Degrees of spaces of holomorphic foliations of codimension one in $\mathbb{P}^n$

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ABSTRACT. Let $\mathcal{F}(d;n)$ be the parameter space of the family of holomorphic foliations of codimension one and degree d in $\mathbb{P}^n$. Gomez-Mont and Lins-Neto have shown that the Zariski closure of the set of foliations defined by a differential 1-form of type $aF\mathbf{d}G - bG\mathbf{d}F$, where F, G denote co-prime homogeneous polynomials of degrees a, b is an irreducible component of $\mathcal{F}(a+b-2;n)$. Our main result gives a formula for the degree of this component for $a = 2$, b odd.

1. Introduction

A holomorphic foliation of codimension one and degree d in $\mathbb{P}^n$ is defined by a 1-form $\omega = \sum_0^n A_i \mathbf{d}x_i$, up to scalar multiple, where the A_i denote homogeneous polynomials of degree $d+1$, satisfying the conditions (i) $\sum A_i x_i = 0$ (projectivity), and (ii) $\omega \wedge \mathbf{d}\omega = 0$ (Frobenius integrability).

The family of such foliations is parameterized by a closed subscheme $\mathcal{F}(d;n)$ of $\mathbb{P}^N$, the projectivization of the space of global sections of the twisted cotangent bundle $\Omega^1_{\mathbb{P}^n}(d+2)$. Condition (i) (resp. (ii)) yields linear (resp. quadratic) equations for the space of foliations $\mathcal{F}(d;n)$ in $\mathbb{P}^N$. Given that any projective scheme is isomorphic to one defined by equations of degree at most 2, it should come as no surprise that the description of the irreducible components of $\mathcal{F}(d;n)$ seems hard to tackle in full generality.

For $d = 0$ or $d = 1$, all components of $\mathcal{F}(d;n)$ are known thanks to Jouanolou [10]. For $d = 2$ and $n \geq 3$, Cerveau and Lins Neto [2] have shown that there are just six irreducible components. For larger degree no such classification is known; for a glimpse on the subject the reader is referred to [3], [1], [4], [9], and more recently, [5].

Our goal is to determine the degrees of certain irreducible components of $\mathcal{F}(d;n)$. Fix positive integers $a \leq b$. Denote by S_a the space of homogeneous polynomials of degree a. Pick coprime $F \in S_a$ and $G \in S_b$. The foliation induced by the 1-form $\omega = aF\mathbf{d}G - bG\mathbf{d}F$ has degree $a+b-2$. Gomez-Mont and Lins Neto have shown in [9] that the closure of the set of such foliations constitutes an irreducible component

$$\mathcal{R}_n(a,b) \subset \mathcal{F}(a+b-2;n).$$

Actually, the natural equations arising from (i) (projectivity) and (ii) (Frobenius) give a generically reduced scheme structure, cf. [6].

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The component $\mathcal{R}_n(a,b)$ is the closure of the image of the rational map

$$(1)\qquad \begin{array}{rccc} \theta_{a,b}: & \mathbb{P}(S_a)\times\mathbb{P}(S_b) & \dashrightarrow & \mathbb{P}^N \\ & (F,G) & \longmapsto & aF\mathbf{d}G-bG\mathbf{d}F \end{array}$$

When a divides b the degree of $\mathcal{R}_n(a,b)$ was found in [6]; ditto for $\mathcal{R}_n(2,3)$ for $n\le 5$, using computer algebra.

Presently, for $a=2, b=2r+1$, $n\ge 2$ our main result gives a closed formula for $\deg\mathcal{R}_n(2,2r+1)$, cf. Theorem 4.3. The case $n=2$ refers to components of the space of foliations in $\mathbb{P}^2$ with center conditions, cf. [12].

We resolve the indeterminacies of the rational map (1) replacing it by a morphism,

$$\mathbb{X}'' \longrightarrow \mathbb{X}' \longrightarrow \mathbb{X} := \mathbb{P}(S_2)\times\mathbb{P}(S_{2r+1}) \dashrightarrow \mathbb{P}^N$$

obtained by a sequence of two explicit blowups. We start with $\mathbb{X}' :=$ blowup of $\mathbb{X}$ along $\mathbb{V}\times\mathbb{P}(S_{2r+1})$, where $\mathbb{V}\overset{\sim}{\to}\mathbb{P}(S_1)\hookrightarrow\mathbb{P}(S_2)$ is the Veronese. The main technical difficulty is to examine the indeterminacy scheme of the induced rational map $\mathbb{X}'\dashrightarrow\mathbb{P}^N$. Though non-reduced, it turns out to be a manageable local complete intersection and its associated reduced scheme is an explicit projective bundle over the image of the bi-Veronese $L\mapsto(L^2,L^{2r+1})$. This renders feasible the application of appropriate tools from intersection theory as in [8, Prop. 4.4, p. 83].

2. Prelims

We denote by $R := x_0\partial_{x_0}+\cdots+x_n\partial_{x_n}$ the radial vector field on $\mathbb{C}^{n+1}$. Let $\mathbf{i}_R$ be the map of contraction (interior product) of differential forms by the radial vector field. We register the following identities for $F\in S_a, G\in S_b$ (cf. [10])

$$(2)\qquad \begin{cases} \mathbf{i}_R(\mathbf{d}F\wedge\mathbf{d}G)=aF\mathbf{d}G-bG\mathbf{d}F \ \text{ in } S_{a+b-1}\otimes S_1^\vee, \\ \mathbf{d}\circ\mathbf{i}_R(\mathbf{d}F\wedge\mathbf{d}G)=(a+b)\mathbf{d}F\wedge\mathbf{d}G \ \text{ in } S_{a+b-2}\otimes\overset{2}{\wedge}S_1^\vee. \end{cases}$$

Let $V_d\subset S_d\otimes\overset{2}{\wedge}S_1^\vee$ be the subspace of closed 2–forms with coefficients homogeneous polynomials of degree d. Thus $\mathbf{d}F\wedge\mathbf{d}G\in V_d, d=a+b-2$. Put $W_d:=\mathbf{i}_R(V_d)\subset S_{d+1}\otimes S_1^\vee$. Then $\mathbf{i}_R: V_d\to W_d$ is a linear isomorphism. We still denote by $\mathbf{i}_R:\mathbb{P}(V_d)\to\mathbb{P}(W_d)$ the projectivization. We have a commutative diagram,

$$(3)\qquad \begin{array}{ccccc} & & \mathbb{P}(V_d) & \subset & \mathbb{P}(S_d\otimes\overset{2}{\wedge}S_1^\vee) \\ & \overset{\rho_{a,b}}{\nearrow} & & & \\ \mathbb{P}(S_a)\times\mathbb{P}(S_b) & & \wr\downarrow \mathbf{i}_R & & \\ & \underset{\theta_{a,b}}{\searrow} & & & \\ & & \mathbb{P}(W_d) & \subset & \mathbb{P}(S_{d+1}\otimes S_1^\vee) \end{array}$$

where $\rho_{a,b}(F,G):=\mathbf{d}F\wedge\mathbf{d}G\in\mathbb{P}(V_d)\subset\mathbb{P}(S_d\otimes\overset{2}{\wedge}S_1^\vee)$.

Since the image of $\rho_{a,b}$ lies in $\mathbb{P}(V_d)$ and $\theta_{a,b}=\mathbf{i}_R\circ\rho_{a,b}$, we see that the degrees of the image closures of $\rho_{a,b}$ and $\theta_{a,b}$ are one and the same. Similarly, it can be easily seen that the base locus schemes of $\rho_{a,b}$ and $\theta_{a,b}$ are equal. We work with the map $\rho_{a,b}$ in the sequel.

2.1. We start with a set-theoretical description of the base locus of $\rho_{a,b}$.

2.1.1. **Lemma.** *Let* $F, G \in \mathbb{P}(S_d)$. *Then* $\mathbf{d}F \wedge \mathbf{d}G = 0$ *if and only if* $F = G$ *in* $\mathbb{P}(S_d)$.

Proof. We have $\partial_{x_i}(F/G) = (G\partial_{x_i}F - F\partial_{x_i}G)/G^2$. The numerator is zero because $0 = \mathbf{i}_R(\mathbf{d}F \wedge \mathbf{d}G) = F\mathbf{d}G - G\mathbf{d}F = \sum(F\partial_{x_i}G - G\partial_{x_i}F)\mathbf{d}x_i$ up to a constant, cf.(2). □

2.1.2. **Lemma.** *Let* $a \leq b$ *be positive integers. Let* p *and* q *denote positive coprime numbers such that* $d := ap = bq$. *Let* $F \in \mathbb{P}(S_a)$, $G \in \mathbb{P}(S_b)$. *Then the* $2-$*form* $\mathbf{d}F \wedge \mathbf{d}G = 0$ *if and only if* $F^p = G^q$ *in* $\mathbb{P}(S_d)$.

Proof. If $\mathbf{d}F \wedge \mathbf{d}G = 0$ then $\mathbf{d}(F^p) \wedge \mathbf{d}(G^q) = pF^{p-1}\mathbf{d}F \wedge qG^{q-1}\mathbf{d}G = 0$. Apply the previous lemma to F^p, G^q. □

The argument below is due to A. Contiero, whom we heartily thank.

2.1.3. **Lemma.** *Notations as above pick* $F \in \mathbb{P}(S_a)$, $G \in \mathbb{P}(S_b)$ *such that* $a < b, \gcd(a,b) = 1$, $F^b = G^a$. *Then* $F = L^a, G = L^b$ *for some* $L \in \mathbb{P}(S_1)$.

Proof. We have $1 = G^a/F^b = (1/F^{b-a})(G/F)^a$ hence $(G/F)^a = F^{b-a}$ is a form, hence so is $H := G/F, \deg H = b - a$. Now $H^a = F^{b-a}$ and $\gcd(a, b-a) = 1$. By induction, we have $H = L^{b-a}, F = L^a$. The assertion follows. □

2.2. **Proposition.** (General injectivity) *Let* a, b, F, G *be as in the previous Lemma. Assume* $a < b$ *and* b *is not a multiple of* a. *Then the map*

$$\rho_{a,b} : (F, G) \mapsto \mathbf{d}F \wedge \mathbf{d}G$$

is generically injective.

Proof. Put $\omega := \mathbf{d}F \wedge \mathbf{d}G$. Assume F, G irreducible. Pick $A \in \mathbb{P}(S_a)$ and $B \in \mathbb{P}(S_b)$ such that $\omega = \mathbf{d}A \wedge \mathbf{d}B$. We proceed to show that $A = F, B = G$. We have $\operatorname{codim} \operatorname{sing}\omega \geq 2$ because the locus of tangency of F, G contains no hypersurface. From $\omega \wedge \mathbf{d}A = 0$, Saito's division Lemma [13] tells us that $\mathbf{d}A = P\mathbf{d}F + Q\mathbf{d}G$ for some homogeneous polynomials P, Q. Comparing degrees, we have $Q = 0$ and $P =$ constant. Hence $\mathbf{d}(A - PF) = 0$ and $A = F$ projectively. Similarly, $\omega \wedge \mathbf{d}B = 0$ implies $\mathbf{d}B = P\mathbf{d}F + Q\mathbf{d}G$, $Q \in \mathbb{C}$. If $P \neq 0$ then $\deg P = b - a$. We have $0 = \mathbf{d}^2 B = \mathbf{d}P \wedge \mathbf{d}F$. Thus $P^a = F^{b-a}$ in $\mathbb{P}\left(S_{a(b-a)}\right)$ by the previous Lemma. Since F is irreducible, we get $P = F^c$ whence $b - a = ac$ and a divides b, contradiction. □

3. Resolution of indeterminacies of the map $\rho_{2,2r+1}$

In order to find the degree of the image closure of a rational map $f : \mathbb{X} \dashrightarrow \mathbb{P}^N$, we may pass to the morphism $\widetilde{f} : \widetilde{\mathbb{X}} \to \mathbb{P}^N$ induced in the blowup $\widetilde{\mathbb{X}} \to \mathbb{X}$ along the base locus of f. Two pieces of information are required:

(i) Find $\deg(\widetilde{\mathbb{X}}/f(\widetilde{\mathbb{X}}))$ assuming $f : \widetilde{\mathbb{X}} \to f(\widetilde{\mathbb{X}})$ generically finite;

(ii) The Segre class $s(\mathbb{B}, \mathbb{X})$ of the base locus $\mathbb{B}$.

Presently (i) is taken care by the general injectivity (Prop. 2.2). To handle (ii), we are allowed to replace our rational map by a "partial resolution" of the indeterminacies. Thus, we take the blowup $\mathbb{X}'$ of $\mathbb{X}$ along the subvariety given by the Veronese *just on the first factor*,

$$(4)\qquad \begin{array}{ccc} \mathbb{E}' & \hookrightarrow & \mathbb{X}' := \mathbb{P}(S_2)' \times \mathbb{P}(S_{2r+1}) \\ \downarrow & & \downarrow \\ \mathbb{V} \times \mathbb{P}(S_{2r+1}) & \hookrightarrow & \mathbb{X} := \mathbb{P}(S_2) \times \mathbb{P}(S_{2r+1}) \\ (L, G) & \longmapsto & (L^2, G) \end{array}$$

where $\mathbb{P}(S_2)' \to \mathbb{P}(S_2)$ is the blowup of the Veronese, cp. [6, p. 713].

3.1. **Base locus scheme.** Following [8, §4.4, p. 83], we set

$\mathbb{B} :=$ base locus scheme of the rational map $\rho_{2,2r+1} : \mathbb{X} \dashrightarrow \mathbb{P}^N$,

$\mathbb{B}' :=$ base locus scheme of the induced rational map $\rho'_{2,2r+1} : \mathbb{X}' \dashrightarrow \mathbb{P}^N$

with $\mathbb{P}^N = \mathbb{P}(V_{2r+1})$, space of closed 2-forms of degree $2r+1$.

3.2. **Affine neighborhood.** Henceforth we set $a = 2, b = 2r+1, r \geq 1$. Let $\mathbb{X}_0$ be the affine open subset of $\mathbb{X}$ given by $(x_0^2 + \alpha, x_0^{2r+1} + \beta)$, where α (resp. β) has no x_0^2 (resp. x_0^{2r+1}) terms. It is clear that any orbit of $\mathbb{G} := \mathrm{GL}_{n+1}(\mathbb{C})$ has a representative in $\mathbb{X}_0$. Our rational map $\rho_{2,2r+1}$ (3) is equivariant under the natural $\mathbb{G}$-actions. Consequently, ditto for the induced map $\rho'_{a,b}$ (3.1). It follows that the scheme of indeterminacy $\mathbb{B}'$ is also invariant. Hence it contains a closed orbit. With this in mind, our local study of the indeterminacy loci may be restricted over the neighborhood $\mathbb{X}_0$ of the sole closed orbit $\mathbb{G} \cdot (x_0^2, x_0^{2r+1}) \subset \mathbb{X}$.

Let us write the polynomials

$$\begin{cases} F := x_0^2 + x_0 F_1 + F_2, \\ G := x_0^{2r+1} + \sum_{k=1}^{2r+1} x_0^{2r+1-i} G_i, \end{cases}$$

where F_i, G_j stand for homogeneous polynomials of degrees i, j, not involving the variable x_0:

$$(5)\qquad \begin{cases} F_1 &= a_{01}x_1 + \cdots + a_{0n}x_n, \; F_2 = \sum_{j \geq i = 1}^{n} a_{ij}x_i x_j, \\ G_1 &= b_1 x_1 + \cdots + b_n x_n, \\ G_2 &= b_{1+n}x_1^2 + b_{n+2}x_1x_2 + \cdots + b_{\nu_2}x_n^2, \\ \cdots & \\ G_{2r+1} &= b_{1+\nu_{2r}}x_1^{2r+1} + \cdots + b_{\nu_{2r+1}-1}x_{n-1}x_n^{2r} + b_{\nu_{2r+1}}x_n^{2r+1} \end{cases}$$

with $\nu_i = \binom{n+i}{i} - 1 = \dim \mathbb{P}(S_i)$. Let

$$(6)\qquad \mathbf{a} = (a_{01}, \ldots, a_{nn}), \; \mathbf{b} = (b_1, \ldots, b_{\nu_{2r+1}})$$

denote the blocks of variables appearing as coefficients in F_i, G_j. The coordinate ring of our affine neighborhood $\mathbb{X}_0$ is the polynomial ring $\mathbb{C}[\mathbf{a}, \mathbf{b}]$ in $\nu_2 + \nu_{2r+1}$ variables. The equations defining the embedding (4) in the neighborhhod $\mathbb{X}_0$ arise by equating coefficients in $F_2 = \frac{1}{4}F_1^2$, to wit,

$$(7)\qquad a_{ii} = \tfrac{1}{4}a_{0i}^2 \text{ and } a_{ij} = \tfrac{1}{2}a_{0i}a_{0j} \text{ for } j > i = 1, \ldots, n.$$

3.3. **Ideal of the base locus.** An element of $\mathbb{P}\big(S_{2r+1}\otimes \overset{2}{\wedge}\, S_1^\vee\big)$ lying in the image of the map $\rho_{2,2r+1}$ can be written as

$$\mathbf{d}F\wedge\mathbf{d}G=\sum_{m>l=0}^{n} A_{lm}\mathbf{d}x_l\wedge\mathbf{d}x_m,\quad \text{with}$$
$$A_{lm}=(\partial_l F)(\partial_m G)-(\partial_m F)(\partial_l G),\ \text{ for } m>l=0,\dots,n$$

where ∂_k denotes partial derivative with respect to x_k. Write

(8) $$A_{lm}=\sum_i A_{lm,i}x^i.$$

The coefficients $A_{lm,i}\in\mathbb{C}[\mathbf{a},\mathbf{b}]$ are generators of

(9) $$J:=\ \text{the ideal of the base locus subscheme } \mathbb{B}\cap\mathbb{X}_0\subset\mathbb{X}_0.$$

We may expand

(10) $$\left[\begin{aligned}\mathbf{d}F\wedge\mathbf{d}G\ &=\mathbf{d}\big(x_0^2+\sum_1^2 x_0^{2-i}F_i\big)\wedge\mathbf{d}\big(x_0^{2r+1}+\sum_1^{2r+1}x_0^{2r+1-i}G_i\big)\\ &=\sum_{k=0}^{2r}x_0^{2r+1-k}\mathbf{d}x_0\wedge\omega_k^{F,G}\\ &\quad+\sum_{k=0}^{2r}x_0^{2r+1-k}(\mathbf{d}F_1\wedge\mathbf{d}G_{k+1}+\mathbf{d}F_2\wedge\mathbf{d}G_k)\\ &\quad+\mathbf{d}x_0\wedge\big(F_1\mathbf{d}G_{2r+1}-G_{2r}\mathbf{d}F_2\big)+\mathbf{d}F_2\wedge\mathbf{d}G_{2r+1},\end{aligned}\right.$$

where we set $G_0=1$, $G_{-1}=0$ and write the 1-form

(11) $$\omega_k^{F,G}:=2\mathbf{d}G_{k+1}+F_1\mathbf{d}G_k-(2r+2-k)G_{k-1}\mathbf{d}F_2-(2r+1-k)G_k\mathbf{d}F_1.$$

The coefficients appearing in the 1-forms $\omega_k^{F,G}, 0\le k\le 2r$, generate a subideal

(12) $$J_1\subset J$$

which plays a special role.

3.3.1. **Lemma.** *Let F_i, G_j be as in (5). Then $\omega_k^{F,G}=0$ if and only if*

(13) $$G_{k+1}=\sum_{\alpha=0}^{\beta}\gamma_{r,\alpha,k}F_1^{k+1-2\alpha}F_2^{\alpha},$$

where $\beta=\frac{k}{2}$ if k is even and $\beta=\frac{k+1}{2}$ if k is odd and

$$\gamma_{r,\alpha,k}:=\frac{\prod_{i=0}^{k-\alpha}(2r+1-2i)}{2^{k+1-\alpha}(k+1-2\alpha)!\alpha!}.$$

Hence, the ideal J_1 is prime, generated by a regular sequence of length $\dim\mathbb{P}(S_{2r+1})$ in the coordinate ring $\mathbb{C}[\mathbf{a},\mathbf{b}]$.

Proof. Since x_0 doesn't occur in F_i, G_j, the condition $\mathbf{d}F\wedge\mathbf{d}G=0$ defining the indeterminacy locus implies the vanishing of the 1-forms (11) for $k\ge 0$. When $k=0$ and $k=1$, we have

(14) $$\left\{\begin{aligned}\omega_0^{F,G}&:=2\mathbf{d}G_1-(2r+1)\mathbf{d}F_1\quad\text{and}\\ \omega_1^{F,G}&:=2\mathbf{d}G_2+F_1\mathbf{d}G_1-(2r+1)\mathbf{d}F_2-2rG_1\mathbf{d}F_1.\end{aligned}\right.$$

Since F_i, G_j are homogeneous, the equations $\omega_0 = 0, \omega_1 = 0$ are equivalent to

$$G_1 = \tfrac{2r+1}{2} F_1 \text{ and } G_2 = \tfrac{(2r+1)(2r-1)}{8} F_1^2 + \tfrac{2r+1}{2} F_2$$

as claimed. Let us assume $k = 2\beta$. The odd case is similar. Equating (11) to zero and using induction on k, we may write for $k \geq 2$,

$$\begin{aligned} 2\mathbf{d}G_{k+1} + F_1 \mathbf{d}\big(\sum_{\alpha=0}^{\beta} \gamma_{r,\alpha,k-1} F_1^{k-2\alpha} F_2^{a}\big) \\ - (2r+2-k)\cdot\big(\sum_{\alpha=0}^{\beta-1} \gamma_{r,\alpha,k-2} F_1^{k-1-2\alpha} F_2^{\alpha}\big)\mathbf{d}F_2 \\ - (2r+1-k)\big(\sum_{\alpha=0}^{\beta} \gamma_{r,\alpha,k-1} F_1^{k-2\alpha} F_2^{\alpha}\big)\mathbf{d}F_1 = 0. \end{aligned}$$

Using Leibniz, we arrive at

$$\begin{aligned} 2\mathbf{d}G_{k+1} - \big[\sum_{\alpha=0}^{\beta} \big(2r+1-2(k-\alpha)\big)\gamma_{r,\alpha,k-1} F_1^{k-2\alpha} F_2^{\alpha}\big]\mathbf{d}F_1 \\ -\big[\sum_{\alpha=0}^{\beta-1} \big(2r+1-2(k-1-\alpha)\big)\gamma_{r,\alpha,k-2} F_1^{k-1-2\alpha} F_2^{\alpha}\big]\mathbf{d}F_2 = 0. \end{aligned} \tag{15}$$

This last equality is equivalent to applying $\mathbf{d}$ to (13). □

We register for later use the following.

3.3.2. **Remark.** Let A be a domain and let $B = A[\mathbf{y}, \mathbf{z}], B' = A[\mathbf{y}, \mathbf{w}]$ be polynomial rings in the blocks of variables

$$\mathbf{y} = y_1, \ldots, y_m,\ \mathbf{z} = z_1, \ldots z_n,\ \mathbf{w} = w_1, \ldots w_k.$$

Pick $p_0(\mathbf{z}), p_1(\mathbf{z}), \ldots, p_k(\mathbf{z})$ in $A[\mathbf{z}], k \leq m$. Let $\varphi : B \to B'$ be a homomorphism of $A[\mathbf{y}]$-algebras. Assume $\langle p_0 \rangle$, $\langle \varphi(p_0) \rangle$ are nonzero prime ideals. Then

$$y_1 - p_1, \ldots, y_k - p_k, p_0$$

is a regular sequence which generates a prime ideal in B; ditto for the sequence $y_1 - \varphi(p_1), \ldots, y_k - \varphi(p_k), \varphi(p_0)$ in B'.

3.3.3. We will use the previous Remark in the following context.

- $B :=$ coordinate ring of the affine open neighborhood $\mathbb{X}_0$ cf. (5), with

$$\begin{cases} \mathbf{z} := \mathbf{a}, \text{ block of } \nu_2 \text{ coefficients of the quadric } F \text{ and} \\ \mathbf{y} := \mathbf{b}, \text{ block of } \nu_{2r+1} \text{ coefficients of } G \text{ as in (6);} \end{cases}$$

- $B' :=$ coordinate ring of an affine open neighborhood $\mathbb{X}'_0 \subset \mathbb{X}'$ specified by a choice of equation of the exceptional divisor, say

$$\varepsilon := a_{11} - \tfrac{1}{4} a_{01}^2,$$

cf. (7). The new block of variables is

$$\mathbf{w} := a_{11}, a_{01}, a_{02}, \ldots, a_{0n}, d_{22}, \ldots, d_{nn}, d_{ij},\ j > i = 1, \ldots, n.$$

The homomorphism φ corresponding to the blowup $\mathbb{X}'_0 \to \mathbb{X}_0$ is given by

$$\begin{cases} \varphi(a_{jj}) = \varepsilon d_{jj} + \tfrac{1}{4} a_{0j}^2 \text{ for } j \neq 1 \text{ and} \\ \varphi(a_{ij}) = \varepsilon d_{ij} + \tfrac{1}{2} a_{0i} a_{0j} \text{ for } j > i = 1, \cdots, n. \end{cases} \tag{16}$$

3.4. **Proposition.** *Let $J_1' \subset J'$ denote the total transforms of the ideals $J_1 \subset J$ to the affine neighborhood $\mathbb{X}_0' \subset \mathbb{X}'$ over $\mathbb{X}_0$ as above. Let ε be a local equation of the exceptional divisor $\mathbb{E}' \subset \mathbb{X}'$. Then we have the following.*

(i) *J_1' is prime, generated by a regular sequence of length ν_{2r+1}.*

(ii) *$J' = J_1' + \langle \varepsilon^{r+1} \rangle$ is the ideal of the base locus $\mathbb{B}' \cap \mathbb{X}_0'$ and is generated by a regular sequence of length $1 + \nu_{2r+1}$.*

(iii) *The radical* $\mathrm{rad} J' = J_1' + \langle \varepsilon \rangle$ *is prime.*

Proof. Let $\overline{G}, \overline{G}_j \in \mathbb{C}[\mathbf{a}, \mathbf{x}]$ be the polynomials defined by the substitutions (13). Explicitly, define for $0 \le k \le 2r$

$$\overline{G}_{k+1} := \sum_{\alpha=0}^{\beta} \gamma_{r,\alpha,k} F_1^{k+1-2\alpha} F_2^{\alpha}. \tag{17}$$

Each $\overline{G}_i$ is a polynomial of degree i in the homogeneous coordinates $\mathbf{x} = x_0, \dots, x_n$ with coefficients polynomials in the a_{ij}. Substituting $\overline{G}_i$ in place of G_i in (10) kills $\omega_k^{F,G}$. It can be seen from (15) that also $\mathbf{d}F_1 \wedge \mathbf{d}\overline{G}_{k+1} + \mathbf{d}F_2 \wedge \mathbf{d}\overline{G}_k = 0$. Thus, we are left just with the bottom row in (10):

$$\mathbf{d}F \wedge \mathbf{d}\overline{G} = \mathbf{d}x_0 \wedge (F_1 \mathbf{d}\overline{G}_{2r+1} - \overline{G}_{2r} \mathbf{d}F_2) + \mathbf{d}F_2 \wedge \mathbf{d}\overline{G}_{2r+1}.$$

Plugging (17) into the right hand side above we find

$$\begin{aligned} \mathbf{d}F \wedge \mathbf{d}\overline{G} = F_1 \mathbf{d}x_0 \wedge \big[\big(\sum_{\alpha=0}^{r} (2r+1-2\alpha)\gamma_{r,\alpha,2r} F_1^{2r-2\alpha} F_2^{\alpha}\big)\mathbf{d}F_1 + \\ \big(\sum_{\alpha=0}^{r-1} \gamma_{r,\alpha+1,2r} F_1^{2r-1-2\alpha} F_2^{\alpha}\big)\mathbf{d}F_2 - \big(\sum_{\alpha=0}^{r} \gamma_{r,\alpha,2r-1} F_1^{2r-2\alpha} F_2^{\alpha}\big)\mathbf{d}F_2\big] \\ + \big(\sum_{\alpha=0}^{r} (2r+1-2\alpha)\gamma_{r,\alpha,2r} F_1^{2r-2\alpha} F_2^{\alpha}\big)\mathbf{d}F_2 \wedge \mathbf{d}F_1. \end{aligned}$$

This simplifies to

$$\mathbf{d}F \wedge \mathbf{d}\overline{G} = 2\gamma_{r,r,2r} \left(F_2 - (\tfrac{F_1}{2})^2\right)^r \mathbf{d}(F_2 - (\tfrac{F_1}{2})^2) \wedge \mathbf{d}(x_0 + \tfrac{F_1}{2}). \tag{18}$$

Write $\mathbf{d}F \wedge \mathbf{d}\overline{G} = \sum \overline{A}_{lm} \mathbf{d}x_l \wedge \mathbf{d}x_m$. Let $\overline{J}$ denote the ideal spanned by the coefficients of the polynomials $\overline{A}_{lm}$. We clearly have $J = J_1 + \overline{J}$. Let $\overline{J}'$ be the total transform of $\overline{J}$, obtained by means of the relations (16). Then we have $J' = J_1' + \overline{J}'$. The assertion (i) now follows from Remark 3.3.2. We take $p_0 = \varepsilon$; the other p_k arise from the expression of each coefficient of $\overline{G}$ in terms of the a_{ij} collected from (13). Presently $\varphi(p_0) = p_0$ is irreducible in the polynomial ring $S' = \mathbb{C}[\mathbf{y}, \mathbf{w}]$, with $\mathbf{w}, \mathbf{y}$ as in 3.3.3. The regular sequences

$$b_1 - \varphi(p_1), \dots, b_{\nu_{2r+1}} - \varphi(p_{\nu_{2r+1}})$$

and

$$b_1 - \varphi(p_1), \dots, b_{\nu_{2r+1}} - \varphi(p_{\nu_{2r+1}}), p_0$$

generate the prime ideals J_1' and $J_1' + \langle p_0 \rangle \subset S'$. Therefore, replacing p_0 by a power p_0^e, we still get a regular sequence. Clearly the radical of the ideal $J_1' + \langle p_0^e \rangle$ is equal to $J_1' + \langle p_0 \rangle$, whence assertion (iii) follows. Since the coordinate ring of $\mathbb{X}_0'$ is a UFD, the ideal of the base locus of the induced rational map $\mathbb{X}'_0 \dashrightarrow \mathbb{P}^N$ is J' because the given generators $\varphi(A_{lm,i})$ (cf. (8)) have no common factor (a subset forms a regular sequence). To finish the proof of assertion (ii) it suffices to show $\overline{J}' = \langle \varepsilon^{r+1} \rangle$. Recalling (5) we may write

$$(19) \quad \varphi(F_2) - (\tfrac{F_1}{2})^2 =$$

$$\varepsilon x_1^2 + \sum_{i=2}^{n} \underbrace{\varphi(a_{ii} - \tfrac{1}{4}a_{0i}^2)}_{\varepsilon d_{ii}} x_i^2 + \sum_{j>i=1}^{n} \underbrace{\varphi(a_{ij} - \tfrac{1}{2}a_{0i}a_{0j})}_{\varepsilon d_{ij}} x_i x_j.$$

The ideal generated by the coefficients of this quadric is equal to $\langle \varepsilon \rangle$. Hence in (18) the ideal $\overline{J}'$ of the total transforms of the coefficients of the 2-form $\mathbf{d}F \wedge \mathbf{d}\overline{G}$ is equal to $\langle \varepsilon^{r+1} \rangle$. This proves (ii). □

We have the following global counterpart.

3.5. **Proposition.** (i) *The subscheme of indeterminacy* $\mathbb{B}' \subset \mathbb{X}'$ *is a local complete intersection contained in* $(r+1)\mathbb{E}'$.
(ii) *The reduced induced subscheme* $\mathbb{B}'_{\mathrm{red}} \subset \mathbb{E}'$ *is irreducible.*
(iii) *We have the formula* $[\mathbb{B}'] = (r+1)[\mathbb{B}'_{\mathrm{red}}]$ *for the fundamental class in the Chow group of* $\mathbb{B}'$.

Proof. The scheme inclusion $\mathbb{B}' \subset (r+1)\mathbb{E}'$ is equivalent to the vanishing of the quotient sheaf $\mathscr{J} := \big(\mathcal{I}(\mathbb{B}') + \mathcal{I}((r+1)\mathbb{E}')\big)/\mathcal{I}(\mathbb{B}')$. If $\mathscr{J}$ were nonzero, its support would be a closed invariant subscheme of $\mathbb{X}'$, whence should contain the unique closed orbit (cf. 3.5.1). This contradicts Prop. 3.4 (ii): the ideal of $\mathbb{B}' \cap \mathbb{X}'_0$ is generated by a regular sequence which includes ε^{r+1}. Thus $\mathbb{B}'$ is l.c.i in a neighborhood of the closed orbit in $\mathbb{X}'$. Recall the condition of l.c.i. is equivalent to the exactness of a Koszul complex, which in turn is detected by the vanishing of cohomology cf. [8, A.5, p. 415]. Thus the l.c.i. locus is open (see also [17]). Since $\mathbb{B}'$ is invariant under the action of $\mathbb{G}$, it follows that $\mathbb{B}'$ is l.c.i. everywhere. For the last assertion, let $\mathbb{W} \subseteq \mathbb{B}'_{\mathrm{red}}$ be an irreducible component. Since $\mathbb{B}'_{\mathrm{red}}$ is $\mathbb{G}$-invariant, so is $\mathbb{W}$ because $\mathbb{G}$ is connected. Hence $\mathbb{W}$ must contain the sole closed orbit of $\mathbb{X}'$, a representative of which appears in $\mathbb{X}'_0$. This implies $\mathbb{W} = \mathbb{B}'_{\mathrm{red}}$. Finally, the coefficient of the cycle $[\mathbb{B}'_{\mathrm{red}}]$ can be read as the length of the local ring at the generic point over any neighborhood, *e.g.*, $\mathbb{B}' \cap \mathbb{X}'_0$ whence the assertion follows from Prop. 3.4 (ii). □

3.5.1. **Remark.** Let us recall a couple of facts about complete quadrics, cf. [14]. The Veronese embedding $\mathbb{P}(S_1) = \mathbb{V} \subset \mathbb{Y} := \mathbb{P}(S_2)$ has normal bundle

$$(20) \qquad \mathscr{N}_{\mathbb{V}|\mathbb{Y}} = \mathrm{Sym}^2 \mathscr{Q} \otimes \mathscr{O}_V(2)$$

where $\mathscr{Q}$ fits in the tautological sequence over $\mathbb{P}(S_1)$,

$$\mathscr{O}_{\mathbb{V}}(-1) \rightarrowtail S_1 \twoheadrightarrow \mathscr{Q}.$$

Let $\mathbb{Y}' \to \mathbb{Y}$ be the blowup along $\mathbb{V}$. Each point in the fiber of the exceptional divisor $\mathbb{P}\big(\mathrm{Sym}^2 \mathscr{Q} \otimes \mathscr{O}_{\mathbb{V}}(2)\big)$ lying over $L \in \mathbb{V}$ can be identified to a choice of a quadric hypersurface $\overline{q}$ in the hyperplane $L \subset \mathbb{P}^n$. There is a unique closed orbit for the induced action of $\mathbb{G}$ on $\mathbb{Y}'$, represented by the choice of $\overline{q}$ as a rank-1 quadric in L.

The Chern classes of $\mathrm{Sym}^2\mathcal{Q}$ can be gotten from the exact sequence

$$\mathcal{O}_{\mathbb{V}}(-1)\otimes S_1 \rightarrowtail S_2 \twoheadrightarrow \mathrm{Sym}^2\mathcal{Q}. \tag{21}$$

Recalling (4), let $\mathbb{E}'_{\mathbb{V}}$ be the restriction of the exceptional divisor to the variety $\mathbb{V}$ now embedded in $\mathbb{V}\times\mathbb{P}(S_{2r+1})$ as the graph of the Veronese of forms of degree $2r+1$.

3.5.2. **Corollary.** *We have* $\mathbb{B}'_{\mathrm{red}} = \mathbb{E}'_{\mathbb{V}} = \mathbb{P}\left(\mathcal{N}_{\mathbb{V}|\mathbb{P}(S_2)}\right)$.

Proof. We know already (cf. 3.5) that $\mathbb{B}'_{\mathrm{red}}$ is irreducible and contained in $\mathbb{E}'$, so it suffices to show that it lies over $\mathbb{V}$, so that $\mathbb{B}'_{\mathrm{red}}\subseteq\mathbb{E}'_{\mathbb{V}}$. We may restrict over $\mathbb{X}'_0$. Substituting (16) into the relations (13) defining J_1, we get for $0\le k\le 2r, \beta$ as in Lemma 3.3.1,

$$\begin{aligned} G_{k+1} &= \sum_{\alpha=0}^{\beta}\gamma_{r,\alpha,k}F_1^{k+1-2\alpha}\varphi(F_2)^{\alpha} \quad \text{(recalling (19))}\\ &= \sum_{\alpha=0}^{\beta}\gamma_{r,\alpha,k}F_1^{k+1-2\alpha}((\tfrac{F_1}{2})^{2\alpha}+\varepsilon F'(\cdots))\\ &= (\tfrac{F_1}{2})^{k+1}\left(\sum_{\alpha=0}^{\beta}2^{k+1-2\alpha}\gamma_{r,\alpha,k}\right)+\varepsilon F'(\cdots)\\ &= \tbinom{2r+1}{k+1}(\tfrac{F_1}{2})^{k+1}+\varepsilon F'(\cdots) \end{aligned}$$

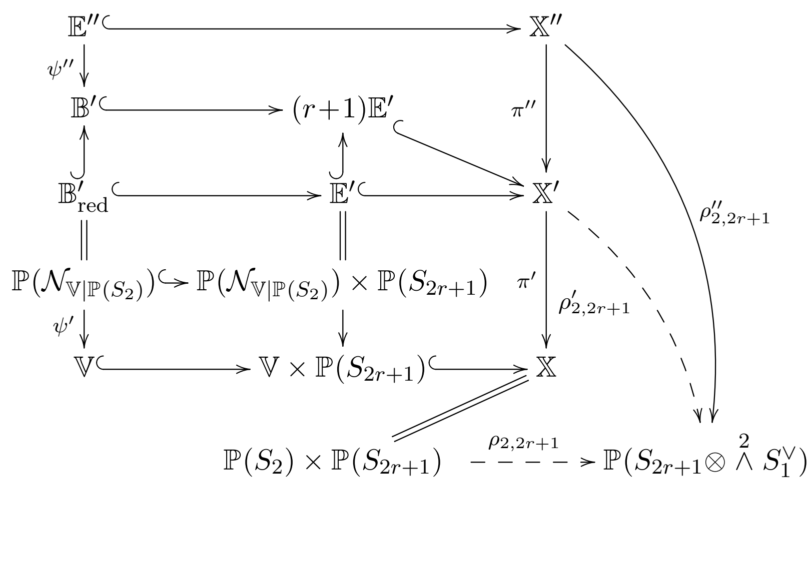
where $F' = x_1^2+\sum d_{ij}x_ix_j$ is the quadric obtained from (19) upon dividing by ε and $(\cdots)$ stands for a polynomial. Thus the relations spanning $\mathrm{rad}J' = J'_1+\langle\varepsilon\rangle$ are given by

$$\left\{G_{k+1} = \tbinom{2r+1}{k+1}(\tfrac{F_1}{2})^{k+1}\right\}_{0\le k\le 2r} \quad \text{and} \quad \varepsilon. \tag{22}$$

Substituting in $G = x_0^{2r+1}+\sum_1^{2r+1}x_0^{2r+1-i}G_i$ we find, lo and behold, $(x_0+\frac{F_1}{2})^{2r+1}$, which represents the embedding $\mathbb{V}\hookrightarrow\mathbb{P}(S_{2r+1})$, $L\mapsto L^{2r+1}$. This proves the inclusion $\mathbb{B}'_{\mathrm{red}}\subseteq\mathbb{E}'_{\mathbb{V}}$. In fact equality holds because they have the same dimension ν_2-1. □

The information gathered thus far yields the following diagram

(23)



where the second exceptional divisor $\mathbb{E}'' = \mathbb{P}(\mathcal{N}_{\mathbb{B}'|X'})$ is a projective bundle. We may now proceed to our goal.

4. THE DEGREE OF $\mathcal{R}_n(2, 2r+1)$.

4.1. **Proposition.** *Notation as in diagram* (23)*, set* $\nu = \dim \mathbb{X}$ *and*
$$h_1 := c_1((\pi'\pi'')^\star \mathcal{O}_{\mathbb{P}(S_2)}(1)),\ h_2 := c_1((\pi'\pi'')^\star \mathcal{O}_{\mathbb{P}(S_{2r+1})}(1)).$$
Then the degree of the component $\mathcal{R}_n(2, 2r+1)$ *is given by the integral*

$$\int_{\mathbb{X}''} (h_1 + h_2 - [\mathbb{E}''])^\nu.$$

Proof. Since the map $\rho''_{2,2r+1} : \mathbb{X}'' \to \mathbb{P}^N$ is generically injective, the assertion follows from Proposition 3.5(i) together with [8, Prop. 4.4, p. 83]. Indeed, denoting by H the hyperplane class of $\mathbb{P}^N$, we may write

$$(\rho''_{2,2r+1})^\star H = m_1 h_1 + m_2 h_2 + m_3[\mathbb{E}'] + m_4[\mathbb{E}'']$$

for suitable integers $m_1, \ldots, m_4$. These coefficients are determined by excision (cf. [8, Prop. 1.8, p. 21]). Over the open subset

$$\mathbb{U} = \mathbb{X} - (\mathbb{V} \times \mathbb{P}(S_{2r+1}))$$

only the classes h_1 and h_2 survive. Then, $(\rho''_{2,2r+1})^\star_{|\mathbb{U}} H = (\rho_{2,2r+1})^\star_{|\mathbb{U}} H = m_1 h_1 + m_2 h_2$. Since the map $\rho_{2,2r+1}$ is bihomogeneous of bidegree (1,1), we get $m_1 = m_2 = 1$. Now Prop. 3.4 tells us that we have the surjections

$$\begin{array}{c} (S_{2r+1} \otimes \bigwedge^2 S_1^\vee)^\vee \otimes \mathcal{O}_{\mathbb{P}(S_2)}(-1) \otimes \mathcal{O}_{\mathbb{P}(S_{2r+1})}(-1) \otimes \mathcal{O}_{\mathbb{X}'} \\ \downarrow \\ \mathcal{I}(\mathbb{B}) \otimes \mathcal{O}_{\mathbb{X}'} \twoheadrightarrow \mathcal{I}(\mathbb{B}') \subset \mathcal{O}_{\mathbb{X}'}. \end{array}$$

Since $\operatorname{codim}(\mathbb{B}', \mathbb{X}') > 1$, it follows that $m_3 = 0$. Next, blowing up the subscheme $\mathbb{B}'$, we find similar surjections over $\mathbb{X}''$ enabling us to get

$$\begin{array}{rrcl} & (\rho''_{2,2r+1})^\star \mathcal{O}_{\mathbb{P}^N}(1) & = & \mathcal{O}_{\mathbb{P}(S_2)}(1) \otimes \mathcal{O}_{\mathbb{P}(S_{2r+1})}(1) \otimes \mathcal{O}_{\mathbb{X}''}(-\mathbb{E}'') \\ \text{whence} & (\rho''_{2,2r+1})^\star H & = & h_1 + h_2 - [\mathbb{E}'']. \end{array}$$

□

The integral in Prop. 4.1 splits into two summands,

$$(24) \qquad \int_{\mathbb{X}''} (h_1 + h_2)^\nu \ + \int_{\mathbb{X}''} \sum_{k=1}^{\nu} \binom{\nu}{k} (-[\mathbb{E}''])^k (h_1 + h_2)^{\nu - k},$$

where the first integral is over cycles off $\mathbb{E}''$ whereas the second one lives in $\mathbb{E}''$, hence over $\mathbb{V}$, cf. diagram (23). We set for short $\nu_i = \dim \mathbb{P}(S_i) = \binom{n+i}{n} - 1$, so $\nu = \dim \mathbb{X} = \nu_2 + \nu_{2r+1}$. The first integral evaluates to the degree of the Segre variety,

$$(25) \qquad \int_{\mathbb{X}''} (h_1 + h_2)^\nu = \int_{\mathbb{X}''} \binom{\nu}{\nu_2} h_1^{\nu_2} h_2^{\nu_{2r+1}} = \binom{\nu}{\nu_2}.$$

In order to compute the second integral in (24), we use projection formula ([8, 3.2(c), p. 50]) for the inclusion $\mathbb{E}'' \overset{j}{\hookrightarrow} \mathbb{X}''$. We have

$$[\mathbb{E}'']^k = c_1\mathcal{O}_{\mathbb{X}''}(\mathbb{E}'')^{k-1} \cap j_\star[\mathbb{E}''] = (-1)^{k-1} j_\star(c_1\mathcal{O}_{\mathbb{E}''}(1)^{k-1} \cap [\mathbb{E}'']).$$

Recalling [8, Def. 1.4, p. 13], we may write

$$\int_{\mathbb{X}''} (-[\mathbb{E}''])^k (h_1 + h_2)^{\nu-k} = -\int_{\mathbb{E}''} c_1(\mathcal{O}_{\mathbb{E}''}(1))^{k-1}(h_1 + h_2)^{\nu-k}. \tag{26}$$

Since the restriction $\pi''_{\mathbb{E}''} : \mathbb{E}'' = \mathbb{P}\left(\mathcal{N}_{\mathbb{B}'/\mathbb{X}'}\right) \to \mathbb{B}'$ is a $\mathbb{P}^{\nu_{2r+1}}$-bundle over the l.c.i. base $\mathbb{B}'$, the right hand side in (26) evaluates to

$$-\int_{\mathbb{B}'} s_{(k-1)-\nu_{2r+1}}(\mathcal{N}_{\mathbb{B}'|\mathbb{X}'})(h_1 + h_2)^{\nu-k}. \tag{27}$$

(cf. [8, § 3.1, p. 47]). By Prop. 3.5 (iii), we have $[\mathbb{B}'] = (2r+1)[\mathbb{B}'_{\text{red}}]$. Since the Chow group of a scheme is equal to that of its associated reduced subscheme, we need the Segre classes

$$\mathrm{s}_k^{\mathbb{B}'} := \mathrm{s}_k(\mathcal{N}_{\mathbb{B}'|\mathbb{X}'})_{|\mathbb{B}'_{\text{red}}}$$

for $k \le \dim(\mathbb{B}') = \nu_2 - 1$. The classes h_1 and h_2 in the integral (27) are both restricted to $\mathbb{V}$, so $h_1 = 2h,\ h_2 = (2r+1)h$, where $h = c_1(\mathcal{O}_{\mathbb{V}}(1))$ stands for the hyperplane class on $\mathbb{V} = \mathbb{P}(S_1)$. Hence the second integral in (24) reads

$$\delta := -(r+1) \int_{\mathbb{B}'_{\text{red}}} \sum_{k=m}^{\nu_2-1} \binom{\nu}{k+1+\nu_{2r+1}} \mathrm{s}^{\mathbb{B}'}_{(k-1)-\nu_{2r+1}} \cdot ((2r+3)h)^{\nu-k}, \tag{28}$$

where we set $m = \nu_2 - n - 1$. The Segre classes $\mathrm{s}_k^{\mathbb{B}'}$ are obtained via the following.

4.2. **Proposition.** *Let Z, D be closed, smooth subvarieties of a smooth variety Y. Suppose $Z \subset D \subset Y$ and D is a divisor of Y. Assume ${}^eZ \subset eD$ is a local complete intersection thickening of Z. Then*
(i) *the conormal modules satisfy*

$$(\check{\mathcal{N}}_{{}^eZ|eD})_{|Z} = \check{\mathcal{N}}_{Z|D};$$

(ii) *there is an exact sequence,*

$$\mathcal{O}_Y(-eD)_{|{}^eZ} = (\check{\mathcal{N}}_{eD|Y})_{|{}^eZ} \rightarrowtail \check{\mathcal{N}}_{{}^eZ|Y} \twoheadrightarrow (\check{\mathcal{N}}_{{}^eZ|eD}).$$

Proof. Denote by A the coordinate ring of an affine neighborhood of the variety Y. Let $a \in A$ be a local equation of D. Put $A' = A/\langle a\rangle$ (resp. ${}^eA' := A/\langle a^e\rangle$) the coordinate ring of D (resp. the thickening eD). Let ${}^eI \subset A$ be the ideal of ${}^eZ$. Let ${}^eI' \subset {}^eA'$ be the ideal of ${}^eZ$ in eD. Let I' denote the image of ${}^eI'$ in the coordinate ring A' of D. The quotients

$${}^e\overline{A} := {}^eA'/{}^eI' = A/{}^eI, \quad \overline{A} := A'/I'$$

are coordinate rings of the schemes ${}^eZ$ and of the reduced scheme Z, respectively. To prove (i) we produce a natural isomorphism

$$\left({}^eI'/({}^eI')^2\right) \otimes_{{}^e\overline{A}} \overline{A} = I'/(I')^2.$$

The module on the left represents the restriction $(\check{\mathcal{N}}_{{}^eZ|eD})_{|Z}$ of the conormal. The surjection of ${}^eA'$-modules ${}^eI' \twoheadrightarrow I'$ induces the surjection ${}^eI'/({}^eI')^2 \twoheadrightarrow I'/(I')^2$ and hence $\left({}^eI'/({}^eI')^2\right) \otimes_{{}^e\overline{A}} \overline{A} \twoheadrightarrow I'/(I')^2$. Since these are locally free

$\overline{A}$-modules (cf. [11, Cor. 5.11, p. 153]) of equal rank, assertion (i) is proven. Similarly, the natural exact sequence

$$\big(\langle a^e\rangle/\langle a^e\rangle^2\big)\otimes_{{}^eA'}{}^e\overline{A}\rightarrowtail {}^eI/({}^eI)^2 \twoheadrightarrow {}^eI'/({}^eI')^2$$

globalizes to yield assertion (ii), cf. [8, §B.7.4]. □

4.2.1. **Corollary.** *We have the identifications*

$$(\mathcal{N}_{(r+1)\mathbb{E}'|\mathbb{X}'})_{|\mathbb{B}'_{\rm red}} = \mathcal{O}_{\mathbb{E}'}(-(r+1))_{|\mathbb{B}'_{\rm red}} = \mathcal{O}_{\mathbb{B}'_{\rm red}}(-(r+1))$$

$$(\mathcal{N}_{\mathbb{B}'|(r+1)\mathbb{E}'})_{|\mathbb{B}'_{\rm red}} = \mathcal{N}_{\mathbb{B}'_{\rm red}|\mathbb{E}'},$$

and the exact sequence

$$\mathcal{N}_{\mathbb{B}'_{\rm red}|\mathbb{E}'} \rightarrowtail (\mathcal{N}_{\mathbb{B}'|\mathbb{X}'})_{|\mathbb{B}'_{\rm red}} \twoheadrightarrow \mathcal{O}_{\mathbb{B}'_{\rm red}}(-(r+1)).$$

□

4.2.2. **Corollary.** *We have the formulas for the Segre classes*

$$\mathrm{s}(\mathcal{N}_{\mathbb{B}'_{\rm red}|\mathbb{E}'}) = \mathrm{s}(T\mathbb{P}(S_{2r+1})_{|\mathbb{B}'_{\rm red}}, \quad \textit{and}$$

$$\mathrm{s}(\mathcal{N}_{\mathbb{B}'|\mathbb{X}'})_{|\mathbb{B}'_{\rm red}} = \mathrm{s}(T\mathbb{P}(S_{2r+1}))\mathrm{s}(\mathcal{O}_{\mathbb{B}'_{\rm red}}(-(r+1))).$$

Proof. Recalling (4) and Cor. 3.5.2 we have

$$\mathbb{E}' = \mathbb{P}(\mathcal{N}_{\mathbb{V}|\mathbb{P}(S_2)}) \times \mathbb{P}(S_{2r+1})$$

and $\mathbb{B}'_{\rm red} = \mathbb{P}(\mathcal{N}_{\mathbb{V}|\mathbb{P}(S_2)})$, hence the exact sequence

$$T\mathbb{B}'_{\rm red} \rightarrowtail (T\mathbb{E}')_{|\mathbb{B}'_{\rm red}} = T\mathbb{B}'_{\rm red}\bigoplus T\mathbb{P}(S_{2r+1})_{|\mathbb{B}'_{\rm red}} \twoheadrightarrow \mathcal{N}_{\mathbb{B}'_{\rm red}|\mathbb{E}'}$$

yields the first formula. The second one comes from Cor. 4.2.1. □

Using Euler's sequence

$$\mathcal{O}_{\mathbb{P}(S_{2r+1})} \rightarrowtail \mathcal{O}_{\mathbb{P}(S_{2r+1})}(1)^{1+\nu_{2r+1}} \twoheadrightarrow T\mathbb{P}(S_{2r+1})$$

we may write the total Segre class

(29) $$\mathrm{s}(T\mathbb{P}(S_{2r+1})) = (1+h_2)^{-(1+\nu_{2r+1})} = \sum \tbinom{i+\nu_{2r+1}}{i}(-h_2)^i,$$

where $h_2 := c_1\mathcal{O}_{\mathbb{P}(S_{2r+1})}(1)$.

We may proceed to the explicit calculation of (28). Write

$$h' := c_1\big(\mathcal{O}_{\mathbb{B}'_{\rm red}}(1)\big)$$

for the hyperplane class of the projective bundle $\mathbb{B}'_{\rm red} = \mathbb{P}(\mathcal{N}_{\mathbb{V}|\mathbb{P}(S_2)})$. Thus, $c_1(\mathcal{O}_{\mathbb{B}'_{\rm red}}(-(r+1))) = -(r+1)h'$ and so

$$\mathrm{s}_i(\mathcal{O}_{\mathbb{B}'_{\rm red}}(-(r+1))) = (r+1)^i(h')^i.$$

Substituting in Cor. 4.2.2, we find

(30) $$\mathrm{s}_k(\mathcal{N}_{\mathbb{B}'|\mathbb{X}'})_{|\mathbb{B}'_{\rm red}} = \sum_{i=0}^{k}(r+1)^i(h')^i\mathrm{s}_{k-i}\big(T\mathbb{P}(S_{2r+1})\big).$$

Recall the direct image (cf. diagram (23)) yields the Segre classes

(31) $$\psi'_\star\big((h')^i\big) = \mathrm{s}_{i-(\nu_2-n-1)}(\mathcal{N}_{\mathbb{V}|\mathbb{P}(S_2)});$$

these are zero for $0 \le i < m := \nu_2 - n - 1$ and $i > \dim\mathbb{B}' = \nu_2 - 1$. The Segre classes of $\mathcal{N}_{\mathbb{V}|\mathbb{P}(S_2)}$ are gotten from (20) and (21). Carrying these informations to (28), together with (24) and (25) completes the proof of our main result, to wit,

4.3. **Theorem.** *The degree of the component* $\mathcal{R}_n(2, 2r+1)$ *is given by*

$$\binom{\nu}{\nu_2} - (r+1)\sum_{k=m}^{\nu_2-1} A_k M_k$$

where we set

$$\begin{cases} \nu_i := \binom{n+i}{n} - 1 = \dim \mathbb{P}(S_i), \ \nu := \dim \mathbb{X} = \nu_2 + \nu_{2r+1}, \\ m := \nu_2 - n - 1, \ M_k := \binom{\nu}{k+1+\nu_{2r+1}} \\ A_k := (2r+3)^{\nu_2-(k+1)} \sum\limits_{i=m}^{k} (r+1)^i (2r+1)^{k-i} C_{ik} \sum\limits_{j=0}^{i-m} 2^j B_{ij}, \\ B_{ij} := (-1)^j \binom{n+1}{i-m-j}\binom{\nu_2+j}{j} \text{ and } C_{ik} := (-1)^{k-i}\binom{k-i+\nu_{2r+1}}{k-i}. \end{cases}$$

□

A script implementing the above formula for SINGULAR [7] can be seen in [15]. Here we list the first few values.

$\deg \mathcal{R}_n(2, 2r+1)$

r	$n=2$
1	770
2	35067
3	528600

r	$n=3$
1	6254612
2	27389258692
3	19054211679360

r	$n=4$
1	481152797320
2	5858642997232446492
3	2734930347184142269264030

These numbers match those conjectured in [6] only for $r = 1$. This is due to an error in a line of code used there, cf.[16].

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