ENUMERATION OF SURFACES CONTAINING A CURVE OF LOW DEGREE

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Abstract. Noether–Lefschetz theory tells us that a very general surface of degree at least 4 in $\mathbb{P}^3$ has Picard group $\mathbb{Z}$, *i.e.*, it contains only curves which are complete intersections with other surfaces. Let W be some irreducible variety of the Hilbert scheme of curves in $\mathbb{P}^3$. For all sufficiently large d, the surfaces of degree d containing a member of W form a subvariety $NL(W,d)$ of $|\mathcal{O}_{\mathbb{P}^3}(d)|$. We give formulas for the degree of $NL(W,d)$, polynomial in d, when the general member $C \in W$ is of one of the following types: a union of up to three general lines; a conic, or a twisted cubic curve.

1. Introduction

Noether–Lefschetz theory tells us that a very general surface of degree at least 4 in $\mathbb{P}^3$ has Picard group $\mathbb{Z}$, *i.e.*, it contains only curves which are complete intersections with other surfaces, [10],[11].

Let W be some irreducible variety of the Hilbert scheme of curves in $\mathbb{P}^3$. The goal of this note is to determine the degree of the closure in $\mathbb{P}^N = |\mathcal{O}_{\mathbb{P}^3}(d)|$ of the locus formed by all surfaces which contain a general member of W.

We give closed formulas (polynomial in d) when the general member $C \in W$ is of one of the following types: a union of up to three general lines; a conic, or a twisted cubic curve. We also find the degree of the locus corresponding to surfaces containing a pair of incident lines, as well as a few other numbers of similar nature. The case of lines rests on a description of the good component of the Hilbert schemes $\mathrm{Hilb}^{m(t+1)}\mathbb{P}^3$, $m \leq 3$, suitable for enumerative purposes. The Hilbert scheme of conics is well known: it is a special case of hypersurfaces in their linear span, see [1]. As for twisted cubics, we profit from [6], [20].

It is a simple exercise to find the degree of the subvariety of $\mathbb{P}^N$ consisting of surfaces of degree d containing one line. It suffices to consider the subvariety $\mathbb{X}_d^|$ of pairs (ℓ, q) such that the surface q contains the line ℓ. Start with the vector bundle, $\mathcal{F}_d^|$, whose fibre over $\ell \in \mathbb{G}$ is the projective subspace of all homogeneous polynomials which vanish on ℓ. Then $\mathbb{X}_d^|$ is the projectivization of $\mathcal{F}_d^|$. The image of the projection $p_2 : \mathbb{X}_d^| \to \mathbb{P}^N$ is the desired locus. Its degree is easily found in terms of the Segre classes of the vector bundle $\mathcal{F}_d^|$. We find that surfaces of degree d

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containing a line are parameterized by a subvariety of codimension $d-3$ and degree

$$\binom{d+1}{4}(3d^4+6d^3+17d^2+22d+24)/24$$

in the parameter space $\mathbb{P}^N$ of surfaces of degree d (with $N=\binom{n+d}{n}-1$).

The well known formula of double points [9],[7] applied to $p_2 : \mathbb{X}^{|}_d \to \mathbb{P}^N$ yields a cycle $\mathbb{D}$ in $\mathbb{X}^{|}_d$ whose support is the set of $x \in \mathbb{X}^{|}_d$ such that there is another point $x' \in \mathbb{X}^{|}_d$ with the same image $p_2(x)$, or else, p_2 is ramified at x. We note however that the support of $\mathbb{D}$ has a component whose general member consists of pairs $(\ell, q) \in \mathbb{X}$ such that there exists a line $\ell' \neq \ell$ *incident* to ℓ and contained in the surface q.

At least morally, the calculation for the case of more lines should be done by means of the Hilbert scheme, $\mathrm{Hilb}^n\mathbb{G}$, of points of $\mathbb{G}$. In fact, $\mathrm{Hilb}^2\mathbb{G}$ is simple to describe since the Grassmannian is isomorphic to a smooth quadric in $\mathbb{P}^5$. We sketch its construction in §5. However, in order to deal with the case of three lines, we found it easier to bring the situation to a more naïve set up. We start with the blowup of the diagonal,

$$\mathbb{G}^{(2)} \to \mathbb{G}\times\mathbb{G},$$

and then construct our parameter space

$$\mathbb{G}^{(3)} \to \mathbb{G}^{(2)} \times_{\mathbb{G}} \mathbb{G}^{(2)}$$

via a composition of three additional blowups.

We show first that $\mathbb{G}^{(2)}$ is a natural parameter space for a flat family of subschemes of $\mathbb{P}^3$ with generic member a union of two lines. Whenever the support of the subscheme is just one line, it carries a double structure with arithmetic genus -1, since the Hilbert polynomial must remain equal to $2t+2$. When the support is the union of two incident lines, the point of intersection becomes an embedded point. Well, any surface containing such a scheme is singular at the (embedded) point of incidence. This tells us how to sort out the good configurations (*i.e.*, flat limits of 2 distinct lines), from the bad ones; we get in this way a correction for the faulty answer given by the aforementioned double point formula.

Let $\mathcal{F}_d = H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ be the vector space of homogeneous polynomials of degree d. We set for short $\mathcal{F}=\mathcal{F}_1$; we have $\mathcal{F}_d = \mathrm{Sym}_d\mathcal{F}$. There is a subbundle

$$\mathcal{F}''_d \subset \mathcal{F}_d \times \mathbb{G}^{(2)}$$

of the trivial bundle such that the fibre over each point (ℓ,ℓ') of $\mathbb{G}^{(2)}$ is the space of equations of surfaces of degree d which contain the subscheme corresponding to (ℓ,ℓ') –be it either the union of two lines in general position, or a flat degeneration thereof. The image of the projective subbundle

$$\mathbb{X}^2_d := \mathbb{P}(\mathcal{F}''_d) \subset \mathbb{G}^{(2)} \times \mathbb{P}(\mathcal{F}_d) \tag{1}$$

in $\mathbb{P}^N = \mathbb{P}(\mathcal{F}_d)$ is the closure of the locus of surfaces which contain two lines in general position.

The case of surfaces with three lines is similar: one arrives at a nice subbundle

$$\mathcal{F}'''_d \subset \mathcal{F}_d \times \mathbb{G}^{(3)}$$

the general fiber of which is the space of surfaces of degre d containing three skew lines. Presently the construction of the space $\mathbb{G}^{(3)}$ requires more blowups, cf.§6.

For the case of surfaces containing a conic, the calculation is straightforward. Only the case $d = 4$ escapes the general formula due to Bézout: if an irreducible quartic surface contains a conic, there must appear a second conic.

To deal with twisted cubic curves, we take the "official" description of the Hilbert scheme component with the ready to use list of fixed points given in [6]. Alternatively, we've also checked calculations using [20], [19].

The sought for numbers are obtained by an application of a $\mathbb{C}^\star$-equivariant version of Bott's formula, cf. [2], [6], [13]. What renders the calculation feasible is the knowledge of the fibers of the appropriate vector bundles over the fixed points under the $\mathbb{C}^\star$-action.

2. LOCAL STUDY

We write

$$\mathcal{S} \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{Q} \tag{2}$$

the tautological exact sequence over the Grassmannian $\mathbb{G}$ of lines in $\mathbb{P}^3$, where $\operatorname{rank} \mathcal{S} = 2$. The fibre of $\mathcal{S}$ over each point ℓ of $\mathbb{G}$ is the space of linear equations defining the line $\ell \subset \mathbb{P}^3$. We abuse notation and write $\ell = \mathcal{S}_\ell$; we use the same symbol both for the line and the corresponding point in $\mathbb{G}$. Likewise, $\mathcal{F}$ stands for the vector space of linear homogeneous polynomials or the trivial bundle.

We fix to begin with the line $\ell_0 = \langle x_0, x_1 \rangle$. It will start to move around in § 4.

Look at the morphism of fiber bundles over of $\mathbb{G}$ defined by multiplication,

$$\ell_0 \otimes \mathcal{S} \xrightarrow{\mu_1} \mathcal{F}_2. \tag{3}$$

The generic rank of μ_1 is 4; it drops to 3 precisely over the point $\ell_0 \in \mathbb{G}$.

Let us consider the open affine subset of $\mathbb{G}$ corresponding to the lines of the form

$$\ell = \left\{ \begin{array}{llllll} x_0 & & +a_1 x_2 & + & a_2 x_3, \\ & x_1 & +a_3 x_2 & + & a_4 x_3. \end{array} \right. \tag{4}$$

The a_i are affine coordinate functions. Taking the multiplication by the space $\ell_0 = \langle x_0, x_1 \rangle$, one finds in general a space spanned by 4 independent quadrics, to wit

$$\left\{ \begin{array}{l} q_1 = x_0(x_0 + (a_2 x_3 + a_1 x_2)), \\ q_2 = x_1(x_0 + (a_2 x_3 + a_1 x_2)), \\ q_3 = x_1(x_1 + (a_4 x_3 + a_3 x_2)), \\ q_4 = x_0\left(a_4 x_3 + a_3 x_2\right) - x_1\left(a_2 x_3 + a_1 x_2\right). \end{array} \right. \tag{5}$$

We see that the quadric q_4 is null at the origin ($a_1 = a_2 = a_3 = a_4 = 0$). That's precisely when the variable line, ℓ, coincides with the line ℓ_0. If $\ell \neq \ell_0$ is incident to ℓ_0, say $\ell = \langle x_0, x_1 + x_2 \rangle$, one checks that the scheme defined by the product $\ell \cdot \ell_0$ acquires an embedded point at the point of intersection, *e.g.,*

$$\begin{aligned} \langle x_0, x_1 \rangle \cdot \langle x_0, x_2 \rangle &= \langle x_0^2, x_0 x_1, x_0 x_2, x_1 x_2 \rangle \\ &= \langle x_1, x_2, x_0^2 \rangle \cap \langle x_0, x_1 \rangle \cap \langle x_0, x_2 \rangle. \end{aligned}$$

2.1. **Blowup $\mathbb{G}$.** The linear system of quadrics (5) defines a rational map of $\mathbb{G}$ to the Grassmannian of subspaces of $\mathcal{F}_2$ of dimension 4,

$$\mathbb{G} = \mathbb{G}\mathrm{r}_2(\mathcal{F}_1) \dashrightarrow \mathbb{G}\mathrm{r}_4(\mathcal{F}_2).$$

The locus of indeterminacy of this map is (scheme theoretically) reduced to the point $\ell_0 \in \mathbb{G}$. We blow it up:

$$\begin{tikzcd} \mathbb{E}'_{\ell_0} \arrow[r, hook] \arrow[d] & \mathbb{G}' \arrow[d] \\ \{\ell_0\} \arrow[r, hook] & \mathbb{G}. \end{tikzcd} \tag{6}$$

We obtain the morphism

$$\mathbb{G}' \longrightarrow \mathbb{G}\mathrm{r}_4(\mathcal{F}_2). \tag{7}$$

This yields, by pullback of the tautological subbundle of $\mathcal{F}_2$ over the Grassmannian $\mathbb{G}\mathrm{r}_4(\mathcal{F}_2)$, a vector subbundle $\mathcal{F}_2''(\ell_0)$ of rank 4 of the trivial bundle $\mathcal{F}_2 \times \mathbb{G}'$,

$$\mathcal{F}_2''(\ell_0) \subset \mathcal{F}_2 \times \mathbb{G}'. \tag{8}$$

2.2. **The exceptional divisor.** How do the fibers of $\mathcal{F}_2''(\ell_0)$ over points on the exceptional divisor $\mathbb{E}'_{\ell_0} \subset \mathbb{G}'$ look like?

One hopes to obtain good tools to produce a subscheme of $\mathbb{P}^3$ supported at the line ℓ_0, of degree 2 and arithmetic genus –1. Let us verify that we get in fact what's needed. We write the blowup $\mathbb{G}' \to \mathbb{G}$ in local coordinates,

$$\begin{cases} a_2 &= a_1 b_1, \\ a_3 &= a_1 b_2, \\ a_4 &= a_1 b_3, \end{cases} \tag{9}$$

where a_1 is chosen as the principal generator of the exceptional divisor. The functions a_1, b_1, b_2, b_3 are local coordinates over $\mathbb{G}'$.

One computes the saturation $\mathcal{F}_2''(\ell_0) \subset \mathcal{F}_2|\mathbb{G}'$ of the subsheaf $\mathcal{F}_2(\ell_0)$ of the free sheaf $\mathcal{F}_2$ spanned by $\ell_0 \cdot \mathcal{S}$ obeying the prescription (3). This means, $\mathcal{F}_2''(\ell_0)$ is the subsheaf of the free sheaf $\mathcal{F}_2|\mathbb{G}'$ formed by the sections of $\mathcal{F}_2$ for which the multiplication by a local equation of the exceptional divisor belong to $\ell_0 \cdot \mathcal{S} \subset \mathcal{F}_2$. One finds local generators for $\mathcal{F}_2''(\ell_0)$,

$$\begin{cases} x_0(x_0 + a_1\,(b_1 x_3 + x_2)), \\ x_1(x_0 + a_1\,(b_1 x_3 + x_2)), \\ x_1(x_1 + a_1\,(b_3 x_3 + b_2 x_2)), \\ q_4' := x_0\,(b_3 x_3 + b_2 x_2) - x_1(b_1 x_3 + x_2), \end{cases} \tag{10}$$

thereby showing that $\mathcal{F}_2''(\ell_0)$ has the good rank 4 everywhere.

Let us examine the restriction of $\mathcal{F}_2''(\ell_0)$ over the exceptional divisor, $a_1 = 0$. We've just adjoined to the ideal of the triple line $\ell_0^2 := \langle x_1^2, x_0x_1, x_0^2\rangle$, the "new" quadric $q_4' := (b_2x_2 + b_1x_3)\,x_0 - (b_3x_3 + x_2)\,x_1$. One achieves in this way a system of quadrics which renders the triple line thinner, namely, the subscheme cut out by $\ell_0^2 + \langle q_4'\rangle$ has degree two. Notice its Hilbert polynomial is indeed correct, $2t+2$. Our q_4' comes from q_4 (see 5) by substitution of the relations (9), followed by division by the local equation (a_1) of the exceptional divisor. This quadric corresponds to the datum of a field of normal directions along ℓ_0 in $\mathbb{P}^3$ or, equivalently, a tangent direction to the Grassmannian $\mathbb{G}$ at ℓ_0, recalling that the exceptional divisor is equal to the projectivization of the tangent space $T_{\ell_0}\mathbb{G} = \mathrm{Hom}(\ell_0, \mathcal{F}/\ell_0)$.

In a more intrinsic manner, pick $\varphi \in \mathbb{P}\left(\mathrm{Hom}(\ell_0, \mathcal{F}/\ell_0)\right) \subset \mathbb{G}'$.

2.2.1. **Lemma.** *Notations as just above, put*

$$q'_\varphi = \varphi(x_1)x_0 - \varphi(x_0)x_1,$$
$$q_\varphi = \langle \varphi(x_1)x_0 - \varphi(x_0)x_1 \rangle \; + \; \ell_0^2 \subset \mathcal{F}_2.$$

Then q_φ is equal to the subspace of rank 4, image of $\varphi \in \mathbb{P}\left(T_{\ell_0}\mathbb{G}\right) \subset \mathbb{G}'$ by the map $\mathbb{G}' \longrightarrow \mathbb{G}\mathrm{r}_4(\mathcal{F}_2)$ induced by the blowup (7).

Proof. Even though the quadric q'_φ depends on choices of representatives for $\varphi(x_i)$, we remark that the space q_φ is well defined. Let us write the infinitesimal curve given by φ,

$$\langle x_0 + t\varphi(x_0),\, x_1 + t\varphi(x_1)\rangle.$$

We take the multiplication by ℓ_0, thus obtaining the space of four quadrics,

$$\begin{gathered}\langle x_0^2 + t\varphi(x_0)x_0,\; x_0x_1 + t\varphi(x_1)x_0,\; x_0x_1 + t\varphi(x_0)x_1,\; x_1^2 + t\varphi(x_1)x_1\rangle \\ = \\ \langle x_0^2 + t\varphi(x_0)x_0,\; x_0x_1 + t\varphi(x_1)x_0,\; x_1^2 + t\varphi(x_1)x_1,\; t(\varphi(x_0)x_1 - \varphi(x_1)x_0)\rangle \\ = \\ \langle x_0^2 + t\varphi(x_0)x_0,\; x_0x_1 + t\varphi(x_1)x_0,\; x_1^2 + t\varphi(x_1)x_1,\; \varphi(x_0)x_1 - \varphi(x_1)x_0\rangle.\end{gathered}$$

Plainly, the limit when $t \to 0$ is the system q_φ in the statement. □

3. SURFACES WITH TWO LINES

Let us define inductively subsheaves $\mathcal{F}''_d(\ell_0) \subset \mathcal{F}_d$, $d \geq 2$, as the image of the multiplication map

$$\mathcal{F}''_d(\ell_0) \otimes \mathcal{F}_1 \longrightarrow \mathcal{F}''_{d+1}(\ell_0) \subset \mathcal{F}_{d+1} \tag{11}$$

We show next that $\mathcal{F}''_d(\ell_0)$ is in fact a vector subbundle of the trivial bundle $\mathcal{F}_d$ over our variety $\mathbb{G}'$. By definition, each fibre spans a homogeneous ideal which cuts out a subscheme of $\mathbb{P}^3$ with the correct Hilbert polynomial, $2t+2$, flat limit of the the union of two skew lines.

3.1. **Lemma.** *The homomorphism (11) has everywhere the good rank*

$$m^{(2)}(d) := \binom{d+3}{3} - (2d+2),\, d \geq 2.$$

Proof. We argue by induction on d. For $d = 2$ the equality results from the construction of $\mathcal{F}''_2(\ell_0)$, cf. (10). Suppose $d \geq 3$. Put $c_{d-1} := 2(d-1)+2$. Then

$$c_{d-1} = \binom{d}{d-1} + \binom{d-1}{d-2} + \binom{d-3}{d-3},\; \forall d \geq 3.$$

This expression is the so called $(d-1)-$*th decomposition of Macaulay of* c_{d-1}. Thus, with the notation of Lopez & C. Maclean [12, § 5], he have

$$c_{d-1}^{\langle d-1\rangle} = \binom{d+1}{d} + \binom{d}{d-1} + \binom{d-2}{d-2} = 2d+2 = c_d.$$

Hence, the induction hypothesis and the theorem of Macaulay-Gotzmann (cf. [12, p.335]) show that

$$m^{(2)}(d) \geq \binom{d+3}{3} - (2d+2).$$

By semi-continuity, we see that the strict inequality is not possible. Indeed, the reversed inequality holds generically: it's a consequence of the fact that the regularity of the ideal of the union of two disjoint lines is equal to 2 (see [5]). $\square$

3.1.1. **Remark.** The rank of the homomorphism $\mathcal{F}''_2(\ell_0) \otimes \mathcal{F}_{d-2} \to \mathcal{F}_d \times \mathbb{G}'$ is equal to the rank prescribed by the polynomial of Hilbert; the regularity is 2 everywhere. The image is therefore a subbundle

$$\mathcal{F}''_d(\ell_0) \subset \mathcal{F}_d|\mathbb{G}'.$$

A general fiber of $\mathcal{F}''_d(\ell_0)$ is the space of surfaces of degree d which contain ℓ_0 as well as another line in general position.

Put

$$\left\{ \begin{array}{r} \mathbb{X}''_d(\ell_0) = \mathbb{P}\,(\mathcal{F}''_d(\ell_0)) \subset \mathbb{P}^N \times \mathbb{G}', \\ \mathbb{Y}''_d(\ell_0) \subset \mathbb{P}^N = \text{image via projection.} \end{array} \right. \tag{12}$$

3.2. **first enumerations.** Without moving ℓ_0 for the the time being, we may get for instance the degree of the image $\mathbb{Y}''_4(\ell_0)$ of $\mathbb{X}''_4(\ell_0)$ in $\mathbb{P}^{34} = \mathbb{P}\,(\mathcal{F}_4)$. It consists of the surfaces of degree 4 which contain ℓ_0 together with another general line. It is a variety of dimension

$$\dim \mathbb{Y}''_4(\ell_0) = 4 + 24 = 28. \tag{13}$$

The degree is given by capping with h^{28}, where h denotes the hyperplane class of $\mathbb{P}^{34}$. It can be checked that the projection map map $\mathbb{X}''_4(\ell_0) \to \mathbb{Y}''_4(\ell_0) \subset \mathbb{P}^N$ is generically injective. One lifts the calculation to $\mathbb{X}''_4(\ell_0)$, then pushes it to $\mathbb{G}'$ and $\mathbb{G}$. Details are as follows. The value of

$$\int h^{28} \cap \mathbb{X}''_4(\ell_0)$$

reduces to the degree of the Segre class

$$\int s_4 \mathcal{F}''_4(\ell_0) \cap \mathbb{G}'$$

(cf.[7, 3.1, p. 47]). Since $\mathcal{F}''_4(\ell_0)$ is a subbundle of the trivial vector bundle $\mathcal{F}''_4$, the Segre class is equal to the Chern class of the quotient bundle,

$$\overline{\mathcal{F}_4}(\ell_0) := \mathcal{F}''_4 / \mathcal{F}''_4(\ell_0). \tag{14}$$

3.2.1. **Applying Bott.** For the explicit evaluation we employ Bott's formula,

$$\int (p(c) \cap [\mathbb{G}']) = \sum_{\ell} \left(\frac{p(c)_{|\ell}}{c_4(T\mathbb{G}')_{|\ell}} \right).$$

cf. [2], [6], [13]. Here $p(c)$ stands for a polynomial in the Chern classes of equivariant vector bundles. The sum is over the set of fixed points in $\mathbb{G}'$ under the action of $\mathbb{C}^\star$ induced by the usual action, $x_i \mapsto t^{w_i} x_i$ in $\mathbb{P}^3$. Taking the weights w_i sufficiently general ensures that the fixed points are all isolated.

The fraction in the right hand side results from evaluation of $\mathbb{C}^\star$–equivariant Chern classes *à la mode de* [6], [13].

We have to start with the 6 fixed points from $\mathbb{G}$, namely, $\ell_{ij} = \langle x_i, x_j\rangle$, $0 \le i < j \le 3$. For each $(i,j) \ne (0,1)$, ℓ_{ij} lifts as a fixed point of $\mathbb{G}'$ since the blowup center is the point $\ell_0 = \ell_{01}$. In order to find the contribution, say at ℓ_{12}, one writes the decomposition of the fiber $\mathcal{F}_4''(\ell_0)_{\ell_{12}}$ as a sum of subspaces spanned by the semi-invariants, and similarly for the tangent space $T_{\ell_{12}}\mathbb{G}' = T_{\ell_{12}}\mathbb{G}$. The fibers here, (off the blowup center ℓ_0), are

$$\begin{aligned} T_{\ell_{12}}\mathbb{G} = &\ \mathrm{Hom}(\langle x_1, x_2\rangle, \langle x_0, x_3\rangle) = \langle x_1, x_2\rangle^\vee \otimes \langle x_0, x_3\rangle \\ = &\ \langle x_0\rangle \otimes \langle x_1\rangle^\vee + \langle x_0\rangle \otimes \langle x_2\rangle^\vee + \langle x_3\rangle \otimes \langle x_1\rangle^\vee + \langle x_3\rangle \otimes \langle x_2\rangle^\vee, \\ \ell_0 \cdot \ell_{12} \cdot \mathcal{F}_2 = &\ x_1x_2x_3^2 + x_0x_2x_3^2 + \cdots + x_0x_1^3 + x_0^2x_1^2 + x_0^3x_1 \text{ (25 terms)}. \\ \overline{\mathcal{F}_4}(\ell_0) = &\ x_0^4 + x_2^4 + x_0^3x_3 + \cdots + x_2x_3^3 + x_3^4 \text{ (35–25 terms)} \end{aligned}$$

The fiber of the quotient $\overline{\mathcal{F}_4}(\ell_0)$ is obtained erasing from the full list of the 35 monomials spanning $\mathcal{F}_4$ the 25 monomials which appear in $\ell_0 \cdot \ell_{12} \cdot \mathcal{F}_2$. We put for short

$$\langle x_i\rangle \otimes \langle x_j\rangle^\vee = x_i/x_j. \tag{15}$$

That's a $\mathbb{C}^\star$-subspace of $T_{\ell_{12}}\mathbb{G}$ associated to the character $t^{w_i - w_j}$. Next we evaluate the polynomial $p(c)$ in the equivariant cohomology of the fixed point. This amounts to substitution of the $i-$th Chern class appearing in $p(c)$ by the $i-$th elementary symmetric function of the weights. Presently, replace $c_4(\overline{\mathcal{F}_4}(\ell_0))_{\ell_{12}}$ by the coefficient of t^4 in the product of $10(= 35 - 25)$ factors

$$\big(1 + t(4w_0)\big)\big(1 + t(4w_2)\big) \cdots \big(1 + t(w_2 + 3w_3)\big)\big(1 + t(4w_3)\big).$$

One finds as the contribution of the fixed point ℓ_{12} the frightening fraction

$$\frac{p(c)_{|\ell_{12}}}{c_4(T\mathbb{G}')_{|\ell_{12}}} = \frac{24w_0^4 + 50w_0^3w_1 + \cdots + 5734w_2w_3^3 + 1897w_3^4}{(w_0 - w_1)(w_0 - w_2)(w_3 - w_1)(w_3 - w_2)}\cdot$$

One also has to cope with the fixed points in the fiber of $\mathbb{G}'$ over the blowup center ℓ_0. The tangent space to $\mathbb{G}$ at $\ell_0 = \ell_{01}$ can be written as

$$T_{\ell_0}\mathbb{G} = \mathrm{Hom}(\langle x_0, x_1\rangle, \langle x_2, x_3\rangle) = \langle x_0, x_1\rangle^\vee \otimes \langle x_2, x_3\rangle.$$

With the notation as in (15), the decomposition of $T_{\ell_0}\mathbb{G}$ under the induced $\mathbb{C}^\star$-action is given by the semi-invariants $\langle x_2/x_0, x_3/x_0, x_2/x_1, x_3/x_1\rangle$. Hence $\mathbb{G}'$ has 4 fixed points in the excepcional divisor $\mathbb{P}(T_{\ell_0}\mathbb{G})$ sitting over ℓ_0.

Each fraction x_i/x_j, say $\varphi := x_2/x_0$, represents the element of $T_{\ell_0}\mathbb{G}$ such that $x_0 \overset{\varphi}{\mapsto} x_2, x_1 \overset{\varphi}{\mapsto} 0$. The space of quadrics associated to φ is equal to

$$\ell_0^2 + \langle \varphi(x_1)x_0 - \varphi(x_0)x_1\rangle = \ell_0^2 + \langle x_2x_1\rangle$$

(cf. 2.2.1). We have adapted the script of Meurer [13] to do the calculation of the total contribution of those 9(=5+4) fixed points, cf. [18]. One finds **231** quartic surfaces which pass through 28 general points (cf. (13)) and contain 2 lines in general position, one of which has been kept fixed. (We can't help being astonished by the appearance of such a positive integer as the sum of a bunch of horrible fractions as above!:-) We register below the answers for the first few degrees.

(16)

4	5	6	7	8	9	10	11	12
231	**1590**	**7365**	**26425**	**79156**	**207186**	**488280**	**1057980**	**2139775**

SURFACES CONTAINING 2 LINES, ONE OF WHICH HAS BEEN FIXED

3.3. **Proposition.** *Notations as above, the degree of the subvariety of* $\mathbb{P}(\mathcal{F}_d)$ *formed by the surfaces containing two lines one of which is fixed is given by*

$$\frac{1}{2^5\cdot 3}\binom{d}{3}(d+3)(3d^4-d^2+18d-32).$$

Proof. We get it by interpolation. One must show the polynomial nature of the formula for the degrees and how to bound the degree. The magic is offered by Grothendieck-Riemann-Roch. Consider the universal subscheme $\mathbb{L}\subset\mathbb{G}'\times\mathbb{P}^3$. That's the total space of the family of subschemes of $\mathbb{P}^3$ parameterized by $\mathbb{G}'$. The generic member of the family is the union of the line ℓ_0 with a line in general position. Let

$$\pi_1:\mathbb{Y}=\mathbb{G}'\times\mathbb{P}^3\longrightarrow\mathbb{G}'\quad\text{and}\quad\pi_2:\mathbb{Y}\longrightarrow\mathbb{P}^3$$

denote the projections. We have the exact sequence of sheaves over $\mathbb{Y}$,

$$\mathcal{I}(d):=\mathcal{I}\otimes\pi_2^\star\mathcal{O}_{\mathbb{P}^3}(d)\hookrightarrow\pi_2^\star\mathcal{O}_{\mathbb{P}^3}(d)\twoheadrightarrow\mathcal{O}_{\mathbb{L}}(d)\ =:\mathcal{R}_d$$

where $\mathcal{I}$ denotes the sheaf of ideals defining the closed subscheme $\mathbb{L}$ of $\mathbb{Y}$. The sheaf $\mathcal{R}_d$ is π_1-flat and the formation of the direct image $\pi_{1\star}(\mathcal{R}_d)$ commutes with base change, given that $R^1\pi_{1\star}(\mathcal{O}_{\mathbb{L}}(d))=0$, $(d\geq 2)$ by regularity as mentioned in Lemma 3.1. We obtain the exact sequence of locally free sheaves,

$$\mathcal{F}''_d(\ell_0)=\pi_{1\star}(\mathcal{I}(d))\rightarrowtail\mathcal{F}_d\twoheadrightarrow\pi_{1\star}(\mathcal{R}_d).$$

Since $\mathcal{F}_d$ is trivial, we see that the Segre class of $s_4\mathcal{F}''_d(\ell_0)$ is equal to $c_4\pi_{1\star}(\mathcal{R}_d)$. The idea is to estimate the degree of $c_4\pi_{1\star}\mathcal{R}_d$ in d avoiding explicit calculation of the direct image. We are rescued by the theorem of Grothendieck-Riemann-Roch, [7, 15.2.8],

$$\operatorname{ch}(\pi_{1\star}(\mathcal{R}_d))=\pi_{1\star}(\operatorname{ch}(\mathcal{R}_d)\cdot\operatorname{td}(T_{\pi_1})).$$

Set

$$h=c_1(\mathcal{O}_{\mathbb{P}^3}(1)).$$

In view of Euler sequence, we have the formula for the Todd class,

$$\operatorname{td}(T_{\pi_1})=\operatorname{td}(\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4})=1+2h+\frac{11}{6}h^2+h^3.$$

One knows that the coherent sheaves over any smooth projective variety admit a finite resolution by locally free sheaves. Hence, writing such a resolution for $\mathcal{O}_{\mathbb{L}}$, one obtains an expression as a linear combination of termes of the form $\mathcal{G}\otimes\mathcal{O}_{\mathbb{P}^3}(e)$, where $\mathcal{G}$ comes from $\mathbb{G}'$ and does not depend on d. Using the fact that the Chern character is a ring homomorphism, we may write $\operatorname{ch}(\mathcal{O}_{\mathbb{L}})=\sum z_ih^i$, with z_i in the Chow group of $\mathbb{G}'$. It follows an expression

$$\operatorname{ch}(\mathcal{R}_d)\cdot\operatorname{td}T_\pi=\sum_0^3 y_ih^i,$$

where the coefficients y_i come from $\mathbb{G}'$ and, as polynomials in d, have degree at most equal to 3. When we calculate the direct image of cycles via $\pi_1 : \mathbb{G}' \times \mathbb{P}^3 \to \mathbb{G}'$ only the coefficient y_3 of h^3 survives. This coefficient can be written as a polynomial in d with degree in d at most equal to 3, and in classes of $\mathbb{Y} = \mathbb{G}' \times \mathbb{P}^3$. Hence,

$$\mathrm{ch}(\pi_{1\star}(\mathcal{R}_d)) = y_3 = k_0 + k_1 + k_2 + \dots .$$

Now, it suffices to recall that the 4th Chern class of $(\pi_1)_\star(\mathcal{R}_d)$ is a weighted polynomial in the coefficients k_i, with total degree equal to 4, and k_i is taken with weight i. Thus, we may deduce that, as a polynomial in d, we obtain degree at most $4 \cdot 3 = 12$. □

3.3.1. **Remark.** The degree found in the proposition 3.3 is smaller, equal to twice the dimension of the family of lines incident to ℓ_0.

3.3.2. **double point formula.** We remark that a direct application of the formula of double points (cf. [9], [7, 9.5, p. 166]) to the morphism

$$p_2 : \mathbb{X} := \{(\ell, q) \in \mathbb{G} \times \mathbb{P}(\mathcal{F}_4) \mid \ell \subset q\} \longrightarrow \mathbb{P}^{34}$$

yields an answer which differs from the one gotten above.

In fact, the class of the double point locus of p_2 in $\mathbb{X}$ is given by

$$\mathbb{D} = 315h - 6c_1\mathcal{Q}.$$

That's by construction a cycle supported on the set of all $x \in \mathbb{X}$ such that ${p_2}^{-1}p_2(x)$ is a subscheme strictly larger than x, either due to the existence of $x' \neq x$ with $p_2(x') = p_2(x)$, or to ramification at x. The intersection of $\mathbb{D}$ with the class, $c_2(\mathcal{Q})^2$, of a fiber of $\mathbb{X}$ over $\ell_0 \in \mathbb{G}$ is a cycle of degree **315**, instead of **231** registerd in (16).

An explanation for such discrepancy is that we also find in the support of the cycle $\mathbb{D}$ elements (ℓ, q) such that there is a line $\ell' \neq \ell$ incident to ℓ and contained in the surface q. In our construction, the surface would be forced to contain the subscheme $\ell \cup \ell'$ *fattened with an embedded point placed at the point of intersection*; in particular, the surface should be singular at said point.

We also note that the set-theoretic union $\ell \cup \ell'$, without embedded point, imposes just nine conditions on the quartic surfaces, whereas *chez-nous*, one has $35 - 25 = 10$ conditions.

Summarizing, in the hypersurface of $\mathbb{P}^{34}$ formed by quartics which contain a line, one finds two subvarieties of dimension 32 at stake: (1) the closure of the set of quartics which contain two incident lines (7+25 by parameters); (2) the closure of the good locus of the quartics which contain two lines in general position (8+24 parameters). We see that the double point formula does not manage to distinguish these two loci.

3.3.3. **incident line.** Let us sketch the determination per se of the degree of the locus of quartics containing ℓ_0 together with a variable incident line, *without embedded point*. This means, a reducible conic having the line ℓ_0 as a component.

Let $\mathbb{V}_{\ell_0} \subset \mathbb{G}$ denote the divisor of lines incident to ℓ_0; one knows that $\mathbb{V}_{\ell_0}$ is a quadratic cone with vertex the point $\ell_0 \in \mathbb{G} \subset \mathbb{P}^5$ via Plücker. In fact $\mathbb{V}_{\ell_0}$ is equal to the intersection of $\mathbb{G}$ with its tangent hyperplane at ℓ_0. Consider the rational map $\mathbb{V}_{\ell_0} \dashrightarrow \mathbb{P}^3 \times \mathbb{V}_{\ell_0} \times \check{\mathbb{P}}^3$ which sends each line ℓ to the triplet $(\mathrm{p}, \ell, \langle \ell, \ell_0 \rangle)$ where $\mathrm{p} \in \ell \cap \ell_0$ and $\langle \ell, \ell_0 \rangle$ is the plane spanned by the pair of incident lines. Let $\mathbb{V}'_{\ell_0}$ be the set of triplets (p, ℓ, h) with the point $\mathrm{p} \in \ell \cap \ell_0$ and the plane $h \supset \ell \cup \ell_0$. Denote by $\check{\ell}_0 \subset \check{\mathbb{P}}^3$ the dual line, pencil of planes containing ℓ_0. Then $\mathbb{V}'_{\ell_0}$ is a $\mathbb{P}^1$–bundle over $\ell_0 \times \check{\ell}_0$: the fiber over (p, h) is the pencil of lines ℓ contained in the plane $h \supset \ell_0$ and passing through the point $\mathrm{p} \in \ell_0$. It can be easily checked that if we blowup the vertex $\ell_0 \in \mathbb{V}_{\ell_0}$, we obtain the $\mathbb{P}^1$–bundle $\mathbb{V}'_{\ell_0} \to \ell_0 \times \check{\ell}_0$.

The exceptional divisor of the blowup $\mathbb{V}'_{\ell_0} \to \mathbb{V}_{\ell_0}$ is the smooth quadric

$$\mathbb{W}'_0 \subset \mathbb{P}\left(T_{\ell_0}\mathbb{G}\right) = \mathbb{E}'_{\ell_0} \simeq \mathbb{P}^3, \tag{17}$$

image of the section of $\mathbb{V}'_{\ell_0} \to \ell_0 \times \check{\ell}_0$ defined by $\ell_0 \times \check{\ell}_0 \ni (\mathrm{p}, h) \mapsto (\mathrm{p}, h, \ell_0)$.

Over $\mathbb{V}'_{\ell_0}$ we have a flat family of conics with fixed component ℓ_0, to wit, $\ell + \ell_0 \subset h$. The conic $2\ell_0 \subset h$ appears ∞^1 times in this family, it does not depend on the choice of $\mathrm{p} \in \ell_0$ (though the double line is taken in some plane $h \supset \ell_0$). We'll compute the degree of the subvariety of $\mathbb{P}^{34}$ of dimension 28 formed by the quartics which contain some $\ell + \ell_0$. That's the image of a $\mathbb{P}^{25}$–bundle over $\mathbb{V}'_{\ell_0}$ whose fiber is the linear system of quartics which contain the reducible conic $\ell + \ell_0 \subset h$. Let $\mathcal{G}(\ell_0)$ be the vector subbundle of $\mathcal{F}_4$ with fiber over $(\mathrm{p}, h, \ell) \in \mathbb{V}'_{\ell_0}$ given by $h\mathcal{F}_3 + \ell_0\ell\mathcal{F}_2$. We have $\operatorname{rank}\mathcal{G}(\ell_0) = 26$. One proceeds just as before: apply Bott's formula, this time to

$$\int s_3(\mathcal{G}(\ell_0)) \cap \mathbb{V}'_{\ell_0} = \int c_3(\overline{\mathcal{G}}(\ell_0)) \cap \mathbb{V}'_{\ell_0},$$

with $\overline{\mathcal{G}}(\ell_0) = \mathcal{F}_4 / \mathcal{G}(\ell_0)$. We have found the total contribution of the 8 fixed points using SINGULAR. We got the number **84** of quartic surfaces which pass through 28 points in general position and contain ℓ_0 together with some line ℓ incident to ℓ_0. As we see, that's precisely the difference **315–231** with the calculation performed using the double point formula.

3.3.4. **Remark.** If an irreducible quartic surface q contains a reducible conic $\ell + \ell_0 \subset h$ with supporting plane h, the plane section $q \cap h$ contains a residual conic, which, in general, is irreducible. In other words, it can be checked that the map $\mathbb{P}\left(\mathcal{G}(\ell_0)\right) \to \mathbb{P}^{34}$ is generically injective.

3.3.5. **embedded point.** One may also get the number **150** of quartic surfaces which pass through 27 points in general position and contain the scheme formed by the union of the line ℓ_0 with a line ℓ incident to ℓ_0, now augmented by an embedded point at the point of incidence. It suffices to take the restriction of $\mathbb{X}''_4(\ell_0)$ over the strict transform, $\mathbb{V}'_{\ell_0} \subset \mathbb{G}'$, of the divisor of incidence $\mathbb{V}_{\ell_0} \subset \mathbb{G}$. The class of this divisor is the Schubert cycle $[\mathbb{V}_{\ell_0}] = c_1\mathcal{Q} \cap [\mathbb{G}]$. Since ℓ_0 is a **double** point of $\mathbb{V}_{\ell_0}$ (cf. 3.3.3), one has the formula

$$[\mathbb{V}'_{\ell_0}] = c_1\mathcal{Q} \cap [\mathbb{G}'] - \mathbf{2}[\mathbb{E}'_{\ell_0}],$$

where $\mathbb{E}'_{\ell_0}$ denotes the exceptional divisor of $\mathbb{G}' \to \mathbb{G}$. The sought for degree is given by

$$\int h^{27} \cap [\mathbb{X}''_4(\ell_0) | \mathbb{V}'_{\ell_0}].$$

With notation as in (14), it reduces to

$$\int c_3(\overline{\mathcal{F}_4}(\ell_0)) \cap [\mathbb{V}'_{\ell_0}].$$

The projection formula yields

$$\int c_3 \overline{\mathcal{F}_4}(\ell_0) c_1(\mathcal{Q}) \cap [\mathbb{G}] = \mathbf{150}.$$

4. Two variable lines

Now we allow ℓ_0 to move. We go to the blowup $\mathbb{G}^{(2)} \to \mathbb{G}\times\mathbb{G}$ along the diagonal. The fiber of $\mathbb{G}^{(2)} \to \mathbb{G}$ over each $\ell_0 \in \mathbb{G}$ is the blowup $\mathbb{G}'$ of $\mathbb{G}$ at ℓ_0 we've seen above, cf. (6), p. 4.

4.1. **quartics with two lines.** We have a subbundle $\mathcal{F}_2'' \subset \mathcal{F}_2 \times \mathbb{G}^{(2)}$ whose restriction to $\mathbb{G}'$ is the bundle $\mathcal{F}_2''(\ell_0)$ of rank 4 introduced before. We also have a subbundle $\mathcal{F}_4'' \subset \mathcal{F}_4 \times \mathbb{G}^{(2)}$ of rank 25 with generic fiber the space of quartics which contain two lines in general position. The image of $\mathbb{P}(\mathcal{F}_4'')$ in $\mathbb{P}^{34}$ is the variety $\mathbb{Y}_4''$ of dimension 32, closure of the locus of quartics which contain two (variable) lines in general position. Observe that the general quartic of $\mathbb{Y}_4''$ comes from exactly two elements of $\mathbb{P}(\mathcal{F}_4'')$. Thus the degree can be found as

$$\tfrac{1}{2}\int s_8(\mathcal{F}_4'') \cap \mathbb{G}^{(2)}.$$

The Segre class again can be written as Chern of the quotient $\overline{\mathcal{F}_4}$.

Over each of the 6 fixed points of $\mathbb{G}$ we get a fiber like $\mathbb{G}'$ with 9 fixed points. One finds that the total contribution of these 54 fixed points is equal to **35640=71280/2**.

For the case of pairs of incident lines, we arrive at **14100**.

Again, the double point formula gives the answer **99480** = **28200+71280**.

The tables below show a few other intersection numbers over $\mathbb{G}^{(2)}$, *e.g.,*

$$s_2 q_{1,1}^3 q_{2,2} q_{2,1} = \mathbf{168}.$$

This accounts for quartic surfaces in $\mathbb{P}^3$ which pass through 26 points and contain two lines, the first one forced to meet three lines and the second one to pass by a point and further meet another line, everything in general position. When the conditions imposed on the two lines are symmetric (*e.g.,* s_8, ou $s_6 q_{1,1} q_{2,1}, \ldots$), we must divide by 2.

s_8	$s_7 q_{1,1}$	$s_7\mathbb{E}$	$s_6 q_{1,1}^2$	$s_6 q_{1,2}$	$s_6 q_{1,1} q_{2,1}$	$s_5 q_{1,2} q_{2,1}$	$s_5 q_{1,1}^2 q_{2,1}$
71280	**49740**	**9060**	**20550**	**12585**	**35350**	**9048**	**14776**

$s_5 q_{1,1}^3$	$s_4 q_{1,1}^4$	$s_4 q_{1,2}^2$	$s_4 q_{1,1}^2 q_{2,1}^2$	$s_4 q_{1,2} q_{2,1}^2$	$s_4 q_{1,2} q_{2,2}$	$s_3 q_{1,1}^4 q_{2,1}$	$s_3 q_{1,1}^3 q_{2,1}^2$
4596	**462**	**231**	**6226**	**3801**	**2316**	**340**	**1428**

$s_3 q_{1,2} q_{2,1}^3$	$s_2 q_{1,2} q_{1,1} q_{2,2} q_{2,1}$	$s_2 q_{1,1}^3 q_{2,2} q_{2,1}$	$s_1 q_{1,1}^3 q_{2,2} q_{2,1}^2$	$s_1 q_{1,1} q_{1,2} q_{2,2} q_{2,1}^2$
870	**84**	**168**	**18**	**9**

4.2. **arbitrary degree.** Needless to say one may deal with the case of surfaces of degrees ≥ 4 in a similar fashion. For example, one finds the number **3027430** of surfaces of degree five which pass through $51(= 8 + \binom{5+3}{3} - 1 - (2\cdot 5 + 2))$ points and contain two lines. Below is a table for the first few degrees. See [18] for a script.

4	5	6	7	8	9
35640	**1513715**	**30593535**	**379960140**	**3332248871**	**22482531750**

As a matter of fact, arguing by interpolation as in Prop. 3.3, p. 8, one deduces

4.2.1. **Proposition.** *Notations as above, the degree of the subvariety of $\mathbb{P}(\mathcal{F}_d)$ formed by the surfaces containing two general lines is given by*

$$\binom{d}{3}\big(45d^{13}+135d^{12}-345d^{11}+2655d^{10}-9115d^9+15657d^8-7371d^7 \\ -157707d^6+716586d^5-1767492d^4+3318376d^3-4244928d^2+ \\ 3723264d-5460480\big)\Big/2^{12}3^35.$$

4.2.2. **Remark.** For $d=3$ the polynomial found above evaluates to **216**. Of course, it's *not* the degree of the image of the map induced by projection,

$$\begin{array}{rcl}\pi:\mathbb{P}(\mathcal{F}_3'') & \longrightarrow & \mathbb{P}(\mathcal{F}_3)=\mathbb{P}^{19}\\ (\ell_1,\ell_2,q) & \mapsto & q,\end{array}$$

which is surjective, generically finite. That number is indeed half the degree of π. This is compatible with the combinatorics of the well known graph of incidence of the 27 lines contained in a smooth cubic, [3], [14, p. 176]. If one picks one of the 27 lines, one finds 16 others which are disjoint from the chosen one. Hence, the set of pairs of skew lines $27\cdot 16=2\cdot 216$.

5. two lines, bis

The goal of this section is to sketch how to perform the calculation working directly with the good component $\mathbb{H}\subset \mathrm{Hilb}^{2t+2}$ of the Hilbert scheme. Since $\mathbb{G}$ is a smooth quadric hypersurface in $\mathbb{P}^5$ (via Plücker), the Hilbert scheme $\mathrm{Hilb}^2\mathbb{G}$ of unordered pairs of points of $\mathbb{G}$ admits the following elementary description.

5.1. **Proposition.** *Let $\mathbb{G}\subset\mathbb{P}^5$ be the Plücker embedding. Let Γ denote the Grassmannian of lines of $\mathbb{P}^5$. Let $\Lambda\subset\Gamma$ be the locus corresponding to lines contained in the quadric $\mathbb{G}$. Let Γ' be the blowup of Γ along Λ. Then:*
(i) Λ *is isomorphic to the flag variety point* $\in$ *plane;*
(ii) Γ' *is isomorphic to* $\mathrm{Hilb}^2\mathbb{G}$*;*
(iii) $\mathrm{Hilb}^2\mathbb{G}$ *is isomorphic to the component of* $\mathbb{H}$ *of the Hilbert scheme whose general point corresponds to two skew lines.*

Proof. One knows that $\mathrm{Hilb}^2\mathbb{G}$ is a closed subscheme of $\mathrm{Hilb}^2\mathbb{P}^5$. The latter is the $\mathbb{P}^2$-bundle of the equations of degree 2 over lines in $\mathbb{P}^5$. More precisely, with notations as in (2), $\mathbb{P}^5=\mathbb{P}(\overset{2}{\wedge}\mathcal{F})$, let

$$\mathcal{A}\rightarrowtail\overset{2}{\wedge}\mathcal{F}$$

be the tautological subbundle of rank $\mathcal{A}=2$ over of $\mathbb{G}\mathrm{r}_2(\overset{2}{\wedge}\mathcal{F})$, the Grassmannian of lines in $\mathbb{P}^5$. Set $\pi:\mathbb{P}(\mathcal{A})\to\mathbb{G}\mathrm{r}_2(\overset{2}{\wedge}\mathcal{F})$, the universal line. Put $\mathrm{S}_2(\mathcal{A}^\vee)=\pi_\star(\mathcal{O}_{\mathcal{A}}(2))$.

Then we have $\mathrm{Hilb}^2\mathbb{P}^5=\mathbb{P}(\mathrm{S}_2(\mathcal{A}^\vee))$. For each $L\in\mathbb{G}\mathrm{r}_2(\overset{2}{\wedge}\mathcal{F})$, $L\cong\mathbb{P}^1$, write the fiber $V=\mathcal{A}_L\cong\mathbb{C}^2$. We have $L=\mathbb{P}(V)\subset\mathbb{P}^5=\mathbb{P}(\overset{2}{\wedge}\mathcal{F})$. Given a quadric $q\in\mathbb{P}(\mathrm{S}_2(V^\vee))$, we may identify it with a symmetric map as in the diagram

$$\begin{tikzcd} V \arrow[r, "q"] \arrow[d, leftarrowtail] & V^\vee \arrow[d, twoheadrightarrow] \\ \mathcal{O}_V(-1) \arrow[r, "\bar q"] & \mathcal{O}_V(1) \end{tikzcd}$$

where $\overline{q} \in |\mathcal{O}_{\mathbb{P}^1}(2)|$. The homogeneous equation of $\mathbb{G}$ in $\mathbb{P}^5$ corresponds to a natural symmetric map,

$$\overset{2}{\wedge}\mathcal{F} \xrightarrow{g} \overset{2}{\wedge}\mathcal{F}^{\vee} \otimes \overset{4}{\wedge}\mathcal{F}$$

arising from multiplication, $\overset{2}{\wedge}\mathcal{F} \otimes \overset{2}{\wedge}\mathcal{F} \twoheadrightarrow \overset{4}{\wedge}\mathcal{F}$. We get for each L a commutative diagram

$$\begin{tikzcd} V \arrow[r, "q"] \arrow[d, tail] & V^{\vee} \otimes \overset{4}{\wedge}\mathcal{F} \\ \overset{2}{\wedge}\mathcal{F} \arrow[r, "g"] & \overset{2}{\wedge}\mathcal{F}^{\vee} \otimes \overset{4}{\wedge}\mathcal{F}. \arrow[u, two heads] \end{tikzcd}$$

This defines a rational map

$$\begin{array}{rcl} \Gamma := \mathbb{G}\mathrm{r}_2(\overset{2}{\wedge}\mathcal{F}) & \dashrightarrow & \mathrm{Hilb}^2\mathbb{G} \subset \mathrm{Hilb}^2\mathbb{P}^5 \\ L & \longmapsto & L \cap \mathbb{G}. \end{array}$$

The scheme of indeterminacy is given by the zeros of the section of $\mathrm{S}_2(\mathcal{A}^{\vee})$ induced by the equation g. One checks that it coincides with Λ and its blowup Γ' maps isomorphically onto $\mathrm{Hilb}^2\mathbb{P}^5$. We leave the remaining details to the reader. $\square$

5.2. **Remark.** The data needed for Bott's formula are easy to describe in the Hilbert scheme setting, totaling 39 fixed points in $\Gamma' = \mathrm{Hilb}^2\mathbb{G}$. Details for the implementation of Bott in this setting can be seen in [16].

6. THREE LINES

We go on to sketch the construction of a parameter space for the family with generic member equal to the union of three skew lines in $\mathbb{P}^3$.

We first consider the blowup

$$\mathbb{G}(3) \longrightarrow \mathbb{G}^{(2)} \times_{\mathbb{G}} \mathbb{G}^{(2)} \tag{18}$$

along the relative diagonal. Recall that $\mathbb{G}^{(2)}$ was in turn built by blowing up $\mathbb{G} \times \mathbb{G}$ along the diagonal.

We consider $\mathbb{G}^{(2)}$ as a scheme over of $\mathbb{G}$ via the first projection,

$$\mathbb{G}^{(2)} \to \mathbb{G} \times \mathbb{G} \to \mathbb{G}.$$

The fiber of $\mathbb{G}^{(2)} \to \mathbb{G}$ over any $\ell_0 \in \mathbb{G}$ is the blowup $\mathbb{G}'$ of $\mathbb{G}$ at ℓ_0, cf. (6).

Let $\mathbb{G}''$ be the fiber of $\mathbb{G}(3)$ over $\ell_0 \in \mathbb{G}$. One checks that $\mathbb{G}''$ identifies with the blowup of $\mathbb{G}' \times \mathbb{G}'$ along the diagonal $\Delta' \subset \mathbb{G}' \times \mathbb{G}'$. Our parameter space will be obtained by a sequence of altogether three blowups, as depicted in the diagram:

$$\begin{tikzcd} \widehat{\mathbb{E}} \arrow[r] \arrow[d, hook] & \mathbb{W}''' \arrow[d, hook] & \mathbb{E}'' \arrow[r] \arrow[d, hook] & \Delta\mathbb{G}^{(2)} \arrow[d, hook] \\ \mathbb{G}^{(3)} \arrow[r] & \mathbb{G}(3)' \arrow[r] & \mathbb{G}(3) \arrow[r] & \mathbb{G}^{(2)} \times_{\mathbb{G}} \mathbb{G}^{(2)} \\ & \mathbb{E}''' \arrow[r] \arrow[u, hook] & \mathbb{Y}'' \arrow[u, hook] & \end{tikzcd}$$

The blowup center $\mathbb{Y}'' \subset \mathbb{G}(3)$ is the strict transform of a smooth subvariety $\mathbb{Y}' \subset \mathbb{G}^{(2)} \times_{\mathbb{G}} \mathbb{G}^{(2)}$ described further down, cf. 6.1, 6.1.1. The last center, $\mathbb{W}''' \subset \mathbb{G}(3)'$, comes from a smooth subvariety $\widetilde{\mathbb{W}} \subset \mathbb{G}^{(2)} \times_{\mathbb{G}} \mathbb{G}^{(2)}$, see (17).

The two centers $\mathbb{Y}'', \mathbb{W}'''$ are certain fibrations over $\mathbb{G}$. This allows us to assert that the fiber of the blowup coincides with the blowup of the fiber. For simplicity, we explain what is going on along the fibers over our fixed ℓ_0:

$$\begin{tikzcd}
\widehat{\mathbb{E}}_0 \arrow[r] \arrow[d, hook] & \mathbb{W}'''_0 \arrow[d, hook] & \mathbb{E}''_0 \arrow[r] \arrow[d, hook] & \Delta\mathbb{G}' \arrow[d, hook] \\
\widehat{\mathbb{G}} \arrow[r] & \mathbb{G}''' \arrow[r] & \mathbb{G}'' \arrow[r] & \mathbb{G}' \times \mathbb{G}' \\
 & \mathbb{E}'''_0 \arrow[r] \arrow[u, hook] & \mathbb{Y}''_0 \arrow[u, hook] &
\end{tikzcd}$$

Each point of $\mathbb{G}' \times \mathbb{G}'$ corresponds to two subschemes, $\ell'_1, \ell'_2 \subset \mathbb{P}^3$ with Hilbert polynomial $2t+2$, and the line $\ell_0 \subset \ell'_i, i = 1, 2$. For (ℓ'_1, ℓ'_2) general in $\mathbb{G}' \times \mathbb{G}'$, we have $\ell'_1 = \ell_0 \cup \ell_1, \ell'_2 = \ell_0 \cup \ell_2$ with the three lines, ℓ_0, ℓ_1, ℓ_2 disjoint. In this case, the homogeneous ideal of their union is spanned by a space of 8 independent cubics. Thus we start with the rational map

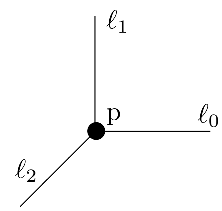
$$\rho' : \mathbb{G}' \times \mathbb{G}' \dashrightarrow \mathbb{G}\mathrm{r}_8(\mathcal{F}_3(\ell_0)) \tag{19}$$

given by

$$\rho'(\ell'_1, \ell'_2) = \ell_0 \cdot \ell_1 \cdot \ell_2 \in \mathbb{G}\mathrm{r}_8(\mathcal{F}_3(\ell_0)),$$

defined by the product of the three subspaces $\ell_i \subset \mathcal{F}_1$.

6.1. **Lemma.** *The scheme of indeterminacy of ρ' is defined by an ideal of the form $J \cap I^2$ where $I = I(\Delta')$ is the ideal of the diagonal and J is the ideal of a smooth subvariety $\mathbb{Y}'_0 \subset \mathbb{G}' \times \mathbb{G}'$ with generic member corresponding to the union of three concurrent lines at an embedded point .*

$\mathbb{Y}'_0 = \Bigg\{$  $\Bigg\}.$

Proof. Using appropriate affine coordinates in $\mathbb{G}' \times \mathbb{G}'$, we have

$$\begin{cases} \ell_0 = \langle x_0, x_1 \rangle, \\ \ell_1 = \langle x_0 + a_1(x_2 + b_1 x_3), x_1 + a_1(b_2 x_2 + b_3 x_3) \rangle, \\ \ell_2 = \langle x_0 + c_4(x_2 + c_1 x_3), x_1 + c_4(c_2 x_2 + c_3 x_3) \rangle. \end{cases} \tag{20}$$

Here a_1, b_1, b_2, b_3 (resp. $c_1, \ldots, c_4$) stand for coordinates in the first (resp. second) factor (cf. (9)). The local generator of the pullback of the exceptional divisor of $\mathbb{G}'$ by the 1st (resp. 2nd) projection is a_1 (resp. c_4). The multiplication $\ell_0 \cdot \ell_1 \cdot \ell_2$ yields a system of 8 cubics, in general independent. One may compute (*e.g.,* with the help of Singular [8]) the 8×20 matrix whose rows represent the coefficients with respect to the base of cubic monomials. Performing elementary row operations and dividing rows by possible common factors, one obtains a matrix M whose ideal of

$8{\times}8$ minors is as in the statement of the lemma. The primary decomposition of this ideal shows that the equations of $\mathbb{Y}'_0$ can be written as

$$b_3 = b_1 b_2, c_1 = b_1, c_3 = c_2 b_1. \tag{21}$$

The equations of the diagonal of $\mathbb{G}' \times \mathbb{G}'$ are, of course,

$$a_1 = c_4, b_1 = c_1, b_2 = c_2, b_3 = c_3.$$

Making the substitution of (21) into the matrix M, one checks that the rank drops to 7 off Δ', where it reduces to 6. □

6.1.1. **Remark.** To shed more light on $\mathbb{Y}'_0$, we recall the $\mathbb{P}^1$–bundle $\mathbb{V}'_{\ell_0} \subset \mathbb{G}'$, desingularisation of the divisor $\mathbb{V}_{\ell_0} \subset \mathbb{G}$ defined by the lines incident to ℓ_0, (cf. 3.3.3). Each point of $\mathbb{V}'_{\ell_0}$ corresponds to the choice of a triplet (p, ℓ, h) with the point $\mathrm{p} \in \ell \cap \ell_0$ and the plane $h \supset \ell \cup \ell_0$. Now $\mathbb{Y}'_0$ is the subvariety of $\mathbb{V}'_{\ell_0} \times \mathbb{V}'_{\ell_0} \subset \mathbb{G}' \times \mathbb{G}'$ formed by the pairs of triplets $(\mathrm{p}_1, \ell_1, h_1), (\mathrm{p}_2, \ell_2, h_2) \in \mathbb{V}'_{\ell_0}$ such that $\mathrm{p}_1 = \mathrm{p}_2$. It can be shown that $\mathbb{Y}'_0$ is smooth of dimension 5, isomorphic to the fiber product $\mathbb{V}'_{\ell_0} \times_{\ell_0} \mathbb{V}'_{\ell_0}$. We have a rational map

$$\mathbb{Y}'_0 \dashrightarrow \mathbb{G}\mathrm{r}_7(\mathcal{F}_3(\ell_0)). \tag{22}$$

The scheme of indeterminacy of the latter is precisely the image of the diagonal,

$$\mathbb{V}'_{\ell_0} \hookrightarrow \mathbb{V}'_{\ell_0} \times \mathbb{V}'_{\ell_0} .$$

We blowup the diagonal of $\mathbb{G}' \times \mathbb{G}'$, thereby producing $\mathbb{G}'' \to \mathbb{G}' \times \mathbb{G}'$. Still with the help of local coordinates we find:

6.2. **Lemma.** **(i)** *The blowup $\mathbb{Y}''_0 \longrightarrow \mathbb{Y}'_0$ along $\Delta' \cap \mathbb{Y}'_0$ is isomorphic to the closure of the graph of the rational map* (22).

(ii) *The strict transform $\mathbb{Y}''_0$ of $\mathbb{Y}'_0$ is the scheme of indeterminacy of the rational map induced by ρ', cf.* (19)

$$\begin{tikzcd} \mathbb{G}'' \arrow[r] \arrow[rr, dashed, bend right] & \mathbb{G}' \times \mathbb{G}' \arrow[r, dashed, "\rho'"] & \mathbb{G}\mathrm{r}_8(\mathcal{F}_3(\ell_0)). \end{tikzcd}$$

□

Now it's time to blowup $\mathbb{G}''$ along $\mathbb{Y}''_0$:

$$\begin{tikzcd} \mathbb{E}'''_0 \arrow[r, hook] \arrow[d] & \mathbb{G}''' \arrow[d] \arrow[dr, "\kappa"] & \\ \mathbb{Y}''_0 \arrow[r, hook] & \mathbb{G}'' \arrow[r, dashed] & \mathbb{G}\mathrm{r}_8(\mathcal{F}_3(\ell_0)). \end{tikzcd}$$

Thus $\mathbb{G}'''$ is the closure of the graph of ρ', cf. (19). So, over $\mathbb{G}'''$ we have acquired a subbundle

$$\mathcal{F}'''_3(\ell_0) \subset \mathcal{F}_3(\ell_0) \times \mathbb{G}''', \tag{23}$$

pullback of the tautological subbundle of rank 8 of $\mathcal{F}_3(\ell_0) \times \mathbb{G}\mathrm{r}_8(\mathcal{F}_3(\ell_0))$ via the morphism $\kappa : \mathbb{G}''' \to \mathbb{G}\mathrm{r}_8(\mathcal{F}_3(\ell_0))$. The generic fiber is a system spanned by 8 cubics having as zeros the union of ℓ_0 with two skew lines.

Local coordinates enables us to get a hold of the fibers of $\mathcal{F}_3'''(\ell_0)$ over $\mathbb{G}'''$. Among these fibers one finds the vector space

$$\underbrace{\ell_0^3}_{\text{rank } 4} + \underbrace{x_1x_2\ell_0}_{\text{rank } 2} + \underbrace{x_1x_3\ell_0}_{\text{rank } 2}, \qquad \text{with } \ell_0 = \langle x_0, x_1\rangle.$$

Unfortunately, the Hilbert polynomial of the ideal it spans is equal to $4t$, instead of $3t+3$. The family of subschemes parameterized by $\mathbb{G}'''$ is not flat. This will be fixed with a final blowup.

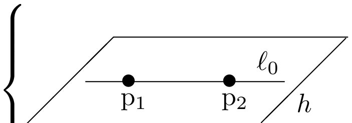
6.3. **last blowup center.** A calculation with local coordinates reveals that the rank of the natural map of vector bundles

$$\mathcal{F}_3'''(\ell_0) \otimes \mathcal{F}_1 \longrightarrow \mathcal{F}_4 \times \mathbb{G}''' \tag{24}$$

is the expected one, *i.e.*, $35 - 3\cdot 4 - 3 = 20$, *almost everywhere.* It drops to 19 along a smooth subvariety $\mathbb{W}_0''' \subset \mathbb{G}'''$ described in the sequel. Notations as in (17), put

$$\widetilde{\mathbb{W}}_0 = \mathbb{W}_0' \times_{\check{\ell}_0} \mathbb{W}_0' \subset \mathbb{G}' \times \mathbb{G}'.$$

Each point of $\widetilde{\mathbb{W}}_0$ represents a configuration as depicted below:

$$\widetilde{\mathbb{W}}_0 = \left\{ \right.$$


$$\left. \right\}.$$

In fact, we have $\mathbb{W}_0' \simeq \ell_0 \times \check{\ell}_0$, so that $\widetilde{\mathbb{W}}_0 \simeq \ell_0 \times \ell_0 \times \check{\ell}_0$: choose two points on ℓ_0 and a plane containing ℓ_0. Using once again local coordinates, one finds that our last blowup center, $\mathbb{W}_0'''$, is the strict transform of $\widetilde{\mathbb{W}}_0$ by the composition of blowups

$$\begin{tikzcd}
 & \mathbb{E}_0''' \arrow[r] \arrow[d, hook] & \mathbb{Y}_0'' \arrow[d, hook] & & \\
\mathbb{W}_0''' \arrow[r, phantom, "\subset"] \arrow[rrrr, bend left] & \mathbb{G}''' \arrow[r] & \mathbb{G}'' \arrow[r] & \mathbb{G}' \times \mathbb{G}' \arrow[r, phantom, "\supset"] & \widetilde{\mathbb{W}}_0 \\
 & & \mathbb{E}_0'' \arrow[r] \arrow[u, hook] & \Delta' = \mathbb{G}' \arrow[u, hook] &
\end{tikzcd}$$

6.3.1. **Proposition.** *Notation as just above, let $\widehat{\mathbb{G}}$ be the blowup of $\mathbb{G}'''$ along $\mathbb{W}_0'''$. Then $\widehat{\mathbb{G}}$ is isomorphic to the closure of the graph of the rational map*

$$\mathbb{G}''' \dashrightarrow \mathbb{G}\mathrm{r}_{20}(\mathcal{F}_4(\ell_0))$$

induced by (24). Let furthermore

$$\mathcal{F}_4'''(\ell_0) \subset \mathcal{F}_4 \times \widehat{\mathbb{G}}$$

be defined by pullback of the tautological subbundle of rank 20 of $\mathcal{F}_4 \times \mathbb{G}\mathrm{r}_{20}(\mathcal{F}_4(\ell_0))$. Then

(i) *the homomorphism $\mu : \mathcal{F}_4'''(\ell_0) \otimes \mathcal{F}_{d-4} \longrightarrow \mathcal{F}_d \times \widehat{\mathbb{G}}$ is everywhere of the correct rank*

$$\tbinom{d+3}{3} - (3d+3),\ d \geq 4.$$

(ii) *Let $\mathcal{F}_d'''(\ell_0) \subset \mathcal{F}_d$ be the subbundle image of μ. Set*

$$\mathbb{X}_{\prime}'''(\ell_0) = \mathbb{P}\left(\mathcal{F}_d'''(\ell_0)\right) \subset \widehat{\mathbb{G}} \times \mathbb{P}^N.$$

Then $\mathbb{X}''''(\ell_0)$ is the set of pairs $(\widehat{\ell}, q) \in \widehat{\mathbb{G}} \times \mathbb{P}^N$ such that the surface q contains the scheme $\widehat{\ell}$, flat specialization of the union of ℓ_0 with two lines in general position.

□

The explicit description of the blowup centers fills in all the information required to determine the fixed points of a general $\mathbb{C}^\star$–action. That's all we need to employ once again Bott's formula.

6.4. **enumeration of surfaces with three lines.** Just as in 4.2.1, we may interpolate and arrive at the following

6.4.1. **Proposition.** *The subvariety $\mathbb{Y}''' \subset \mathbb{P}^N$ defined as closure of the set of surfaces of degree d containing three lines in general position is of codimension $3d+3-12$ and degree given by*

$$\begin{gathered}
\tfrac{1}{2^{18}\cdot 3^7\cdot 5\cdot 7\cdot 11}\tbinom{d-1}{2}\Big(10395d^{22}+31185d^{21}-405405d^{20}+1590435d^{19}+679140d^{18}\\
-47122614d^{17}+288217622d^{16}-990632874d^{15}+1925104115d^{14}-776459211d^{13}\\
-6122385577d^{12}+35263846215d^{11}-317356063490d^{10}+2348419505664d^{9}\\
-11794584945440d^{8}+41910000473328d^{7}-109450732226336d^{6}\\
+214044651812736d^{5}-312731335941120d^{4}+332706442622976d^{3}\\
-240386042732544d^{2}+69285020958720d+61802702438400\Big)\cdot
\end{gathered}$$

For $d = 3$ one finds **720**, again in agreement with the combinatorics of triplets of skew lines contained in a smooth cubic surface.

7. CONICS

The case of conics is almost as simple as that of imposing just one line. The parameter space is a well known $\mathbb{P}^5$-bundle over the dual space $\check{\mathbb{P}}^3$. There are 4×6 fixed points: pick a plane x_i and a quadratic monomial $x_j x_k, i \notin \{j, k\}$.

7.1. **Proposition.** *For all $d \geq 5$ the locus of surfaces in $\mathbb{P}^3$ of degree d containing a conic is a variety of codimension $2d-7$ and degree*

$$\begin{gathered}
\tbinom{d}{4}(d^2-d+8)(d^2-d+6)\big(207d^8-288d^7+498d^6+5068d^5\\
-15693d^4+31732d^3-37332d^2+9280d-47040\big)/967680.
\end{gathered}$$

7.2. **Remarks.** (i) For $d = 4$, the correct degree is $5016/2$ due to Bézout, as mentioned before.
(ii) If we restrict the conics to move on a given plane, we get the following formula for the degree of the corresponding locus:

$$\frac{1}{1920}d\,(d-1)\left(d^2-d+2\right)\left(d^2-d+4\right)\left(d^2-d+6\right)\left(d^2-d+8\right).$$

The codimension is $2d-4$. For $d = 2$ the formula gives 1 as expected. For $d = 3$, we get 21, which coincides with the degree of the locus of cubic surfaces bitangent to a fixed plane, same as the number of binodal plane cubics through 7 general points...

8. TWISTED CUBICS

The case of twisted cubics can be handled using the material in [6]. The Hilbert scheme component is also described in [15]. Altogether, there are 130 fixed points. Everything is sufficiently explicit to feed into Bott's formula. A detailed script is availabel in [17].

We find, again with an argument of interpolation, the formula for the degree of the subvariety of $\mathbb{P}^N = |\mathcal{O}_{\mathbb{P}^3}(d)|$ consisting of surfaces which contain a twisted cubic curve:

$$\begin{gathered}\tbinom{d-1}{3}\big(67635d^{21} - 875772d^{20} + 6291351d^{19} - 24755238d^{18} + 14958054d^{17}\\ +534566808d^{16} - 4415655154d^{15} + 21482361732d^{14} - 73280138161d^{13}\\ +161558718372d^{12} - 52247446981d^{11} - 1541272979406d^{10} + 9060084624568d^{9}\\ -33544134606864d^{8} + 94153450914112d^{7} - 208561082534208d^{6}\\ +365344201610944d^{5} - 496020233998464d^{4} + 502377293509632d^{3}\\ -359676049397760d^{2} + 169831821312000d - 6621718118400\big)/(2^{16}3^{7}5^{2}7 \cdot 11).\end{gathered}$$

9. CONCLUDING REMARKS

One can also get similar formulas for the loci of surfaces containing a plane curve of arbitrary degree k. Beware however of the fact that for $k < d$, there is a residual curve of degree $d - k$. For instance, when $k = 3$, any quartic surface containing one plane cubic, must contain infinitely many, cut out by the pencil of planes with axis the residual line. The case of quartic surfaces containing an elliptic quartic curve is subtler, [4]. Indeed, if a smooth quartic surface contains one such curve, it is in fact spanned by a pencil.

In all cases we've met so far, the polynomial formula obtained is of degree twice the dimension of the family of curves.

Perhaps Kontsevich's spaces of stable maps could be used to handle the loci of surfaces of sufficiently high degree containing a rational curve of fixed degree. We hope to report on these matters elsewhere.

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