c_i(E) = 0$ for all $i > \operatorname{rank} E$. Finally, for the proof of 9.7(2), we may build a filtration for E joining filtrations of E' and E''. The formula of additivity results from the calculation of $c(E'), c(E'')$ and $c(E)$ using 9.9(2). □

The splitting principle may be rephrased roughly as follows: in order to prove formulas involving Chern classes of vector bundles $E_1, \ldots, E_n$, it suffices to prove it whenever each E_i is a direct sum of line bundles. In fact, we have the following.

9.10. Proposition. *Let $E_1, \ldots, E_n$ be vector bundles over a scheme X. Then there exists a flat map $f : X' \to X$ of some relative dimension m such that $f^\star : \mathcal{A}_\star(X) \to \mathcal{A}_\star(X')$ is injective and there are line bundles $L_{i,j}$ such that $c(f^\star E_i) = c(\oplus L_{i,j})$ for each $i = 1, \ldots, n$.*

Proof. The assertion follows easily from prop. 9.7 (p. 61) together with 9.8 and 9.9. □

The key to many interesting geometrical applications of Chern classes is the following generalization of (7.1) and of 9.9(1).

9.11. Proposition. *Let E be a vector bundle of rank e over a scheme X of pure dimension n. Let s be a regular section of E* (see 6.13). *Then*

$$c_e(E) \cap [X] = [\mathcal{Z}(s)] \quad \textit{in } \mathcal{A}_{n-e}(X).$$

Proof. For $e = 1$ the formula is a special case of (7.1). Assume now $e \geq 2$. By the splitting principle, we may pick a flat map $f : X' \to X$ of some relative dimension m such that $f^\star : \mathcal{A}_\star(X) \to \mathcal{A}_\star(X')$ is injective and $f^\star E$ fits into an exact sequence of vector bundles,

$$F \rightarrowtail f^\star E \twoheadrightarrow L$$

with $\operatorname{rank} L = 1$. The section $f^\star s$ induces a section s' of L and both are regular. Let $i : \mathcal{Z}(s') \hookrightarrow X'$ be the inclusion map. The restriction $i^\star f^\star s$ over $\mathcal{Z}(s')$ factors through a section $\widetilde{s}$ of $i^\star F$ as in (9.5) which is also regular. By induction, we have

$$c_{e-1}(i^\star F) \cap [\mathcal{Z}(s')] = [\mathcal{Z}(\widetilde{s})] \ \text{ in } \mathcal{A}_{n-e}(\mathcal{Z}(s')).$$

Note that we have $\mathcal{Z}(\widetilde{s}) = \mathcal{Z}(f^\star s) = f^{-1}\mathcal{Z}(s)$. Write $g : \mathcal{Z}(\widetilde{s}) \to \mathcal{Z}(s)$ for the (flat) map induced by f. Let $j : \mathcal{Z}(\widetilde{s}) \hookrightarrow \mathcal{Z}(s')$ and $k : \mathcal{Z}(s) \to X$ be the inclusion maps. We have

$$\begin{aligned}
f^\star k_\star[\mathcal{Z}(s)] &= i_\star j_\star g^\star[\mathcal{Z}(s)] \\
\text{(by 4.4, 4.7)} & \\
&= i_\star j_\star[\mathcal{Z}(\widetilde{s})] \\
\text{(by induction)} & \\
&= i_\star c_{e-1}(i^\star F) \cap [\mathcal{Z}(s')] \\
\text{(by 7.3)} & \\
&= c_{e-1}(F) \cap c_1(L)[X'] \\
\text{(9.7 (2))} & \\
&= c_e(f^\star E) \cap f^\star[X] \\
\text{(7.2)} & \\
&= f^\star\left(c_e(E) \cap [X]\right).
\end{aligned}$$

Since $f^\star$ is injective, the verification is complete. $\square$

9.12. Example. Let $E \rightarrowtail F \twoheadrightarrow Q$ be an exact sequence of vector bundles over a pure dimensional scheme X. Write $p : \mathbb{P}(F) \to X$ for the structure map. Study the diagram of natural vector bundle homomorphisms,

$$\begin{tikzcd}
& \mathcal{O}_F(-1) \arrow[d, tail] \arrow[dr] & \\
p^\star E \arrow[r, tail] & F \arrow[r, two heads] & p^\star Q.
\end{tikzcd}$$

It yields a section s of $p^\star Q \otimes \mathcal{O}_F(1)$. The scheme of zeros of s is equal to $\mathbb{P}(E) \subseteq \mathbb{P}(F)$. Restricting to an open subset where the bundles become trivial, it can be seen that s is regular. Therefore we have

$$[\mathbb{P}(E)] = c_r\left(p^\star Q \otimes \mathcal{O}_F(1)\right) \cap [\mathbb{P}(F)]$$

in $\mathcal{A}_\star(\mathbb{P}(F))$, with $r = \operatorname{rank} Q$.

For instance, if X is just a point, we retrieve the class of a subprojective space $\mathbb{P}^{n-r} \subseteq \mathbb{P}^n$ as $c_r(\mathcal{O}(1)^{\oplus r}) \cap [\mathbb{P}^n] = c_1(\mathcal{O}(1))^r \cap [\mathbb{P}^n]$.

We end this chapter giving a characterization of Chern classes.

9.13. Proposition. *The Chern classes operators are uniquely determined by the following requirements:*

(1) (normalization) *If L is a line bundle over a variety X and D is Cartier divisor such that $\mathcal{O}(D) \simeq L$, then*

$$c_1(L) \cap [X] = [D].$$

(2) (projection formula) *If $p : X' \to X$ is a proper map and E is a vector bundle over X, then for each i we have*

$$p_\star\left(c_i(p^\star E) \cap z'\right) = c_i(E) \cap p_\star z' \ \forall\, z' \in \mathcal{A}_\star(X').$$

(3) (naturality) *If $f : X' \to X$ is flat of some relative dimension and E is a vector bundle over X, then for each i we have*

$$f^\star\left(c_i(E) \cap z\right) = c_i(f^\star E) \cap f^\star z \ \forall\, z \in \mathcal{A}_\star(X).$$

is commutative in view of the remark 9.6. Now α_{E_Y} (resp. α_{E_U}) is surjective by induction (resp. by the previous case), thereby implying that α_E is surjective too.

For the proof of injectivity, suppose we have a relation

$$\sum_0^{\epsilon} h^i \cap p^\star a_i = 0.$$

Applying $p_\star$ and recalling 9.4 (1),(2), we find $a_\epsilon = 0$. Multiplying by h and repeating the argument, injectivity follows. □

The next result is fundamental for the construction of intersection classes for codimensions bigger than one.

10.2. Corollary. *Let $f : E \to X$ be a vector bundle of rank e. Then*

$$f^\star : \mathcal{A}_k(X) \longrightarrow \mathcal{A}_{k+e}(E)$$

is an isomorphism for all k.

Proof. Put $F = E \oplus \mathcal{O}$. Consider the diagram,

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[dr, "f"'] & \mathbb{P}(F) \arrow[d, "g"] & \mathbb{P}(E) \arrow[l, hook', "i"'] \arrow[dl, "p"] \\ & X & \end{tikzcd} \tag{10.2}$$

where j (resp. i) denotes the natural open (resp. closed) immersion. Pick $z \in \mathcal{A}_{k+e}(E)$. There exists $y \in \mathcal{A}_{k+e}(\mathbb{P}(F))$ such that $z = j^\star y$. Put

$$h = c_1(\mathcal{O}_F(1)).$$

Let us express y according to the recipe (10.1), namely,

$$y = g^\star x_k + h \cap (g^\star x_{k+1} + \cdots + h^{e-1} \cap g^\star x_{k+e}), \quad \text{with } x_i \in \mathcal{A}_i(X).$$

Invoking (10.1.1), we see that the term $h \cap (\cdots)$ in the above expression can be written as $i_\star w$ for some $w \in \mathcal{A}_\star(\mathbb{P}(E))$. Therefore, since $j^\star i_\star = 0$ (5.1), we have $j^\star y = j^\star g^\star x_k = f^\star x_k$. This shows that $f^\star$ is surjective. Finally, if $f^\star x_k = 0$ then $g^\star x_k = i_\star w = h \cap w'$ as above. Hence $x_k = 0$ in view of 10.1 applied to F. □

We list below some useful formulas for the inverse of $f^\star$.

44. Let E be a vector bundle of rank $e = \epsilon + 1$ and let L be a line bundle over a scheme X. Let $p : \mathbb{P}(E) \to X$, $q : \mathbb{P}(E \otimes L) \to X$ be the structure maps. Show that there exists an isomorphism $i : \mathbb{P}(E) \simeq \mathbb{P}(E \otimes L)$ such that $qi = p$ and $i^\star \mathcal{O}_{E \otimes L}(1) = \mathcal{O}_E(1) \otimes L^\star$. Deduce the formula

$$s_j(E \otimes L) = \sum (-1)^{j-k} \binom{\epsilon + j}{\epsilon + k} s_k(E) c_1(L)^{j-k}.$$

Chapter 10

Computing some Chow groups

We describe in this chapter the Chow groups of vector bundles and of projective bundles. We also study the Grassmann bundles.

10.1. Proposition. *Let $f : E \to X$ be a vector bundle of rank $e = \epsilon + 1$ and let $p : \mathbb{P}(E) \to X$ be the associated $\mathbb{P}^\epsilon$–bundle. Put $h = c_1(\mathcal{O}_E(1))$. Then the map*

$$\begin{array}{rccc} \alpha_E : & \bigoplus_0^\epsilon \mathcal{A}_{k-\epsilon+i}(X) & \longrightarrow & \mathcal{A}_k(\mathbb{P}(E)) \\ & (a_i) & \longmapsto & \sum h^i \cap p^\star a_i\,. \end{array} \tag{10.1}$$

is an isomorphism.

Proof. Assume at first $E = \mathbb{A}^e_X$, the trivial bundle of rank e. Let

$$\iota : \mathbb{P}^{\epsilon-1}_X \hookrightarrow \mathbb{P}^\epsilon_X \quad \text{and} \quad j : \mathbb{A}^\epsilon_X \hookrightarrow \mathbb{P}^\epsilon_X$$

denote the natural inclusions. Look at the excision exact sequence (5.1),

$$\begin{tikzcd} \mathcal{A}_k(\mathbb{P}^{\epsilon-1}_X) \arrow[r, "\iota_\star"] & \mathcal{A}_k(\mathbb{P}^\epsilon_X) \arrow[r, "j^\star"] & \mathcal{A}_k(\mathbb{A}^\epsilon_X) \\ & \mathcal{A}_{k-\epsilon}(X) \arrow[u, "p^\star"] \arrow[ur, "f^\star"'] & \end{tikzcd}$$

Let us show that α_E is surjective. Given $z \in \mathcal{A}_k(\mathbb{P}^\epsilon_X)$, we have $j^\star z = f^\star a$ for some $a \in \mathcal{A}_{k-\epsilon}(X)$ by virtue of 5.2. Hence $z - p^\star a = \iota_\star b$ for some

$b \in \mathcal{A}_k(\mathbb{P}^{\epsilon-1}_X)$. By induction, we may write

$$b = \sum_0^{\epsilon-1} c_1(\mathcal{O}_{\mathbb{P}^{\epsilon-1}}(1))^i \cap q^\star a_i$$

for suitable $a_i \in \mathcal{A}_r(X)$, $r = k - (\epsilon - 1) + i$, with $q = pi : \mathbb{P}^{\epsilon-1}_X \to X$ the structure map. Hence we get

$$\begin{aligned} z &= p^\star a + \iota_\star \sum_0^{\epsilon-1} c_1(\mathcal{O}_{\mathbb{P}^{\epsilon-1}}(1))^i \cap q^\star a_i \\ &= p^\star a + \sum_0^{\epsilon-1} h^i \cap \iota_\star q^\star a_i \\ &= p^\star a + \sum_1^{\epsilon} h^i \cap p^\star a_i. \end{aligned}$$

The last equality is justified by the following

10.1.1. Claim: $\qquad \iota_\star q^\star a = h \cap p^\star a, \ \forall\, a \in \mathcal{A}_\star(X).$

Indeed, it suffices to verify the formula just stated for $a = [V]$, class of a subvariety. In this case, we have $q^\star[V] = [q^{-1}V] = [\mathbb{P}^{\epsilon-1}_V]$. The formula now reads $\iota_\star[\mathbb{P}^{\epsilon-1}_V] = h \cap [\mathbb{P}^\epsilon_V]$, which follows from (7.1).

Back to the proof of the surjectivity of α_E in the general case, we argue by induction on $m = \dim X$. If $m = 0$ then $X = \mathrm{Spec}\,(A)$ for some artinian ring A. Since every locally free $A-$module of constant rank is free, it follows that E is trivial. For the inductive step, we note that there exists an open dense subscheme $U \subseteq X$ such that $E_U \simeq \mathcal{O}e_U$. Indeed, let $U' \subseteq X$ be a (non empty) open subscheme trivializing E. Suppose there is an irreducible component X' of X disjoint from U'. Take an open subscheme U'' of X contained in X' trivializing E. It is clear that $U' \cup U''$ trivializes E. This way, we eventually achieve U dense. Put $Y = X \setminus U$. Then we have $\dim Y < m$. The diagram,

$$\begin{tikzcd} \mathcal{A}_k(\mathbb{P}(E)_Y) \arrow[r] & \mathcal{A}_k(\mathbb{P}(E)) \arrow[r, two heads] & \mathcal{A}_k(\mathbb{P}^\epsilon_U) \\ \bigoplus_0^\epsilon \mathcal{A}_{k-\epsilon+i}(Y) \arrow[u, "\alpha_{E_Y}"] \arrow[r] & \bigoplus_0^\epsilon \mathcal{A}_{k-\epsilon+i}(\mathbb{P}(E)) \arrow[u, "\alpha_E"] \arrow[r, two heads] & \bigoplus_0^\epsilon \mathcal{A}_{k-\epsilon+i}(\mathbb{P}^\epsilon_U) \arrow[u, "\alpha_{E_U}"] \end{tikzcd}$$

We argue by induction on s.

The verification of the case $s = 1$ consists in comparing (10.1) with the new recipe. In fact, (10.1) can be written in the form

$$\begin{array}{rcl} \alpha : (\mathcal{A}_\star(X))[t] \Big/ \langle t^e + c_1 t^{e-1} + \cdots + c_e \rangle & \longrightarrow & \mathcal{A}_\star(\mathbb{P}\,(E)) \\ \sum z_i t^i & \longmapsto & \sum h^i \cap p^\star z_i. \end{array}$$

To see this, notice that now we have $S = \mathcal{O}_E(-1)$, $h = c_1(S^\star)$ and

$$c_e(Q \otimes S^\star) = h^e + c_1 h^{e-1} + \cdots + c_e = 0$$

holds because $\operatorname{rank} Q = e-1$ (see also 9.6). The Euclidean division algorithm allows us to write any element of $(\mathcal{A}_\star(X))[t]$ as a polynomial of degree $\leq e-1$ modulo $t^e + c_1 t^{e-1} + \cdots + c_e$. On the other hand, the relation (10.5) now reads $c_k = b_k + b_{k-1} a_1$, $(k = 0, 1, \dots)$. We have

$$\begin{aligned} &(\mathcal{A}_\star()X)[a_1, b_1, \dots, b_r] \Big/ \langle b_1 + a_1 - c_1, b_2 + b_1 a_1 - c_2, \dots, b_r + b_{e-1} a_1 - c_e \rangle \\ &\simeq (\mathcal{A}_\star(X))[a_1, b_2, \dots, b_r] \Big/ \langle b_2 + (c_1 - a_1) a_1 - c_2, \dots, b_r + b_{e-1} a_1 - c_e \rangle \\ &\qquad \vdots \\ &\simeq (\mathcal{A}_\star(X))[a_1] \Big/ \langle a_1^e - c_1 a_1^{e-1} + \cdots + (-1)^e c_e \rangle. \end{aligned}$$

The isomorphisms are gotten by successively replacing

$$\begin{array}{rcl} b_1 & \longmapsto & c_1 - a_1, \\ b_2 & \longmapsto & c_2 - a_1 b_1, \ \ etc\dots \end{array}$$

Let us indicate briefly how the inductive step works, leaving the details for the reader. Assuming $e \geq 2$, we construct the projective bundle

$$q : \mathbb{P}\,(S_E) \longrightarrow \mathbb{G}\mathrm{r}_s(E)\,.$$

Write the corresponding tautological exact sequence,

$$L \rightarrowtail q^\star S_E \twoheadrightarrow Q_{S_E},$$

10.3. Corollary. *Let $V \subseteq X$ be a subscheme of pure dimension k and let $s : V \to E$ be a section. We have*

$$(f^\star)^{-1} s_\star[V] = c_e(E) \cap [V] \quad in\ \mathcal{A}_{k-n}(V).$$

Proof. We may assume $V = X$. The formula now reads

$$\begin{aligned} s_\star[X] &= f^\star(c_e(E) \cap [X]) \\ &= c_e(f^\star E) \cap [E]. \end{aligned}$$

in $\mathcal{A}_k(E)$. Well, s induces a section $\widetilde{s}$ of $f^\star E = E \underset{X}{\times} E$ defined by

$$v \longmapsto (v, v - s(f(e))).$$

One checks at once that $\widetilde{s}$ is regular and is zero precisely on $s(X)$. The result now follows from (9.11). □

10.4. Corollary. *With the notation as in (10.2), let Q denote the tautological quotient of rank e of $g^\star F$. We have*

$$g_\star\,(c_e(Q) \cap y) = (f^\star)^{-1} j^\star y, \quad for\ all\ \ y \in \mathcal{A}_\star(\mathbb{P}\,(F)).$$

Proof. As in the proof of (10.2), we may write

$$y = g^\star x + i_\star c.$$

On the other hand, by definition of Q and using additivity of Chern classes, we have

$$c_e(Q) = \sum_0^e h^i c_{e-i}(g^\star E).$$

Therefore, using 9.4,

$$g_\star\,(c_e(Q) \cap g^\star x) = g_\star\,(h^e \cap g^\star x) = x.$$

Now notice that if we restrict to $\mathbb{P}\,(E) \subset \mathbb{P}\,(F)$ the tautological exact sequence of $\mathbb{P}\,(F)$, we find the following diagram of vector bundles,

$$\begin{tikzcd} \mathcal{O}_E(-1) \arrow[r, tail] \arrow[d, equal] & p^\star E \arrow[r, two heads] \arrow[d, tail] & T \arrow[d, tail] \\ i^\star \mathcal{O}_F(-1) \arrow[r, tail] & p^\star F \arrow[r, two heads] \arrow[d, two heads] & i^\star Q \arrow[d, two heads] \\ & \mathbb{A}^1_{\mathbb{P}(E)} \arrow[r, phantom, "="] & \mathbb{A}^1_{\mathbb{P}(E)} \end{tikzcd}$$

wherefrom we deduce $c_e\,(i^\star Q) = 0$. Hence, $c_e\,(Q)\,i^\star c = 0$. Piecing it all together, we finally get

$$g_\star\,(c_e(Q) \cap y) = x = (f^\star)^{-1} j^\star y.$$

□

We describe in the sequel the Chow group of a Grassmann bundle.

We recall that the Grassmann variety,

$$G = \mathbb{G}\mathrm{r}_s(\mathbb{A}^e)$$

parameterizes the family of vector subspaces of dimension s of $\mathbb{A}^e$. Equivalently, the points of G can be thought of as projective subspaces $\mathbb{P}^{s-1} \subset \mathbb{P}^{e-1}$.

By construction, the variety G is endowed with a tautological vector bundle

$$S = \{(W, w) \in G \times \mathbb{A}^e \mid w \in W\},$$

subbundle of the trivial bundle $G \times \mathbb{A}^e$.

The fiber S_W of S over a point $W \in G$ is naturally identified to the subspace of $\mathbb{A}^n$ that W represents. In symbols,

$$S_W = W.$$

(Strictly speaking, we should write $S_W = \{W\} \times W \subseteq \{W\} \times \mathbb{A}^e \ldots$)

More generally, let be given a vector bundle $f : E \to X$ of rank e. Then we have the Grassmann bundle

$$p : \mathbb{G}\mathrm{r}_s(E) \longrightarrow X. \tag{10.3}$$

If $U \subseteq X$ is an open subset trivializing E, then we have

$$p^{-1}(U) \simeq \mathbb{G}\mathrm{r}_s(\mathbb{A}^e) \times U.$$

We have likewise a tautological vector subbundle S of $p^\star E$. It fits into the tautological exact sequence

$$S_E \rightarrowtail p^\star E \twoheadrightarrow Q_E. \tag{10.4}$$

We write for short $S = S_E, Q = Q_E$ if no confusion is possible. The fiber of S over a point $y \in \mathbb{G}\mathrm{r}_s(E)$ is the subspace of $f^{-1}(p(y))$ that y represents. The rank of S is of course s.

We also recall that the scheme structure of the Grassmann bundle is characterized by the following universal property. Let $g : Y \to X$ be a morphism. Let $h : Y \to \mathbb{G}\mathrm{r}_s(E)$ be a morphism such that $ph = g$ holds. Then we get a vector subbundle of rank s of $g^\star E$, to wit, $h^\star S \rightarrowtail g^\star E$. The correspondence $h \longmapsto h^\star S$ is a functorial bijection onto the the set of vector subbundles of rank s of $g^\star E$.

$$\left\{\begin{array}{ccc} Y & & \\ g\downarrow & \searrow h & \\ X & \underset{p}{\leftarrow} & \mathbb{G}\mathrm{r}_s(E) \end{array}\right\} \rightsquigarrow \{h^\star S \rightarrowtail g^\star E\}$$

10.5. Proposition. *The Chow group $\mathcal{A}_\star(\mathbb{G}\mathrm{r}_s(E))$ is isomorphic to the quotient of the group*

$$(\mathcal{A}_\star(X))\,[\mathbf{a}, \mathbf{b}] := (\mathcal{A}_\star(X)) \otimes \mathbb{Z}[\mathbf{a}, \mathbf{b}]$$

by the relation

$$a_\star \cdot b_\star = c_\star(E). \tag{10.5}$$

Here $\mathbf{a} := (a_1, \ldots, a_s)$, $\mathbf{b} := (b_1, \ldots, c_r)$, $(r = e - s)$ denote vectors of indeterminates and $a_\star = 1 + a_1 + \cdots + a_s$, $b_\star = 1 + b_1 + \cdots + b_r$ stand for operators described below.

Sketch of proof. We start by making explicit the meaning of the statement! The elements of the group $(\mathcal{A}_\star(X))\,[\mathbf{a}, \mathbf{b}]$ are polynomials in $\mathbf{a}$, $\mathbf{b}$ with coefficients in $\mathcal{A}_\star(X)$. We think of $a_\star, b_\star, c_\star = c(E)$ as operators on $(\mathcal{A}_\star(X))\,[\mathbf{a}, \mathbf{b}]$ in a most natural way: the a_i and b_j act by multiplying variables whereas $c_\star$ operates on the coefficients. The a_i will play in fact the role of $c_i(S)$ and likewise for the b_j vis-à-vis $c_j(Q)$.

The quotient of $(\mathcal{A}_\star(X))\,[\mathbf{a}, \mathbf{b}]$ modulo the relation $a_\star \cdot b_\star = c_\star$ means modulo the subgroup generated by the images of the operators $c_k - \sum a_i b_{k-i}$.

Having said all that, let us consider the homomorphism of groups,

$$\begin{array}{rcl} \theta : (\mathcal{A}_\star(X))\,[\mathbf{a}, \mathbf{b}] & \longrightarrow & \mathcal{A}_\star(\mathbb{G}\mathrm{r}_s(E)) \\ z\, a_1^{i_1} \cdots a_s^{i_s} b_1^{j_1} \ldots & \longmapsto & c_1(S)^{i_1} \cdots c_r(Q)^{j_r} \cap p^\star z. \end{array} \tag{10.6}$$

We proceed to show that θ is surjective and its kernel is equal to the subgroup just described.

We notice at once that $\theta(ab - c) = 0$ as we see from the relation (9.7)(2),

$$c(p^\star E) = c(S)c(Q).$$

The homomorphism σ defined in the diagram just above is of rank ≤ 1 on the fiber over $\ell \in G$ if and only if there is a nonzero vector lying in both S_ℓ, S_{ℓ_0}. It follows that $Z_3 = \mathcal{Z}(\overset{2}{\wedge}\sigma)$, at least as sets. But it can easily be checked that in fact that scheme of zeros is a variety. By 9.11, p. 65 (or by the very construction of c_1!), we have

$$[Z_3] = c_1(\overset{2}{\wedge} Q) \cap [G].$$

Since $c_1(\overset{2}{\wedge} Q) = c_1(Q)$ (cf. exc. 40, p. 67), the formula (10.11) follows. We also note that $\mathcal{Z}(s) = Z_0 = \{\ell_0\}$ holds. Since $s : \mathbb{A}^2_G \to Q$ is a section of the vector bundle,

$$\mathrm{Hom}_G(\mathbb{A}^2, Q) \simeq (\mathbb{A}^2_G)^\star \otimes Q = Q \oplus Q,$$

it follows from (9.11) and (9.7)(2) that

$$\begin{aligned} [Z_0] &= c_4(Q \oplus Q) \cap [G] \\ &= c_2(Q)^2 \cap [G]. \end{aligned}$$

Next, let the linear immersion $\pi : \mathbb{A}^3 \rightarrowtail \mathbb{A}^4$ represent a plane. It is clear then that for each $\ell \in G$ we have

$$\ell \subset \pi \Leftrightarrow S_\ell \subset \pi(\mathbb{A}^3).$$

Therefore, studying the diagram

$$\begin{tikzcd} & \mathbb{A}^3_G \arrow[d, tail] & \\ S \arrow[ur, dashed] \arrow[r, tail] \arrow[dr, "\sigma"'] & \mathbb{A}^4_G \arrow[r, two heads] \arrow[d] & Q \\ & \mathbb{A}^1_G & \end{tikzcd} \tag{10.13}$$

a moment of reflection should convince us that $Z_2 = \mathcal{Z}(\sigma)$. We also have $\mathcal{Z}(\sigma) = \mathcal{Z}(\sigma^\star)$ where $\sigma^\star$ denotes the dual section. The latter is a section of $S^\star$. Therefore, we have

$$[Z_2] = c_2(S^\star) \cap [G] = c_2(S) \cap [G]$$

(cf. exc. 40).

where $L = \mathcal{O}_S(-1)$. We put it together with (10.4) thus obtaining the fundamental diagram,

$$\begin{tikzcd} L \arrow[r, tail] \arrow[d, equal] & q^\star S_E \arrow[r, two heads] \arrow[d, tail] & Q_{S_E} \arrow[d, tail] \\ L \arrow[r, tail] & q^\star p^\star E \arrow[r, two heads] \arrow[d, two heads] & g^\star T \arrow[d, two heads] \\ & q^\star Q_E \arrow[r, phantom, "="] & q^\star Q_E. \end{tikzcd} \tag{10.7}$$

Here the middle horizontal sequence comes from pullback of the tautological sequence of $\mathbb{P}(E)$ via the map g indicated in the diagram below:

$$\begin{tikzcd} & \mathbb{P}(S_E) \arrow[r, hook] \arrow[dl, "q"'] \arrow[drr, "g"] & \mathbb{G}\mathrm{r}_r(E) \underset{X}{\times} \mathbb{P}(E) \arrow[dr] & \\ \mathbb{G}\mathrm{r}_r(E) \arrow[dr, "p"'] & & & \mathbb{P}(E) \arrow[dll, "h"'] \\ & X & & \end{tikzcd}$$

Thus L is identified as $g^\star\mathcal{O}_E(-1)$. The vector bundle T is equal to the quotient $h^\star E / (\mathcal{O}_E(-1))$. Now the main point is to see that the exact sequence

$$Q_{S_E} \rightarrowtail g^\star T \twoheadrightarrow q^\star Q_E$$

defines a map

$$\mathbb{P}(S_E) \xrightarrow{\sim} \mathbb{G}\mathrm{r}_{s-1}(T) \tag{10.8}$$

which is in fact an isomorphism, thanks to the universal property of Grassmann bundles. Thus, we may identify $\mathbb{P}(S_E)$ and $\mathbb{G}\mathrm{r}_{s-1}(T)$, so that the diagram (10.7) exhibits the relations between the respective tautological vector bundles. In particular, the tautological subbundle S_T of $g^\star T$ is identified to Q_{S_E} and the quotient bundle Q_T is the same as $q^\star Q_E$.

By induction, the structure of $\mathcal{A}_\star(\mathbb{G}\mathrm{r}_{s-1}(T))$ is known to be given by the recipe (10.6) in terms of the Chow group $\mathcal{A}_\star(\mathbb{P}(E))$ of the new base and the

Chern classes of T. In order to dig out the structure of $\mathcal{A}_\star(\mathbb{G}\mathrm{r}_s(E))$, we look at the diagram of homomorphisms,

$$\begin{tikzcd}
\mathcal{A}_\star(\mathbb{G}\mathrm{r}_s(E)) \arrow[r, tail, "q^\star"] & \mathcal{A}_\star(\mathbb{P}(S)) = \mathcal{A}_\star(\mathbb{G}\mathrm{r}_{s-1}(T)) \\
\mathcal{A}_\star(X) \arrow[u, "p^\star"] \arrow[r, tail, "h^\star"] & \mathcal{A}_\star(\mathbb{P}(E)). \arrow[u, two heads, "g^\star"']
\end{tikzcd}$$

We recall that in the diagram above $q : \mathbb{P}(S) \to \mathbb{G}\mathrm{r}_s(E)$ is a projective bundle whereas $g : \mathbb{G}\mathrm{r}_{s-1}(T) \to \mathbb{P}(E)$ is a Grassmann bundle with lower subspace dimension.

To see that θ (10.6) is onto, pick a cycle $y \in \mathcal{A}_\star(\mathbb{G}\mathrm{r}_s(E))$. By induction on s, we may write

$$q^\star y = P\left(c_1(S_T), \ldots, c_r(Q_T)\right) \cap q^\star p^\star z,$$

where P denotes a polynomial with integer coefficients in

$$(s-1) + r = e - 1$$

variables. Recalling the identifications $S_T = Q_{S_E}$ and $Q_T = q^\star Q_E$, the above expression can be expressed as

$$q^\star y = P\left(c_1(q^\star S_E) - c_1(L), \ldots, c_r(q^\star Q_E)\right) \cap q^\star p^\star z,$$

Next, setting $t = c_1(L^\star)$ we compute,

$$y = q_\star(t^{s-1} \cap q^\star y). \tag{10.9}$$

Using projection formula and the identity $c(S_T) = c(q^\star S_E)s(L)$, one checks that the right hand side above lies indeed in the image of θ.

The proof that the kernel of θ is precisely as described for the general case requires more combinatorial subtleties than we wish to do here. The reader may consult [?].

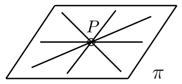
10.6. Example. Let us work out in details the Chow group of the grassmannian $G = \mathbb{G}\mathrm{r}_2(\mathbb{A}^4)$. Recall that the family of 2 dimensional vector subspaces of $\mathbb{A}^4$ is the same as that of projective lines in $\mathbb{P}^3$. The proposition tells us that

$$\mathcal{A}_\star(G) = \mathbb{Z}[a_1, a_2, b_1, b_2] \Big/ \langle a_1 + b_1, a_1 b_1 + a_2 + b_2, a_1 b_2 + a_2 b_1, a_2 b_2 \rangle. \tag{10.10}$$

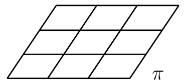
The above relations simply express the identity $c(S)c(Q) = c(\mathbb{A}^{\oplus 4}) = 1$. We will render explicit the isomorphism, describing natural generators for $\mathcal{A}_\star(G)$.

Fix a point, a line and a plane in $\mathbb{P}^3$, denoted by O, ℓ_0 and π. Fix in addition some point $P \in \pi$. We define subvarieties of G as follows:

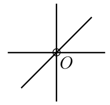
$Z_1 = \{\ell \in G \mid \ell \subset \pi \text{ and } P \in \ell\}$



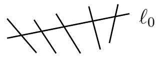
$Z_2 = \{\ell \in G \mid \ell \subset \pi\}$



$Z_2' = \{\ell \in G \mid \ell \ni O\}$



$Z_3 = \{\ell \in G \mid \text{ the line } \ell \text{ meets } \ell_0\}$



We also set $Z_0 = \{\ell_0\}$ and $Z_4 = G$. We proceed to show that

$$\mathcal{A}_\star(G) \simeq \mathbb{Z}^{\oplus 6}$$

is the free abelian group generated by the classes of Z_0, Z_1, Z_2, Z_2', Z_3 and Z_4.

We start by showing that, with the notation as in (10.4) with $E = \mathbb{A}^4$, we have

$$[Z_3] = c_1(Q) \cap [G]. \tag{10.11}$$

To see this, let us represent the line ℓ_0 by a linear immersion, still denoted $\ell_0 : \mathbb{A}^2 \rightarrowtail \mathbb{A}^4$. For each $\ell \in G$, the condition that ℓ be incident to the line ℓ_0, *i.e.*, $\ell \cap \ell_0 \neq \emptyset$, is equivalent that the corresponding 2 dimensional subspaces S_ℓ, S_{ℓ_0} (fibers of S over ℓ, ℓ_0) contain both a nonzero subspace. This can be better expressed by the following diagram of maps of vector bundles over G,

$$\begin{tikzcd}
 & \mathbb{A}^2_G \arrow[d, "\ell_0"] \arrow[dr, "\sigma"] & \\
S \arrow[r, tail] & \mathbb{A}^4_G \arrow[r, two heads] & Q.
\end{tikzcd} \tag{10.12}$$

$$= \int c_1(\mathcal{L}) \cap C = -d$$

(by (8.8)) whence we find

$$\int_{C\times C} c_2(f^\star S) = d^2 - d$$

as asserted.

In order to determine n in (10.16), we invoke (10.14) thereby obtaining

$$n = \int c_2(\mathcal{Q}) \cap C'.$$

By the previous discussion, we transfer the calculation to $C \times C$, where we find

$$\begin{array}{rcc} 2n & = & \int c_2(\mathcal{Q}') \cap [C \times C] \\ & = & \int (c_1(f^\star S)^2 - c_2(f^\star S)) \cap [C \times C] \\ & = & \int (c_1(\mathcal{L}_2) + c_1(\mathcal{L}_1) + c_1\mathcal{O}(\Delta))^2 \cap [C \times C] \\ & = & 2d^2 - 4d + 2 - 2g - (d^2 - d) \\ & = & (d-1)(d-2) - 2g. \end{array}$$

We finally get

$$n = \frac{(d-1)(d-2)}{2} - g. \tag{10.18}$$

This is the so called number of apparent double points of the curve C. It is equal to the number of double points of a general plane projection of C (cf. 11.3).

Exercises

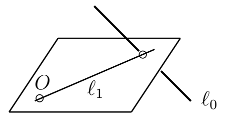
45. Let C be a curve of degree d in $\mathbb{P}^3$. Let $C' \subset \mathbb{G}\mathrm{r}_2(\mathcal{O}^4)$ be the collection of lines incident to C. Show that C' is a hypersurface and that, with the notation as in (10.11), $[C'] = d[Z_3]$. Deduce that there exist $2d$ lines incident to C and to three other given lines in general position.

46. Find natural generators for $\mathbb{G}\mathrm{r}_2(5)$, the grassmannian of projective lines in $\mathbb{P}^4$. Compute the class of the variety of chords of a smooth curve in $\mathbb{P}^4$.

Arguing in a similar way, we obtain

$$\begin{array}{rcl} [Z_2'] & = & c_2(Q) \cap [G], \\ [Z_1] & = & c_1(S)c_2(Q) \cap [G]. \end{array}$$

It follows from 10.5 and 10.10 that $\mathcal{A}_i(G) = \mathbb{Z} \cdot [Z_i]$ for $i = 0, 1, 3, 4$ and $\mathcal{A}_2(G) = \mathbb{Z} \cdot [Z_2] + \mathbb{Z} \cdot [Z_2']$. The same proposition implies that $\mathcal{A}_\star(G)$ is in fact freely generated by these cycles. We give also a direct argument to show that $\mathcal{A}_i(G)$ is free. For $i = 0$, this follows from the epimorphism given by degree, $\int : \mathcal{A}_0(G) \twoheadrightarrow \mathbb{Z}$. For $i = 1$ we invoke the operator $c_1(Q) : \mathcal{A}_1(G) \to \mathcal{A}_0(G)$, and use the fact that $c_1(Q) \cap [Z_1] = [Z_0]$. Indeed, with the notation as in the diagram (10.12), the restriction of $\overset{2}{\wedge}\sigma$ to Z_1 vanishes precisely on a unique $\ell_1 \in G$, to wit, the line joining O and the point of intersection of ℓ_0 and π as indicated in the picture below.



By the same token, the relation

$$c_2(Q) \cap [Z_3] = [Z_1]$$

shows that $\mathcal{A}_3(G)$ is free. Finally, the relations

$$\begin{array}{c} c_2(Q) \cap [Z_2] = c_2(S) \cap [Z_2'] = 0 \\ c_2(Q) \cap [Z_2'] = c_2(S) \cap [Z_2] = [Z_0] \end{array} \tag{10.14}$$

imply that $[Z_2], [Z_2']$ are linearly independent. In short, we have seen that $\mathcal{A}_\star(G)$ is free of rank 6. We leave as an exercise for the reader the verification of the formulas

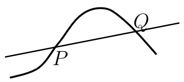
$$\int c_1(Q)^4 \cap [G] = 2,$$
$$\int c_1(Q)^2 c_2(Q) \cap [G] = \int c_2(S)^2 \cap [G] = \int c_2(Q)^2 \cap [G] = 1.$$

10.7. Application: chords of a curve in $\mathbb{P}^3$. Let $C \subset \mathbb{P}^3$ be a smooth projective curve. We consider the variety $C' \subset G = \mathbb{G}\mathrm{r}_2(\mathbb{A}^4)$ which is the image of the map $f : C \times C \to G$ which associates to each pair of points P, Q in C the bi-secant line (or chord) $\overline{AB}$. We clearly have $\dim C' = 2$. Taking into account the previous example, we may write the cycle

$$[C'] = m[Z_2] + n[Z_2'] \ \text{ in } \mathcal{A}_2(G). \tag{10.15}$$

Let us show that

$$\begin{cases} m = \text{number of chords contained in a general plane;} \\ n = \text{number of chords passing through a general point.} \end{cases} \tag{10.16}$$



To compute the coefficient m, we restrict the diagram (10.13) over C'. We deduce that $c_2(S) \cap C'$ is the class of the zero cycle formed by the points of C' that represent chords of C contained in a given plane π. By (10.14) and (10.16), we get

$$\int c_2(S) \cap C' = m.$$

Thus, it is plausible to expect that

$$m = \binom{d}{2}, \ (d = \deg(C))) \tag{10.17}$$

since a general plane π intersects C in d distinct points which, combined two at a time, yield the chords of C in π.

To give a rigorous argument for (10.17) with the tools at our disposal, we bring the calculation to $C \times C$ via $f : C \times C \to C'$. Note to start with that $\deg(f) = 2$ because the general chord is not tri-secant. The projection formula allows us to write

$$f_\star(c_2(f^\star S) \cap C \times C) = 2c_2(S) \cap C'.$$

It remains to calculate $S' = f^\star S$. We are required to find a subbundle S' of rank 2 of $\mathcal{O}^4$ whose fiber over $(P, Q) \in C \times C$ is the subspace $P' + Q'$ of $\mathcal{O}^4$, where $P', Q' \subset \mathcal{O}^4$ denote the one-dimensional subspaces represented by $P, Q \in C \subset \mathbb{P}^3$. Let $\mathcal{L}_i$ (resp. $\mathcal{T}_i$) be the pull back via the projection $p_i : C \times C \to C \subset \mathbb{P}^3$ $(i = 1, 2)$ of the tautological subbundle $\mathcal{O}(-1)$ (resp. quotient bundle $\mathcal{T}$) of $\mathcal{O}^4_{\mathbb{P}^3}$. Off the diagonal, the image of $\mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{O}^4_{C\times C}$

is equal to the sought for bundle S'. The map of vector bundles $\mathcal{L}_1 \to \mathcal{T}_2$ defined by composition,

$$\begin{tikzcd} \mathcal{L}_1 \arrow[r, tail] \arrow[dr, "\sigma"'] & \mathcal{O}^4_{C\times C} \arrow[d] \\ & \mathcal{T}_2 \end{tikzcd}$$

vanishes exactly along the diagonal $\Delta \subset C \times C$. This means that the image of the map $\mathcal{T}_2^\star \otimes \mathcal{L}_1 \to \mathcal{O}^1_C$ is equal to the line bundle $\mathcal{O}(-\Delta)$. Dualizing, we get an injective map

$$\mathcal{L}_1 \otimes \mathcal{O}(\Delta) \rightarrowtail \mathcal{T}_2.$$

This produces a quotient vector bundle $\mathcal{Q}'$ of rank two fitting into the following diagram.

$$\begin{tikzcd} & & \mathcal{L}_1 \otimes \mathcal{O}(\Delta) \arrow[d, tail] \\ \mathcal{L}_2 \arrow[r, tail] \arrow[d, tail] & \mathcal{O}^4_{C\times C} \arrow[r, two heads] \arrow[d, equal] & \mathcal{T}_2 \arrow[d, two heads] \\ f^\star S \arrow[r, tail] \arrow[uurr, two heads, bend left] & \mathcal{O}^4_{C\times C} \arrow[r, two heads] & \mathcal{Q}'. \end{tikzcd}$$

In short, the pullback bundle $f^\star S$ appears as an extension of $\mathcal{L}_1(\Delta)$ by $\mathcal{L}_2$. Hence we may compute,

$$c(f^\star S) = c(\mathcal{L}_2)c(\mathcal{L}_1(\Delta)).$$

In particular, we have

$$c_2(f^\star S) = c_1(\mathcal{L}_2)\left(c_1(\mathcal{L}_1) + c_1\mathcal{O}(\Delta)\right).$$

Thus we may write, using exc. 43

$$\int_{C\times C} c_1(\mathcal{L}_2)c_1(\mathcal{L}_1) = (\int_C c_1(\mathcal{L}))^2,$$

$$\int_{C\times C} c_1(\mathcal{L}_2)c_1(\mathcal{O}(\Delta)) = \int c_1(\mathcal{L}_2) \cap \Delta$$

11.3. Definition. The *intersection class of V by X in Y*, written

$$X \cdot V \quad (\text{or } X \underset{f}{\cdot} V \quad \text{or} \quad i^! V)$$

is the only class in $\mathcal{A}_\star(W)$ such that

$$\pi^\star(X \cdot V) = [C] \quad \text{in } \mathcal{A}_\star(\mathcal{N}).$$

The existence and uniqueness come from the corollary 10.2.

In fact $X \cdot V$ lies in $\mathcal{A}_{k-r}(W)$ because C is of pure dimension k as the next lemma tells us.

11.4. Lemma. *Let V be a scheme of pure dimension k and let $W \subseteq V$ be a subscheme. Then the cone $C_W V$ is of pure dimension k.*

Proof. We may assume V is affine. Let J be the ideal of W in V. Set

$$R = \oplus J^n / J^{n+1}.$$

We have $C = \operatorname{Spec}(R)$. Form the product $C \times \mathbb{A}^1 = \operatorname{Spec}(R[t])$. We look at the projectivization $\mathbb{P}(C \times \mathbb{A}^1) := \operatorname{Proj} R[t]$. Note that the standard affine open subset of $\operatorname{Proj} R[t]$ corresponding to the dehomogeneization $(R[t])_{((t))} \simeq R$ is isomorphic to C. This is dense in $\operatorname{Proj} R[t]$. Indeed, recall that for $r(t) \in J^n/J^{n+1} \subset R$ we have $\operatorname{Spec}\big((R[t])_{(r(t))}\big)$ is a nonempty open subset of $\operatorname{Proj} R[t]$ if and only if $r(t)$ is not nilpotent, and therefore the intersection $\operatorname{Spec}\big((R[t])_{(r(t)t)}\big)$ is nonempty. Thus $\dim C = \dim \mathbb{P}(C \times \mathbb{A}^1)$ holds. Now $\mathbb{P}(C \times \mathbb{A}^1)$ is the exceptional divisor of the blowup $\widetilde{M}$ of $V \times \mathbb{A}^1$ along the nowhere dense subscheme $W \times \{0\}$. It follows that $\widetilde{M}$ is of pure dimension $k+1$ and the exceptional divisor is of pure dimension k as desired. $\square$

11.5. Examples. (1) Suppose X is of pure dimension k and take $V = X$ and $f = i$ in (11.1). Then $W = X$ and C is the image of the zero section of $\mathcal{N}$. By corollary 10.3, we get the

Self-intersection formula: $X \cdot X = c_r(\mathcal{N}) \cap [X]$ in $\mathcal{A}_{k-r}(X)$.

(2) If also $W \hookrightarrow V$ is a regular embedding of the same codimension r, then $C = \mathcal{N}$ holds. Indeed, the epimorphisms (11.2) have source and target locally free sheaves of the same rank. It follows that

$$X \cdot V = [W] \text{ in } \mathcal{A}_{k-r}(W).$$

Chapter 11

The Chow ring

We describe in this chapter the construction due to Fulton for the intersection class of a cycle by a regularly embedded subscheme of arbitrary codimension. It enables us, in particular, to define a product

$$\mathcal{A}_\star(Y) \otimes \mathcal{A}_\star(Y) \longrightarrow \mathcal{A}_\star(Y)$$

for the Chow group of a nonsingular variety Y, which endows $\mathcal{A}_\star(Y)$ with a ring structure satisfying the expected properties.

11.1. We will work with the basic diagram,

$$\begin{array}{ccccc} C \subseteq \mathcal{N} & \xrightarrow{\pi} & W & \overset{j}{\hookrightarrow} & V \\ & & g \downarrow & & \downarrow f \\ & & X & \underset{i}{\hookrightarrow} & Y \end{array} \qquad (11.1)$$

where

(1) i denotes a regular immersion of codimension r (*i.e.*, the ideal of X in Y is locally generated by a regular sequence of length r);

(2) V denotes a scheme of pure dimension k;

(3) $W = f^{-1}(X)$

(4) $\mathcal{N} = g^\star \mathcal{N}_X Y$, the pullback of the normal bundle of X in Y and

(5) $C = C_W V$ denotes the normal cone of W in V.

With these data, we shall define the *intersection class of* V *by* X *in* Y, written

$$X \cdot V \quad \text{in } \mathcal{A}_{k-r}(W).$$

Note that, whenever W is of the expected pure dimension $k-r$, the class $X \cdot V$ is represented by a well defined cycle, to wit, a linear combination of the irreducible components of W, simply because $\mathcal{C}_{k-r}W = \mathcal{A}_{k-r}(W)$.

In particular, if $f : V \hookrightarrow Y$ is the inclusion of a subscheme of Y and the intersection $X \cap V$ is proper, we'll have $X \cdot V$ written as a linear combination of the irreducible components of $X \cap V$.

11.2. The normal cone.

Let I denote the sheaf of ideals of X in Y and write $J = f^{-1}I$ for the idealsheaf of W in V. The normal cone of W in V is the scheme

$$C_W V = \operatorname{Spec}\left(\oplus J^n / J^{n+1}\right).$$

We have the epimorphisms of $\mathcal{O}_W$−modules,

$$g^\star\left(I^n/I^{n+1}\right) = g^\star i^\star I^n \twoheadrightarrow j^\star J^n = J^n/J^{n+1}, \quad n = 0, 1, \ldots \qquad (11.2)$$

It yields the closed embedding

$$C_W V \subseteq (C_X Y) \underset{X}{\times} W$$

into the pullback of the normal cone of X in Y under g.

Now the hypothesis of regular immersion implies that

$$\bigoplus I^n/I^{n+1} = \bigoplus S_n\left(I/I^2\right)$$

(cf. [?] p. 110) so that the cone $C_X Y$ is equal to the normal bundle

$$\mathcal{N}_X Y = \operatorname{Spec}\left(\oplus S_n\left(I/I^2\right)\right).$$

Let us set for short

$$\mathcal{N} = g^\star \mathcal{N}_X Y = C_X Y \underset{X}{\times} W.$$

This is a vector bundle of rank $r = \operatorname{codim}(X, Y)$. Thus $C := C_W V$ is a subcone of $\mathcal{N}$.

$$\begin{array}{ccc} C & \subseteq & N \\ & \searrow \; \swarrow & \\ & W & \end{array} \qquad (11.3)$$

(2) A sufficient condition for the equality $i = \ell$ in 11.10 (3) is that $\mathcal{O}_{V,Z}$ be Cohen-Macaulay (*e.g.*, V smooth).

(3) The lower bound for $\dim W$ given in 11.10 (1) may fail in the absence of regularity, *e.g.*, take X, Y, V as the subvarieties of $\mathbb{A}^4$ defined by $x = y = 0$, $xz = yw$ and $z = w = 0$ respectively.

11.12. Proposition. (Criterion for multiplicity one.) *Let Z be a proper component of the intersection of V by X in Y. Put $A = \mathcal{O}_{V,Z}$ and denote by M the maximal ideal of A. Let J be the ideal of X in A. Then $i(Z) = 1$ holds if A is regular and $J = M$.*

Proof. With the notation as in 11.10, we have $\mathcal{O}_{W,Z} = A/J$. Hence, if $J = M$ it follows that $i(Z) = \ell(A/J) = 1$. Note that the requirement that A be regular follows from $J = M$, since we are then saying that the maximal ideal is generated by $r = \dim A$ elements. □

11.13. Remarks. (1) The converse is true if V is a variety. See [?] for a proof.

(2) If X and V are subvarieties of Y and the maximal ideal of $\mathcal{O}_{Y,Z}$ is the sum of the ideals of X and V then $i(Z) = 1$. If $\mathcal{O}_{Y,Z}$ is regular (*e.g.*, Y nonsingular), this last condition is equivalent to the requirement that X and V be generically transversal along Z.

The next topic is to show that the formation of intersection class preserves rational equivalence.

Keeping the notation as in (11.1), we define homomorphisms

$$i^! : \mathcal{C}_m V \longrightarrow \mathcal{A}_{m-r}(W)$$

by the formula

$$i^! \sum n_k [Z_k] = \sum n_k X \cdot Z_k$$

for subvarieties Z_k of dimension m in V. Note that each $X \cdot Z_k$ is in fact a class that comes from $\mathcal{A}_{m-r}(X \underset{Y}{\times} Z_k)$.

11.14. Theorem. *$i^!$ preserves rational equivalence,* i.e., *$i^! \mathcal{R}_\star V = 0$ thereby inducing homomorphisms $i^! : \mathcal{A}_m(V) \longrightarrow \mathcal{A}_{m-r}(W)$. In particular, for $V = Y$ and $f =$ identity, we get homomorphisms $i^! : \mathcal{A}_m(Y) \longrightarrow \mathcal{A}_{m-r}(X)$.*

That is, in the present case, the intersection class coincides with the class of scheme intersection. This occurs notably whenever $\dim W = k - r$ and V is Cohen-Macaulay along each component of W.

11.6. Lemma. *If in the basic diagram (11.1) $X = D$ is a Cartier divisor in Y and V is a variety, then the intersection class $X \cdot V$ defined in (11.3) is equal to $c_1(f^\star \mathcal{O}(D)) \cap V$. In particular, it is consistent with (6.4).*

Proof. Assume first $W = V = C$. Thus W embeds in $\mathcal{N}$ by the zero section. It follows that C is the scheme of zeros of the regular section of

$$\pi^\star \mathcal{N} = \mathcal{N} \underset{W}{\times} \mathcal{N}$$

given by $v \mapsto (v, 0)$, $v \in \mathcal{N}$. Therefore, we may write

$$\begin{array}{rcll} [C] & = & c_1(\pi^\star \mathcal{N}) \cap [\mathcal{N}] \quad \text{in } \mathcal{A}_k(\mathcal{N}) \ (k = \dim V) & (7.1) \\ & = & \pi^\star \left(c_1(\mathcal{N}) \cap [W] \right) & (7.2) \end{array}$$

whence, by the definition 11.3, we have

$$X \cdot V = c_1(\mathcal{N}) \cap [W] \quad \text{in } \mathcal{A}_{k-1}(W),$$

just as in (6.4).

Next, if $V \neq W$ then W is a Cartier divisor in V. This is a situation as envisaged in 11.5 (2), so that

$$\begin{array}{rcl} X \cdot V & = & [W] \quad \text{in } \mathcal{A}_{k-1}(W) \\ & = & c_1 \left(\mathcal{O}(X) \cap V \right), \end{array}$$

matching the prescription of (6.4) (with $X = D$). □

11.7. Lemma. *With the notation as in (11.1), let $\overline{C} := \mathbb{P}(C \times \mathbb{A}^1) \subseteq \mathbb{P}(\mathcal{N} \times \mathbb{A}^1)$ denote the projective closure of the cone C. Let*

$$\overline{\pi} : \mathbb{P}(\mathcal{N} \times \mathbb{A}^1) \to W$$

be the structure map and let $\mathcal{Q}$ denote the tautological quotient of rank r of $\overline{\pi}^\star \mathcal{N} \times \mathbb{A}^1$. Then we have the formula

$$X \cdot V \;=\; \overline{\pi}_\star \left(c_r(\mathcal{Q}) \cap [\overline{C}] \right) \quad \text{in } \mathcal{A}_{k-r}(W).$$

Proof. The assertion follows from corollary 10.4 (applied to $\mathcal{E} = \mathcal{N}$) once we observe that

$$\overline{C} \cap \mathcal{N} = C.$$

Indeed, with the notation as in (10.4), we have $y = [\overline{C}]$, $j^\star y = [C]$. □

11.8. Remark. Even if $\dim W$ is correct ($=k-r$), the intersection class $X \cdot V$ in general, will be a cycle "smaller" than $[W]$, as shown by the example 1.4 (2) (with $t = 0$).

11.9. Definition. Each irreducible component of W with dimension $k - r$ is called a *proper component* of the intersection of V by X.

We may write

$$X \cdot V = \sum i(Z)Z \quad \text{in } \mathcal{A}_{k-r}(W), \tag{11.4}$$

where Z runs through the subvarieties of dimension $k - r$.

Recalling exerc. 11, p. 8, we see that the coefficient of a proper component in the above expression is unambiguous. Accordingly, we may define the *multiplicity* or *index of intersection of V by X in Y along a proper component* Z as the coefficient of Z in $X \cdot V$, written $i(Z, X \cdot V; Y) = i(Z)$ (for short).

11.10. Proposition. (1) *With the notation and assumptions as in* (11.1)*, we have that each irreducible component of W is of dimension $\geq k - r$.*

(2) *If Z is a proper component of W then we have*

$$1 \leq i(Z) \leq \ell(\mathcal{O}_{W,Z}).$$

(3) $i(Z) = \ell(\mathcal{O}_{W,Z})$ *holds with Z as in* (2) *if and only if the ideal of W is generated, locally at $\mathcal{O}_{V,Z}$, by a regular sequence of length* $r = \operatorname{codim}(X, Y)$.

Proof. (1) Let Z be an irreducible component of W. Let V° denote the open subset of V obtained by taking the complement of the remaining components of W. Put $Z^\circ = Z \cap V^\circ$, $W^\circ = W \cap V^\circ$. Thus Z° is the sole component of W°. Furthermore, it can be easily verified that

$$C^\circ := C_{W^\circ}V^\circ = C_W V \underset{W}{\times} W^\circ.$$

(Quite generally, the formation of normal cones commutes with flat base change.) Now the inclusion $C^\circ \subseteq \mathcal{N}^\circ := \mathcal{N}_{|W^\circ}$ immediately implies the inequality $k = \dim C^\circ \leq \dim Z + r$. This proves (1).

(2) Preserving the above notation, look at the commutative diagram

$$\begin{tikzcd}
\mathcal{A}_{k-r}(W) \arrow[r, "g^\star", "\sim"'] \arrow[d, "\mathrm{w}^\star"'] & \mathcal{A}_k(\mathcal{N}) \arrow[d] \\
\mathcal{A}_{k-r}(W^\circ) \arrow[r, "\sim", "(g_{|W^\circ})^\star"'] & \mathcal{A}_k(\mathcal{N}^\circ)
\end{tikzcd}$$

where $\mathrm{w} : W^\circ \hookrightarrow W$ denotes inclusion. One checks at once that

$$\mathrm{w}^\star(X \cdot V) = X \cdot V^\circ$$

and

$$i(Z, X \cdot V) = i(Z^\circ, X \cdot V^\circ).$$

This allows us to suppose that Z is the sole component of W. Consequently, $\mathcal{N}_{|Z}$ is the sole component of $\mathcal{N}$ and of C as well. Therefore, we have,

$$\begin{aligned} [C] &= i \cdot [\mathcal{N}_{|Z}], \text{ for some } i \in \mathbb{Z}, \\ [N] &= \ell \cdot [\mathcal{N}_{|Z}]. \end{aligned}$$

We obviously have

$$1 \leq i \leq \ell.$$

Since $g^\star[Z] = [\mathcal{N}_{|Z}]$, we must have $i = i(Z)$. Finally, we also have $\ell = \ell(\mathcal{O}_{W,Z})$ in view of (4.4) (with $f = \pi : \mathcal{N} \to W$) or by an easy direct inspection.

(3) If the ideal J of W in $\mathcal{O}_{V,Z}$ is generated by a regular sequence of length r, the maps (11.2) are isomorphisms in a neighborhood of the generic point of Z. Hence $C = \mathcal{N}$ holds over some open subset of W, thereby implying $i = \ell$. The converse is left for the reader. □

11.11. Remarks. (1) The previous result indicates to what extent the "naïve" definition of intersection multiplicity, $\ell(\mathcal{O}_{W,Z})$, differs from the "correct" one, via normal cones. The actual calculation of multiplicities of the components of the normal cone is non-trivial, in general.

$(c/1)(t/c)$ in $J_{(c)}$ shows that $\widetilde{M}_0 = \mathbb{P}\,(C \times \mathbb{A}^1) + \widetilde{V}$ thus proving (i). Assertion (ii) is clear from the fact that the image of $c/1$ in $I_{(c)}$ is the local equation for the exceptional divisor $\mathbb{P}\,(C)$ of the blowup of $V \times \{0\}$ along $W \times \{0\}$ (see also (**??**)). The total transform of $W \times \mathbb{A}^1$ in Spec $\left(J_{(c)}\right)$ corresponds to the ideal $IJ_{(c)} = \langle c/1 \rangle$, that is, the full exceptional divisor restricted to the present neighborhood. Hence the strict transform, image of ψ, is empty there. However, choosing now the affine neighborhood for $c = t$, we find in turn $\widetilde{V} \cap \mathrm{Spec}\left(J_{(t)}\right)$ empty. This shows that $\psi(W \times \mathbb{A}^1)$ is disjoint from $\widetilde{V}$. In particular, $\psi(W \times \{0\})$ is disjoint from $\mathbb{P}\,(C)$; hence it sits in the complement $C = \mathbb{P}\,(C \times \mathbb{A}^1) \setminus \mathbb{P}\,(C)$. To see how it sits therein, we check affine.

Put $B = A/I$. The embedding ψ corresponds in this neighborhood to a ring map

$$J_{(t)} \to R := B[t].$$

Here we use the canonical identification of the Rees algebra $R \oplus \langle t \rangle \oplus \langle t \rangle^2 \cdots$ with the polynomial ring $R[u]$ in a fresh variable. Recall that we have to start with $\psi^\# : \oplus J^n \twoheadrightarrow R[u] = \oplus \langle t \rangle^n$, the Proj of which is ψ; for $f \in J^n$ we have $\psi^\#(f) = \overline{f} u^n$ where $\overline{f}$ is the image under the natural surjection $A[t] \twoheadrightarrow B[t]$. Note that $f = a_0 t^n + \cdots + a_n$, where $a_i \in (I[t])^i$, so that $\psi^\#(f) = \overline{a}_0 t^n u^n$. We have $\psi^\#(t) = tu$ in degree one. Look next at the map induced at the corresponding dehomogeneizations. We get an induced map $\psi^\# : J_{(t)} \twoheadrightarrow R$, sending $a_0/1 + a_1/t + \cdots a_n/t^n$ to $\overline{a}_0$. The equation of the exceptional divisor, $t/1$, is sent to t. The induced map

$$\psi^\# : J_{(t)}/\langle t/1 \rangle \simeq \oplus I^n/I^{n+1} \twoheadrightarrow R/\langle t \rangle \simeq B$$

corresponds to the zero section embedding of W into its cone C.

The assertions (1) … (5) of the lemma 11.15 follow from the construction of $\widetilde{M}$. For instance, formula (5) comes from the fact that $\widetilde{M}_{Z\cap W} Z$ embeds in $\widetilde{M}_W Z$ in such a way that

$$\left(\widetilde{M}_{Z \cap W} Z\right) \cap \mathbb{P}\left(C \times \mathbb{A}^1\right) = \mathbb{P}\left(C' \times \mathbb{A}^1\right),$$

where $C' = C_{Z\cap W} Z$, in view of general properties of blowups recalled just below. □

For each pure-dimensional subscheme $Z \subseteq V$ we have that the normal cone $C_{Z\cap W} Z$ is a subscheme of $C = C_W V$ of pure dimension $m = \dim Z$. We define the homomorphism

$$\begin{array}{rccc} \sigma : & \mathcal{C}_m V & \longrightarrow & \mathcal{C}_m C \\ & \sum n_i Z_i & \longmapsto & \sum n_i [C_{Z_i \cap W} Z_i]. \end{array}$$

One checks at once that the composition

$$\mathcal{C}_m V \longrightarrow \mathcal{C}_m C \longrightarrow \mathcal{C}_m \mathcal{N} \longrightarrow \mathcal{A}_m(\mathcal{N}) \xrightarrow{\sim} \mathcal{A}_{m-r}(W)$$

coincides with

$$i^! : \mathcal{C}_m V \to \mathcal{A}_{m-r}(W).$$

The crucial point is to show that σ preserves rational equivalence. The case $r = 1$ is taken care of by (8.4). The general case will be dealt with using the important technique of deformation to the normal cone.

11.15. Lemma. (Deformation to the normal cone) *Let W be a closed subscheme of a pure dimensional scheme V. Then there are maps*

$$\begin{tikzcd} W \times \mathbb{A}^1 \arrow[rr, hook, "\psi"] \arrow[dr, "\pi"'] & & M = M_W V \arrow[dl, "\phi"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

such that

(1) *ψ is a closed immersion;*

(2) *ϕ is a flat of relative dimension $\dim V$;*

(3) *for $t \neq 0$ in $\mathbb{A}^1$ the immersion $\psi_t : W \times \{t\} \hookrightarrow M_t := \phi^{-1}(t)$ is isomorphic to the given inclusion $W \hookrightarrow V$, whereas ψ_0 is isomorphic to the embedding of $W = W \times \{0\}$ as the zero section of the normal cone $C := C_W V = M_0$;*

(4) *the complement of C in M is isomorphic to $V \times \mathbb{A}^1 \setminus \{0\}$;*

(5) *for each pure dimensional subscheme $Z \subseteq V$ we have that $M_{Z\cap W} Z$ is a subscheme of $M_W V$ which intersects $C_W V$ in $C_{Z\cap W} Z$.*

Assume the lemma for the moment and let us proceed with the proof of the theorem 11.14.

Let

$$\alpha : C \hookrightarrow M, \quad \beta : M \smallsetminus C = V \times \mathbb{A}^1 \setminus \{0\} \hookrightarrow M, \quad \gamma : V \times \mathbb{A}^1 \setminus \{0\} \longrightarrow V$$

denote the natural maps. Consider the diagram with top exact sequence (5.1)

$$\begin{tikzcd}
\mathcal{C}_{m+1}C \arrow[r, "\alpha_\star"] & \mathcal{C}_{m+1}M \arrow[r, "\beta^\star"] \arrow[d, "\alpha^\star"] & \mathcal{C}_{m+1}V \times \mathbb{A}^1 \setminus \{0\} \arrow[dl, dashed, "\alpha'"] \\
& \mathcal{A}_m(C) & \mathcal{C}_m V \arrow[l, dashed] \arrow[u, "\gamma^\star"']
\end{tikzcd} \tag{11.5}$$

The homomorphism $\alpha^\star$ is intersection by the Cartier divisor C (cf. 6.4; see also 8.4). The dotted arrow α' arises from the fact that $\alpha^\star\alpha_\star = 0$. This is due to (8.6) given that $\mathcal{O}(C)$ is trivial since $C \subset M$ is principal. Now it is clear that $\alpha'\gamma^\star$ preserves rational equivalence. It remains to be seen that $\alpha'\gamma^\star$ coincides with the map induced by σ. For this end, pick a pure dimensional subscheme $Z \subset V$. We then have

$$\gamma^\star[Z] = \beta^\star[M_{Z\cap W}Z]$$

in view of (4) and (5) of the lemma. Therefore,

$$\begin{array}{rcll}
\alpha'\gamma^\star[Z] & = & \alpha^\star[M_{Z\cap W}Z] & \\
& = & C \cdot [M_{Z\cap W}Z] & \\
& = & [C_{Z\cap W}Z] & \text{(by (5) of the lemma)} \\
& = & \sigma[Z] \quad \text{in } \mathcal{A}_\star(C) &
\end{array}$$

as desired. □

11.16. Proof of the lemma 11.15.

As in the proof of lemma 11.4, let

$$\mu : \widetilde{M} := \widetilde{M}_W V \longrightarrow V \times \mathbb{A}^1$$

be the blowup of $V \times \mathbb{A}^1$ along $W \times \{0\}$. We present in the diagram below the main players; the description of their respective roles will give us the sought for properties (1)…(5).

$$\begin{tikzcd}
& & M := \widetilde{M} \setminus \widetilde{V} \arrow[d, phantom, "\cap"] & & & \\
\mathbb{P}(C \times \mathbb{A}^1) \arrow[rr, hook, "\alpha"] \arrow[d] & & \widetilde{M} \arrow[d, "\mu"'] \arrow[ddr, "\phi"] & & \widetilde{M}_0 = \mathbb{P}(C \times \mathbb{A}^1) + \widetilde{V} \arrow[ll, hook'] \arrow[ddr] & \\
W \times \{0\} \arrow[r, hook] & W \times \mathbb{A}^1 \arrow[r, hook] \arrow[ur, hook, dashed, "\psi"] & V \times \mathbb{A}^1 \arrow[dr] & & & \\
& & & \mathbb{A}^1 & \ni & 0
\end{tikzcd} \tag{11.6}$$

The normal cone of $W \times \{0\}$ in $V \times \mathbb{A}^1$ is equal to $C \times \mathbb{A}^1$. The exceptional divisor of the blowup $\widetilde{M} \to V \times \mathbb{A}^1$ is the projectivization $\mathbb{P}(C \times \mathbb{A}^1)$ of that cone. This sets the role of α in the above diagram. Since $W \times \{0\}$ is a Cartier divisor in $W \times \mathbb{A}^1$ the blowup of the latter is naturally isomorphic to itself, thereby explaining the embedding $\psi : W \times \mathbb{A}^1 \hookrightarrow \widetilde{M}$. We show next that ψ factors through $M := \widetilde{M} \setminus \widetilde{V}$, where $\widetilde{V}$ denotes the blowup of $V \times \{0\}$ along $W \times \{0\}$. Indeed, we have $\psi^{-1}(\mathbb{P}(C \times \mathbb{A}^1)) = W \times \{0\}$, since the exceptional divisor of a blowup along a Cartier divisor coincides with that Cartier divisor. The pullback to $\widetilde{M}$ of the coordinate function t on $\mathbb{A}^1$ is a nonzero divisor, hence the fiber $\widetilde{M}_0$ is a Cartier divisor. Clearly $\widetilde{M}_0$ contains the exceptional divisor $\mathbb{P}(C \times \mathbb{A}^1)$. We see that $\widetilde{V}$ sits over $0 \in \mathbb{A}^1$ and embeds in $\widetilde{M}$ (cf. **??**). We claim that

(i) $\widetilde{M}_0 = \mathbb{P}(C \times \mathbb{A}^1) + \widetilde{V}$, sum of Cartier divisors;
(ii) $\widetilde{V} \cap \mathbb{P}(C \times \mathbb{A}^1) = \mathbb{P}(C)$;
(iii) $\mathbb{P}(C \times \mathbb{A}^1) \setminus \mathbb{P}(C) = C$.
(iv) $\psi(W \times \mathbb{A}^1) \cap \mathbb{P}(C \times \mathbb{A}^1) \simeq W \hookrightarrow C$, embedded by the zero section.

For the verification of the claim, we may go affine. Say $V = \mathrm{Spec}(A)$ and $I \subsetneq A$ denotes the ideal of W. Thus $V \times \mathbb{A}^1 = \mathrm{Spec}(A[t])$. Now the ideal of $W \times \{0\} \subset V \times \mathbb{A}^1$ is $J := \langle I, t \rangle$. The open sets of the form $\mathrm{Spec}\left(J_{(c)}\right)$ (with $c \in J$ running through a set of generators) cover $\widetilde{M}$. We have also that $\{\mathrm{Spec}\left(I_{(c)}\right)\}_{c \in I}$ cover $\widetilde{V}$. Furthermore, the natural map $J_{(c)} \to I_{(c)}$ is surjective, with kernel given by the principal ideal $\langle t/c \rangle$. Recalling that $c/1$ is the local equation of the exceptional divisor, the relation $t/1 =$