Counting Canonical Curves in $\mathbb{P}^3$

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Contents

Introduction

Our goal is to describe the use of Bott's residue formula for the problem of counting the number of canonical curves in $\mathbb{P}^3$ incident to 24 general lines. The lectures are intended as training ground for similar questions we hope to address in this century: e.g., how many genus 2 quintic curves meet 20 general lines; canonical curves in $\mathbb{P}^4$,.... These notes summarize the material of [10], except for:
(1) the added section on triangles, based on [14],[13] and
(2) comments on MAPLE codes.

Here is a rough outline, lecture by lecture.

1. Torus actions and the $T-$equivariant Chow groups.
2. The localization theorem and Bott's residue formula.
3. Warming up with the intersection of two lines in $\mathbb{P}^2$.
4. The 27 lines on a cubic surface.
5. Triangles.
6. The family of canonical curves in $\mathbb{P}^3$.

We assume throughout these notes that $X \subset \mathbb{P}^N$ is a quasi-projective subscheme with an action of the torus $T = \mathbb{C}^\star$ induced by a representation of T in $\mathbb{C}^{N+1}$.

1 The usual Chow group

This is the “modern” way to state and prove theorems like Bézout’s theorem in greater generality.

1.1 Group of Cycles

Let X be a scheme and set $n = \dim(X)$. The *group of cycles of dimension* k, or k*-cycles* in X is the free abelian group generated by the set of closed irreducible subvarieties of dimension k in X.

It is denoted by $Z_k(X)$. The *group of cycles* of X is the graded group

$$Z_*(X) \ := \ \bigoplus_{k=0}^{n} \ Z_k(X)$$

By definition, each k-cycle c in $Z_k(X)$ can be written uniquely as a linear combination with integer coefficients,

$$c \ = \ \sum_{V} \ n_V \cdot V$$

where V ranges in the collection of (closed and irreducible) subvarieties of X of dimension k, with $n_V \neq 0$ for at most finitely many V's.

The *support* of a cycle $c = \sum n_V V$ is defined by

$$|c| \; = \bigcup_{n_V \neq 0} V.$$

Let $X_1, \cdots, X_m$ be the irreducible components of a scheme X. The *fundamental cycle* of X is defined by

$$[X] \; = \sum_{i=1}^{m} m_i X_i$$

where $m_i = l(\mathcal{O}_{X,X_i})$ is the length of the local ring of X along X_i.

Since the local ring $\mathcal{O}_{X,X_i}$ is artinian, the length is a positive integer, called the geometric multiplicity of X at X_i.

1.1.1 Rational equivalence

This is the most popular way to generalize the idea of deforming cycles yet preserving intersection numbers.

Let V be a variety and let $R(V)$ be the field of rational functions of V. Let $r \in R(V)$ be a nonzero rational function. We define the *order* of r along a subvariety $W \subset V$ of codimension 1 by

$$\mathrm{ord}_W(r) \;:=\; l(A/(a)) \;-\; l(A/(b)),$$

where $A = \mathcal{O}_{V,W}$ and $r = a/b$ with $a, b \in A$.

We note that ord_W is well defined cf. [4] and

$$\mathrm{ord}_W(r \cdot s) = \mathrm{ord}_W(r) \;+\; \mathrm{ord}_W(s),\; \forall r, s \in R(V).$$

We define the *the divisor of a rational function r on a variety V* as

$$\mathrm{div}(r) \;:=\; \sum_W \mathrm{ord}_W(r)\cdot W$$

where W ranges in the collection of closed and irreducible subvarieties of V of codimension 1. Declare zero all such cycles, i.e., mod. out the group of cycles by the subgroup generated by divisors of rational functions.

The intuition behind such formalism is that

$$\underbrace{\left(\sum_{\mathrm{ord}_W(r)>0} \mathrm{ord}_W(r)\cdot W\right)}_{0} \quad \text{and} \quad \underbrace{\left(\sum_{\mathrm{ord}_W(r)<0} \mathrm{ord}_W(r)\cdot W\right)}_{\infty}$$

might be thought of as $r^{-1}(0)$ and $r^{-1}(\infty)$ for a “map” $r: V \to \mathbb{P}^1$. The interpretation can be put on solid ground replacing V by the closure in $V\times\mathbb{P}^1$ of the graph of the rational function.

The above formal sum is in fact a cycle on V since $\text{ord}_W(r) \neq 0$ only for finitely many subvarieties of V, cf. [4].

Let X be a scheme. The group of *k-cycles rationally equivalent to zero* on X is defined as the subgroup $R_k(X)$ of $Z_k(X)$ spanned by divisors of rational functions of subvarieties of X of dimension $k+1$. The *group of cycles rationally equivalent to zero* is the graded group

$$R_*(X) \ := \ \bigoplus_{k=0}^{n} R_k(X)$$

The graded quotient group

$$A_*(X) \ := \ Z_*(X)/R_*(X) \ = \ \bigoplus_{k=0}^{n} Z_k(X)/R_k(X)$$

is called *the Chow group of X*.

1.2 The T-Invariant Chow Group

The T-invariant Chow groups have motivated the study of the T-equivariant groups. It so happens that in many interesting cases the two groups practically coincide.

1.2.1 Cycles and T-invariant rational equivalence

Let X be a T-space. The *T-invariant Chow group* of X is the quotient group $A_k(X,T) = Z_k(X,T)/R_k(X,T)$, where $Z_k(X,T) \subset Z_k(X)$ is the subgroup generated by the closed irreducible subvarieties of X that are T-invariant. The subgroup $R_k(X,T) \subset R_k(X)$ is generated by all divisors of rational eigenfunctions on T-invariant subvarieties of X of dimension $k+1$.

A rational function f on a T-invariant subvariety, $W \subset X$, is an *eigenfunction* if $g \circ f = \chi(g) \cdot f$ for all $g \in T$ and some character $\chi = \chi_f$ of T.

The inclusion $Z_k(X,T) \subset Z_k(X)$ induces a natural homomorphism $A_k(X,T) \to A_k(X)$.

We have the following.

1.3 Theorem of Hirschowitz

The natural homomorphism $A_k(X,T) \longrightarrow A_k(X)$ *is bijective.*

The above theorem was originally proven by André Hirschowitz [7] in 1984 in the case when X is a projective variety. In 1995, W. Fulton, R. MacPherson, F. Sottile and B. Sturmfels [5] proved the general case.

1.4 The T-equivariant Chow group

In this section we introduce the T-equivariant Chow group of a T-space X.

Details will be omitted; most of them are consequence of (more or less) well known results about quotients of varieties by algebraic group actions. The canonical references are Borel [1] and Mumford [12].

1.4.1 T-principal bundle

Let X be a T-space, $n = \dim X$.

We choose an l-dimensional representation V of T such that V contains an invariant open dense subset U where the action is free, i.e., trivial stabilizers. Such $U \subset V$ will be explicitly described in the main examples.

Let $\pi : U \to \overline{U} := U/T$ be the quotient T-principal bundle.

This means that there exists an open cover $\{\overline{U}_i\}$ of $\overline{U}$ such that $\pi^{-1}\overline{U}_i \simeq \overline{U}_i \times T$, with transition functions $\varphi_{ij} : \overline{U}_{ij} \to T$.

Such a quotient always exists as an algebraic space, since T acts freely on U.

For the cases we shall have a closer look at, the quotient U/T is in fact a projective spaces and the construction of he quotient is elementary, cf. 1.4.3.

The diagonal action $\gamma\circ(x,u) = (\gamma \cdot x, \gamma \cdot u)$ over $X \times U$ is also free. Hence there exists a quotient $X \times U \to (X \times U)/T$ in the category of algebraic spaces which is a T-principal bundle.

Denote the quotient $(X \times U)/T$ by $X \times^T U$, or X_T for short.

Again, in all cases we are particularly interested, the quotient X_T is a projective scheme.

Henceforth,

$$U \subset V$$

means an open dense subset U of a representation V of T on which T acts freely, and X_T denotes the base of the T-principal quotient bundle, also written

$$X \times U \longrightarrow X_T = X \times^T U.$$

The choice of $U \subset V$ can always be made in such a way that the map of restriction of cycles of $X \times V$ to the open subset $X \times U$ is bijective for *any pre-assigned dimension.*

1.4.2 An instructive exercise

If dim$V$=1 then the quotient $X \times^T U$ is simply X.

1.4.3 A very instructive example

Let T act on $X = \mathbb{P}^1$ via $t \circ [x_0, x_1] = [x_0, t \cdot x_1]$. Fix $l > 1$ and look at the diagonal representation of T in $V = \mathbb{C}^l$, $(v \to t \cdot v)$. Now set $U = V \setminus \{0\}$. It is clear that T acts freely on U.

Our T-principal bundle $U \to U/T$ is nothing but the familiar construction $\mathbb{C}^l \setminus \{0\} \to \mathbb{P}^{l-1}$.

Continuing, $X \times U \rightarrow X_T$ also is a T-principal bundle.

Its base X_T is described next.

Examine the map

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{C}^l \setminus \{0\} & \stackrel{\psi}{\rightarrow} & \mathbb{P}^{l-1} \times \mathbb{P}^l \\ (x,y) = \big([x_0, x_1], (y_1, \ldots, y_l)\big) & \mapsto & ([y_1, \ldots, y_l], [x_1,\, x_0 y_1, \ldots, x_0 y_l]). \end{array}$$

We have

$$\psi\big(t\circ(x,y)\big) = \psi(x,y),\ \forall t \in T, x \in \mathbb{P}^1, y \in \mathbb{C}^l \setminus \{0\}.$$

In fact, $\psi^{-1}(\psi(x,y)) = T\circ(x,y) \cong T$.

Taking $z_0, \ldots, z_l$ as homogeneous coordinates for $\mathbb{P}^l$, we see that the image of ψ is the subvariety $W \subset \mathbb{P}^{l-1} \times \mathbb{P}^l$ given by

$$y_i z_j = y_j z_i,\ 1 \le i, j \le l.$$

The projection $W \rightarrow \mathbb{P}^{l-1}$ identifies W with the $\mathbb{P}^1$-bundle $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^{l-1}} \oplus \mathcal{O}_{\mathbb{P}^{l-1}}(-1)\big)$. Thus, $X_T \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^{l-1}} \oplus \mathcal{O}_{\mathbb{P}^{l-1}}(-1)\big) \rightarrow \mathbb{P}^{l-1}$.

1.4.4 Proposition-Definition: $A_i^T(X)$

We define the ith T-equivariant Chow group of X by

$$A_i^T(X) = A_{i+l-1}(X_T),$$

where $l = \dim(V)$ *and* A_* *denotes the usual Chow group.*

The group is independent from the chosen representation, provided $V - U$ *is of codimension sufficiently big, i.e.,* $> \dim X - i$.

Whenever we write $A_i^T(X) = A_{i+l-1}(X_T) = A_{i+l-1}(X \times^T U)$ it is always assumed that a representation V and the open subset $U \subset V$ were chosen in such a way that $V - U$ is of codimension bigger than $n - i$ in V.

1.4.5 Non-triviality of $A_i^T(X)$.

It may well happen that the usual Chow group $A_i(X)$ be trivial, but $A_i^T(X)$ be nonzero for some $i \leq n$, including negative i. Take for example $X = pt$, yes, just a single point.

Presently, X_T is just U/T. Choosing a representation as in (1.4.3), we see that $A_i^T(X) = A_{i+l-1}(\mathbb{P}^{l-1})$ is zero for $i \geq 1$ and isomorphic to $\mathbb{Z}$ for all $i \leq 0$.

We will see below that in fact

$$A_*^T(pt) = \bigoplus_n \mathbb{Z} \cdot h^n = \mathbb{Z}[h].$$

1.4.6 Invariant cycles

If $Y \subset X$ is a T-invariant subvariety of X of dimension m, then Y gives rise to a fundamental T-equivariant class,

$$[Y]_T = [Y \times^T U] \in A_m^T(X).$$

In general, if V is an l-dimensional representation of T and $S \subset X \times V$ is an invariant subvariety of dimension $m + l$, then S admits a *T-equivariant fundamental class* $[S]_T \in A_m^T(X)$ given by

$$[S]_T = [\big(S \cap (X \times U)\big)/T].$$

1.4.7 The ring structure

Whenever X is a smooth variety, the T-equivariant Chow group

$$A_*^T(X) = \bigoplus A_i^T(X)$$

inherits an intersection product from the ordinary Chow groups. This endows $A_*^T(X)$ with the structure of a graded ring. In this case, it is more convenient to take the grading given by codimension, writing

$$A_T^j(X) = A_{1-j}^T(X) \text{ and } A_T^*(X) = \bigoplus A_T^i(X).$$

1.5 The T-equivariant ring of a point

For each $i \geq 0$ pick $l > i$ and set $V = \mathbb{C}^{\,l}$ and U as in the above instructive example 1.4.3. With these choices, the codimension of $V - U = \{0\}$ in V is equal to l and we have $U/T = \mathbb{P}^{l-1}$. We may write,

$$A^i_T(pt) = A^i(U/T) = A^i(\mathbb{P}^{l-1}) = \mathbb{Z} \cdot h^i,$$

where $h = c_1(\mathcal{O}(1)) =$ class of a hyperplane section of $\mathbb{P}^{l-1}$. Hence we get,

$$A^*_T(pt) = \bigoplus \mathbb{Z} \cdot h^i = \mathbb{Z}[h].$$

The equivariant ring of a point acted on by T plays a central role. We denote it by R_T. We see that

$$R_T := A^*_T(pt) = \mathbb{Z}[h] \tag{1}$$

is a ring of polynomials with integer coefficients. The indeterminate h represents a hyperplane section in some $\mathbb{P}^{l-1}$.

1.6 Functorial properties

All morphisms are assumed T-equivariant.

Given a morphism $f : X \to Y$ of T-schemes, $(f \times id) : X \times U \to Y \times U$ induces $f_T : X_T \to Y_T$ rendering the following diagram commutative:

$$\begin{array}{ccc} & f \times id & \\ X \times U & \longrightarrow & Y \times U \\ \downarrow & & \downarrow \\ X_T & \longrightarrow & Y_T. \\ & f_T & \end{array}$$

The projections in the diagram above are surjective and flat.

It follows that if $f : X \to Y$ is either smooth, or proper, or flat of relative codimension k or an embedding, then $f_T : X_T \to Y_T$ will have the same property.

The *proper pushforward* $f_* : A_i^T(X) \to A_i^T(Y)$ is given by

$$f_{T*} : A_{i+l-1}(X_T) \to A_{i+l-1}(Y_T).$$

If $f : X \to Y$ is a flat morphism of relative dimension k, the *flat pullback* $f^* : A_i^T(Y) \to A_{i+k}^T(X)$ is defined by

$$f_T^* : A_{i+l-1}(X_T) \to A_{i+k+l-1}(Y_T).$$

1.6.1 Proposition on push–pull

The maps f_* and f^* are well defined.

1.6.2 T-equivariant Chern classes

Let E be a T-equivariant vector bundle on X. For each pair i, j we define a map $c_j^T(E) : A_i^T(X) \to A_{i-j}^T(X)$ as follows.

Let V be an l-dimensional representation of T and choose an open subset $U \subset V$ such that $V - U$ has sufficiently big codimension and the T-principal bundle $X \times U \to X_T$ exists. Then there exists a quotient E_T of $E \times U$. It can be shown that $E_T \to X_T$ is in fact a vector bundle. We give details for this later on in the cases when X is a T-space and either E is a trivial bundle (1.7.1), or when T acts trivially on X (2.2.1).

1.6.3 Definition-Proposition: $c_j^T(E)$

The j-th T-equivariant Chern class

$$c_j^T(E) : A_i^T(X) \to A_{i-j}^T(X)$$

is defined as the operator

$$\alpha \in A_{i+l-1}(X_T) \longmapsto c_j^T(E) \cap \alpha = c_j(E_T) \cap \alpha \in A_{i-j+l-1}(X_T).$$

This definition is independent from the choice of representation.

1.6.4 Equivariant self-intersection

Let $i : Y \hookrightarrow X$ be an equivariant regular embedding of T-spaces of codimension d. The usual self-intersection formula,

$$i^* i_* \alpha = c_d(\mathcal{N}_{Y/X}) \cap \alpha, \ \alpha \in A_*(Y)$$

induces a similar equivariant formula,

$$i_T^* i_{T*} \alpha = c_d^T(\mathcal{N}_{Y/X}) \cap \alpha, \ \alpha \in A_*^T(Y) \tag{2}$$

This follows from the fact that under the given hypotheses, the normal bundle of the quotient $Y_T \hookrightarrow X_T$ is the quotient $(\mathcal{N}_{Y/X})_T$ of the normal $\mathcal{N}_{Y \times U/X \times U}$.

1.7 The line bundle of a character

The lemma below is a simplified version of the general construction of a vector bundle associated to a T-principal bundle.

1.7.1 Lemma: $\mathcal{L}_\chi$

Let $U \to U/T$ *be as before and let* χ *be a character of* T*. Let* $\varphi : U \times_\chi \mathbb{C} \to U$ *be the trivial line bundle endowed with the* T*-action*

$$t \cdot (u, v) \mapsto (t \cdot u, \chi(t) \cdot v).$$

Then φ *is a* T*-equivariant bundle and induces, passing to the quotient, a line bundle* $\mathcal{L}_\chi \to U/T$*.*

Proof. The trivial bundle in the statement is obviously T-equivariant. We now describe a local trivialization of the quotient $\mathcal{L}_\chi$ with transition functions that yield a line bundle on U/T.

Let $\{(U_i, \varphi_{ij})\}$ be a trivialization of the T-principal bundle $U \to U/T$. That is, we have

$$U = \bigsqcup (U_i \times T)/\sim$$

where $(u,t) \sim (u',t') \;\Leftrightarrow\; u = u' \in U_{ij} = U_i \cap U_j$ and $t' = \varphi_{ij}(u) \cdot t$. This implies,

$$\mathcal{L}_{\chi \mid U_i} = (U_i \times T \times_\chi \mathbb{C})/T = U_i \times \mathbb{C}.$$

The gluing in U_{ij} is done by first identifying representatives $(u,t,v) \in U_i \times T \times \mathbb{C}$ with $(u',t',v') \in U_j \times T \times \mathbb{C}$, and then passing to the quotient. Thus, we have

$$u' = u \in U_{ij},\; t' = \varphi_{ij}(u) \cdot t \text{ and } v' = \chi(\varphi_{ij}(u)) \cdot v.$$

Hence, $\{(U_i, \chi\circ\varphi_{ij})\}$ gives a local trivialization of $\mathcal{L}_\chi$ as a line bundle on U/T. □

1.8 Character versus Chern class

1.8.1 The multiplication by a character

In the diagram of natural maps

$$\begin{array}{ccccc} X \times U \times_\chi \mathbb{C} & \longrightarrow & \mathcal{L}_{\chi|X_T} & \longrightarrow & \mathcal{L}_\chi \\ \downarrow & & \downarrow & & \downarrow \\ X \times U & \longrightarrow & X \times^T U & \longrightarrow & U/T \end{array}$$

the horizontal arrows on the left are the quotient maps by the action of T. For each cycle $\alpha \in A_*(X_T)$, write

$$\chi \cdot \alpha := c_1(\mathcal{L}_{\chi|X_T}) \cap \alpha. \tag{3}$$

This operation of the group $\widehat{T}$ $(= \mathrm{Hom}(\mathbb{C}^\star, \mathbb{C}^\star) = \mathbb{Z})$ of characters of the torus T in $A_*(X_T)$ plays a very important role.

1.8.2 Structure of R_T-module

The structure morphism $X \to pt$ induces a morphism

$$X_T \to U/T = \mathbb{P}^l$$

that turns $A_*^T(X)$ into an R_T-module.

Here

$$V = \mathbb{C}^{l+1}, \qquad U = V \setminus \{0\}.$$

The action of T is given by $t \circ v := t^{-1}v$, multiplication with all the weights equal to -1. As in example (1.4.3), we have $U/T = \mathbb{P}^l$ and $A_*(U/T) = A_*(\mathbb{P}^l)$. Letting $l \to \infty$, we have that

$$R_T = \mathbb{Z}[h],$$

where h represents a hyperplane section of some $\mathbb{P}^l$.

The map $R_T \to A_*^T(X)$ is simply pullback of cycles in R_T. Since in each dimension R_T is generated by $c_1(\mathcal{L}) \cap [U/T]$, where $\mathcal{L}$ is a line bundle on $U/T = \mathbb{P}^l$, we see that the multiplication

$$R_T \times A_*^T(X) \to A_*^T(X)$$

is obtained from generators of R_T as multiplication by characters of T, in the form described in (3).

1.8.3 The bundle $\mathcal{O}_{\mathbb{P}^l}(a)$

Continue with the action of T on $V = \mathbb{C}^{l+1}$, $U = V - \{0\}$ as just above. We get the familiar T-principal bundle ,

$$\begin{array}{ccc} U & \longrightarrow & U/T \\ || & & || \\ \mathbb{C}^{l+1} \setminus \{0\} & \longrightarrow & \mathbb{P}^l. \end{array}$$

The latter can be described by the usual local chart, $\{(U_i, \varphi_{ij})\}_{i=0\ldots l}$ where

$$U_i = \{[x_0, \ldots, x_l] \in \mathbb{P}^l \mid x_i \neq 0\}$$

and the transition functions are

$$\varphi_{ij} : U_{ij} \to \mathbb{C}^*$$
$$[x_0, \ldots, x_l] \mapsto x_i/x_j.$$

Any character $\chi : T = \mathbb{C}^* \to \mathbb{C}^*$ is of the form $t \mapsto t^a$ for some $a \in \mathbb{Z}$. We have an isomorphism

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \widehat{T} \\ a & \mapsto & (\chi_a : T \to \mathbb{C}) \end{array}$$

where χ_a denotes the character of T, $t \to t^a$. Consider the line bundle induced by χ_a,

$$\mathcal{L}_a = \mathcal{L}_{\chi_a} \to U/T = \mathbb{P}^l.$$

Recalling the proof of (1.7.1), $\mathcal{L}_a$ is given by transition functions

$$\begin{array}{ccc} \psi_{ij} : U_{ij} & \longrightarrow & GL_1 = \mathbb{C}^* \\ x = [x_0, \ldots, x_l] & \mapsto & \chi_a(\varphi_{ij}(x)) = (x_j/x_i)^{-a}. \end{array}$$

These are the transition functions of the line bundle $\mathcal{O}_{\mathbb{P}^l}(a)$. It follows that

$$\mathcal{L}_a = \mathcal{O}_{\mathbb{P}^l}(a) \longrightarrow U/T = \mathbb{P}^l. \tag{4}$$

1.8.4 Divisors of eigenfunctions

Let $f \in \mathbb{C}(X)$ be a nonzero rational function. Assume f is an eigenvector of T with character χ. Then the support of f is a T-invariant divisor. Hence it defines a class $\mathrm{div}^T(f)$ in the equivariant Chow ring $A_*^T(X)$. Precisely, write the principal divisor

$$\mathrm{div}(f) = \Sigma m_Z Z \in Z^1(X),$$

where Z ranges through all components of the support and m_Z denotes the respective multiplicity. We define

$$\mathrm{div}^T(f) = \Sigma m_Z [Z]_T \in A_*^T(X), \tag{5}$$

where $[Z]_T$ denotes the class (1.4.6) of the invariant subvariety Z in the T-equivariant ring of X.

The rational function f induces a rational section $f_{|X\times U}$ of the equivariant trivial bundle $X \times U \times_\chi \mathbb{C}$. Equivariance follows from the definition of $U \times_\chi \mathbb{C}$:

$$\begin{aligned} f_{|X\times U}(t\cdot(x,u)) &= (t\cdot x, t\cdot u, f(t\cdot x)) \\ &= (t\cdot x, t\cdot u, \chi(t)\cdot f(x)) \\ &= t\cdot(x,u,f(x)). \end{aligned}$$

This T-equivariant rational section, $f_{|X\times U}$, passes to the quotient. More precisely, let $\mathcal{L}_{\chi|X_T}$ be the pullback of the line bundle $\mathcal{L}_\chi$ under the map $X \times^T U \to U/T$. We have an induced rational section,

$$\begin{aligned} s_f : \quad X \times^T U &\quad \cdots\longrightarrow \quad \mathcal{L}_{\chi|X_T} \\ [x,u] &\quad \mapsto \quad [(x,u), f(x)]. \end{aligned}$$

1.8.5 Lemma: $\mathrm{div}^T(f) = \chi \cdot [X_T]$.

Notation as above, $\mathrm{div}^T(f)$ *represents in* $A_*^T(X)$ *the value of the operator* $c_1(\mathcal{L}_\chi)$ (3) *on the fundamental class of* $X_T = X \times^T U$,

$$\mathrm{div}^T(f) = \chi \cdot [X_T].$$

Note that, even though the class of the divisor of the rational function f is zero in the usual Chow group, the equivariant class $\mathrm{div}^T(f)$ is not necessarily zero in $A_*^T(X)$, cf. 1.8.6.

The relation $\mathrm{div}^T(f) = \chi \cdot [Y]_T$ enables us to compare the equivariant class of Y with a class of a T-invariant divisor contained in Y. That is, given a T-invariant subvariety $Y \subset X$, suppose that there is a rational eigenfunction f with non trivial character χ and such that f does not vanish on all of Y. Then, inverting the character χ in $A_*^T(X)$ if needed, we may compare the class $[Y]_T$ with an invariant cycle with support of dimension less than $\dim Y$. This is the key point in the proof of the localization theorem.

1.8.6 T-invariant divisors in $\mathbb{P}^n$

Let T act on $\mathbb{P}^n$ with weights $a_0, \ldots, a_n$, i.e.,

$$t\circ[x_0, \ldots, x_n] = [t^{a_0}x_0, \ldots, t^{a_n}x_n].$$

It can be shown that $(\mathbb{P}^n)_T = \mathbb{P}(\mathcal{O}(\oplus a_i)) \longrightarrow \mathbb{P}^l$.

Let us explain the computation of

$$\mathrm{div}^T(f) \in A_*^T(\mathbb{P}^n)$$

for a rational function $f : \mathbb{P}^n \cdots \to \mathbb{C}$ which we assume is an eigenfunction with weight $a \in \mathbb{Z}$, i.e,

$$f(t\circ x) = t^a \cdot f(x)\ \forall x \in \mathbb{P}^n \text{ (in the domain of } f) \text{ and } \ t \in T.$$

Say $f = x_i/x_j$, then f has weight $a = a_i - a_j$. We may write

$$\mathrm{div}(f) = H_i - H_j \in Z^1(\mathbb{P}^n)$$

with $H_k \subset \mathbb{P}^n$ denoting the T-invariant hyperplane given by $x_k = 0$. Thus, we get by (5)

$$\mathrm{div}^T(f) = [H_i]_T - [H_j]_T \ \text{ in } \ A_*^T(\mathbb{P}^n).$$

The lemma 1.8.5 tells us that

$$\mathrm{div}^T(f) = c_1(\mathcal{L}_\chi) \cap [\mathbb{P}^n_T]$$

with $\chi = \chi_a$. In this manner, recalling (4), we see that

$$[H_i]_T - [H_j]_T = c_1(\mathcal{O}_{\mathbb{P}^l}(a)) \cap [\mathbb{P}^n_T] = a \cdot t \cap [\mathbb{P}^n_T]$$

where, by deliberate abuse, we also write $t = c_1(\mathcal{O}_{\mathbb{P}^l}(1))$ for the pullback of the hyperplane section of $\mathbb{P}^l$. In particular, we see that $\mathrm{div}^T(f)$ is not necessarily zero in $A_*^T(\mathbb{P}^n)$.

1.9 Invariant cycles suffice

The invariant cycles are enough to handle many interesting cases.

1.9.1 Proposition: $A_*^T(X) = R_T[A_*(X,T)]$

The T-equivariant Chow group $A_*^T(X)$ is generated as R_T-module by the T-invariant subvarieties of X.

Proof.

We start descending induction at $A_n^T(X) = A_n(X_T)$, cf. 1.4.4 (with $i = n$, $l = 1$). We may take a 1-dimensional representation V such that $V \setminus U = \{0\}$. Now $X_T = X \times^T U$. We have in fact $X \cong X_T$ (cf. 1.4.2) so that a generator for $A_n^T(X)$ is just (the class of) X, as invariant as it could be.

For the inductive step, recall that a generator for $A_i^T(X) = A_{i+l-1}(X \times^T U)$ is given by a T-invariant subvariety

$$Y \subset X \times U \subset X \times V \text{ of dimension } i+l$$
$$\text{for a suitable choice of } U \subset V,$$
$$\text{provided } V - U \text{ is of codimension} > \dim X - i \ (\star).$$

We have a natural isomorphism between $A_i(X)$ and $A_{i+l}(X \times V)$, because V is a vector space.

We also have $A_{i+l}(X \times U) \cong A_{i+l}(X \times V)$ valid whenever $i > \dim(X) + \dim(V \setminus U)$, which is just $(\star)$.

Hence Y is rationally equivalent to $\sum m_i(Y_i \times U)$. Using the generalization of the theorem of Hirschowitz 1.3 to the (non necessarily projective) variety X, we may assume that each subvariety $Y_i \subset X$ is T-invariant.

Apply once more the theorem, this time to $X \times U$. Thus the rational equivalence between Y and $\sum m_i(Y_i \times U)$ can also be achieved in a T-invariant way: there exist T-invariant subvarieties

$$W \subset X \times U, \qquad \text{such that} \qquad \dim(W) = \dim(Y) + 1$$

and rational eigenfunctions $f_W : W \cdots \to \mathbb{C}$ with characters χ_W, with

$$Y - \sum m_i(Y_i \times V) = \sum \operatorname{div}(f_W).$$

Passing to the equivariant classes we have

$$\begin{aligned}[Y]_T - \sum m_i [Y_i]_T &= \sum \operatorname{div}^T(f_W) \\ &= \sum c_1(\mathcal{L}_{\chi_W}) \cap [W]_T = \sum \chi_W \cdot [W]_T,\end{aligned}$$

where the dimension of each W is bigger than $\dim(Y)$. It follows by induction that $A_*^T(X)$ is generated as R_T module by T-invariant subvarieties $Y \subset X$. $\square$

2 The theorem of localization

2.1 The case of a trivial action: $A_*^T(X) = A_*(X) \otimes R_T$.

Recall from (1) that the T-equivariant Chow ring of a point is

$$R_T = \mathbb{Z}[t],$$

the ring of polynomials with integer coefficients. It appears in fact as the group algebra of $\widehat{T} = \mathbb{Z}$, the group of characters of T. The explicit isomorphism comes from the following recipe.

For each character $\chi \in \widehat{T}$, consider the line bundle $\mathcal{L}_\chi$ on U/T constructed in 1.7.1. We identify χ with the operator $c_1(\mathcal{L}_\chi)$.

Lemma

If T acts trivially on X, then

$$A_*^T(X) = A_*(X) \otimes R_T.$$

Proof. Due to the triviality of the action of T on X, it follows that $X \times^T U = X \times (U/T)$. Thus, choosing U so that the quotient U/T is a projective space (cf. 1.4.3), we have the isomorphism

$$A_*\big(X \times (U/T)\big) = A_*(X) \otimes A_*(U/T).$$

Increasing the dimension of the representation we get the assertion.

□

2.2 Decomposition into eigensubbundles

The T-action on X is still assumed trivial.

Given a T-equivariant vector bundle $E \to X$ we get a canonical decomposition

$$E = \bigoplus_{\chi \in \widehat{T}} E^{\chi}$$

into a direct sum of subbundles, where E^{χ} denotes the eigensubbundle consisting of vectors in E on which T acts with the character χ:

$$E^{\chi} = \{v \in E \,|\, t \circ v = \chi(t)v, \, \forall t \in T\} \cdot$$

It follows that the T-equivariant Chern classes of E can be expressed in term of the classes of eigensubbundles E^{χ}.

So assume now $E = E^\chi$. We describe next the vector bundle E_T on X_T induced by E, profiting from the fact that $X_T = X \times (U/T)$ by the triviality of the action on the factor X.

2.2.1 Lemma: $(E^\chi)_T = E^\chi \otimes \mathcal{L}_\chi$

Notation and hypotheses as above, the quotient bundle $(E^\chi)_T$ *on* $X \times (U/T)$ *is isomorphic to the tensor product of the pullback of the bundle* E^χ *by the pullback of the line bundle* $\mathcal{L}_\chi$.

For a proof, just look at E_T in terms of local charts. $\square$

2.2.2 Corollary: $c_i^T(E^\chi) = \dots$

Let X be a T-space with trivial action. Let $E = E^\chi \to X$ be a T-equivariant vector bundle of rank r on X such that the action of T on each fiber is given by a character χ. Then, for all i, we have

$$c_i^T(E^\chi) = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(E^\chi)\chi^{i-j}.$$

Proof. The assertion follows from example (3.2.2) of [4], where the (usual) Chern class of a tensor product like $(E^\chi)_T \simeq E^\chi \otimes \mathcal{L}_\chi$ is computed. □

2.3 Fixed points locus

The following result ensures nontriviality of the weights of the normal bundle of the locus of fixed points.

2.3.1 Lemma: $(\mathcal{N}_{F/X})_x$

If X is a smooth T-variety then the locus X^T of fixed points is also smooth. If F is a component of X^T, then the normal bundle $\mathcal{N}_{F/X}$ is T-equivariant over F. Furthermore, we have $(\mathcal{T}_x X)^T = \mathcal{T}_x F$ for all $x \in F$, and therefore the T-action on $(\mathcal{N}_{F/X})_x$ is non trivial.

Proof. See B. Iversen, [8]. A word as to the nontriviality: Write the decomposition $\mathcal{T}_x X = \bigoplus_\chi (\mathcal{T}_x X)^\chi$.

We have for $x \in F$ $\mathcal{T}_x F = (\mathcal{T}_x X)^T = (\mathcal{T}_x X)^{(\chi=1)}$, so

$$(\mathcal{N}_{F/X})_x = \bigoplus_{\chi \neq 1} (\mathcal{T}_x X)^{\chi}. \quad \square$$

2.3.2 Localize in order to invert

Let $F \subseteq X^T$ be a component of the fixed points locus. Since T acts trivially on F, we have by 48 that

$$A_T^*(F) = R_T \otimes A^*(F).$$

Let E be an equivariant bundle on X. The restriction $E_{|F}$ decomposes as a sum of eigensubbundles $E^{\chi}_{|F}$.

The lemma 2.2.2 tells us that the component of $c_i^T(E^{\chi}_{|F})$ in R_T^i is given (setting $j = 0$) by $\binom{r}{i}\chi^i$.

Since $A^N(F) = 0$ for $N > \dim(F)$, we have that, for $j > 0$, the elements of $A^j(F)$ are nilpotent in the ring $A_T^*(F)$. It follows that $c_i^T(E^{\chi}_{|F}) \in (R_T \otimes A^*(F))^i$ is invertible if and only if its component in $R_T^i \otimes A^0(F) \cong R_T^i$ is invertible.

Hence, $c_i^T(E^{\chi}_{|F})$ becomes invertible in the localization

$$\mathbb{Q} \otimes R_T \otimes A^*(F)[\chi^{-1}].$$

2.3.3 Lemma: invert $c_d^T(\mathcal{N}_{F/X})$.

Let X be a smooth T-variety and let F be a component of codimension d of the fixed points locus X^T. Then there exist finitely many nontrivial characters $\lambda_1, \ldots, \lambda_r$ such that $c_d^T(\mathcal{N}_{F/X})$ becomes invertible in the ring of fractions

$$A_T^*(F)[1/\lambda_1, \ldots, 1/\lambda_r] \otimes \mathbb{Q}.$$

Proof. By the previous lemma, we know that T acts with nontrivial weights on the normal space $(\mathcal{N}_{F/X})_x = \mathcal{T}_x X/\mathcal{T}_x F$. Hence, the characters λ_i that occur in the decomposition of the normal space $(\mathcal{N}_{F/X})$ into eigenbundles are all nontrivial. By the previous remark, we see that the component of $c_d^T(\mathcal{N}_{F/X})$ in R_T^d is nonzero. Hence, the class $c_d^T(\mathcal{N}_{F/X})$ becomes invertible in the ring $A_T^*(F)[1/\lambda_1, \ldots, 1/\lambda_r]$, as claimed. □

2.4 The theorem of localization

We present in this section the version of M. Brion [2] for the theorem of localization. The principal point in favor of his approach is to avoid the need of the higher Chow groups as required in [3].

2.4.1 Lemma: eigenfunction $f_{|Y} \neq 0$

Let X be an affine T-scheme. Let Y be a T-invariant subvariety. If Y is not fixed pointwise, then there exists a regular eigenfunction f on X with nontrivial weight whose restriction $f_{|Y} \neq 0$.

Proof. Pick $y \in Y \setminus X^T$. Hence, $\exists\, t \in T$ such that $t \circ y \neq y$. Since T is a torus, we know that the coordinates ring of X is generated by eigenfunctions. Hence there exists an eigenfunction f, say associated to the character χ, which separates those points: $f(t \circ y) \neq f(y)$. Hence, $f(t \circ y) = \big(\chi(t) \cdot f\big)(y) \neq f(y)$. This implies at once that $f(y) \neq 0$ and $\chi(t) \neq 1$. □

2.4.2 Lemma: $X^T \subset X$

Let X an affine T-scheme. The fixed points locus $X^T \subset X$ is an intersection of schemes of zeros of the regular eigenfunctions on X with nontrivial weights.

Proof. Let $x \in X^T$, and let f be a regular eigenfunction with weight $\chi \neq 1$. We have $f(x) = f(t \circ x) = \chi(t) \cdot f(x)$, $\forall t \in T$. Hence, if $\chi(t) \neq 1$ then $f(x) = 0$. Conversely, if $x \notin X^T$, we apply the argument of the previous lemma to find an eigenfunction which is zero on X^T and $\neq$0 on x. □

Denote by

$$i_T : X^T \hookrightarrow X$$

the map of inclusion of the fixed points locus. We know that i_T induces a homomorphism of R_T-modules

$$i_{T*} : A_*^T(X^T) \longrightarrow A_*^T(X)$$

for the T-equivariant Chow groups (cf. 1.6.1).

Recall also that we have a natural isomorphism 48

$$A_*^T(X^T) \simeq R_T \otimes A_*(X^T),$$

since the action of T in X^T is trivial.

2.4.3 Theorem of localization

Let X be a T-space. Then the R_T-linear map

$$i_{T*} : A_*^T(X^T) = A_*(X^T) \otimes R_T \longrightarrow A_*^T(X)$$

becomes an isomorphism after inverting finitely many nontrivial characters.

Proof. By our blanket assumption, X can be covered by finitely many T-invariant affine open subsets X_i.

By the previous lemma, each fixed points locus $X_i^T \subset X_i$ is an intersection of zeros of the regular functions on X_i which are eigenvectors of the action of T on X_i with nontrivial weights.

By quasi-compactness, we may extract a finite intersection.

That is,

$$\begin{gathered}\text{there exists a finite set of eigenfunctions } \{f_{ij}\} \\ \text{with respective weights } \{\chi_{ij}\}, \textit{ all nontrivial}, \\ \text{such that } x \in X_i \text{ is in } X_i^T \text{ if and only if } f_{ij}(x) = 0, \ \forall j.\end{gathered} \tag{6}$$

In order to show that i_{T*} is surjective, we invoke the theorem 1.3: the usual Chow ring of X is generated by the cycles of T-invariant subvarieties.

An immediate consequence is the fact that $A_*^T(X)$ is generated, as R_T-module, by the cycles of the form $[Y \times^T U]$ with $Y \subset X$ some T-invariant closed subvariety.

Let now $Y \subset X$ be a T-invariant subvariety.

If $Y \subset X^T$ then its cycle clearly is in the image of i_{T*}.

Suppose that Y is not pointwise fixed by T. Then one of the f_{ij} (6) defines a nonzero rational function on Y. This rational function defines in turn a rational section of the pullback of the line bundle $\mathcal{L}_{\chi_{ij}}$ (1.7.1) by the map $Y \times^T U \to U/T$, still denoted by $\mathcal{L}_{\chi_{ij}}$.

Therefore,

$$\chi_{ij} \cdot [Y]_T = c_1(\mathcal{L}_{\chi_{ij}}) \cdot [Y]_T = \mathrm{div}^T(f_{ij}) \in A_*^T(X),$$

which implies

$$[Y]_T = \chi_{ij}^{-1} \mathrm{div}^T(f_{ij}) \in A_*^T(X) \otimes R_T[1/\chi_{ij}].$$

Well, the support of $\mathrm{div}^T(f_{ij})$ is of dimension smaller than $\dim(Y)$ and is made up of invariant subvarieties. By noetherian induction it follows that, upon inverting finitely many χ_{ij}'s, the induced map i_{T*} becomes surjective.

For injectivity, notice that if $X^T = X$ then i_{T*} is the identity map.

Hence, we may assume that X is not fixed pointwise by T.

Let Y be an irreducible component of X that is not contained in X^T.

Choose f_{ij} as before, namely, $f_{ij\,|Y} \neq 0$.

Denote by $|D|$ the union of the support of the divisor of f_{ij} in Y and the irreducible components of X distinct from Y. Then, by construction, $|D|$ contains all the fixed points of X. Let

$$\iota : |D| \to X$$

be the map of inclusion.

Consider the T-principal bundle $U \to U/T$ given as in 1.4.3, and let $\mathcal{L}_{\chi_{ij}}$ be the line bundle on U/T associated to the weight χ_{ij}. Let

$$p : X \times^T U \to U/T$$

be the projection.

We have a pseudo-divisor on $X \times^T U$ (cf. [4], 2.2),

$$(\ p^*\mathcal{L}_{\chi_{ij}}, |D| \times^T U, f_{ij} \ )$$

which defines a homogeneous map of degree -1,

$$\iota^* : A_*^T(X) \longrightarrow A_*^T(|D|)$$

such that the composition $\iota^* \circ \iota_*$ is multiplication by χ_{ij}.
Examining the diagram,

$$\begin{array}{ccc} A_*^T(X^T) = A_*(X^T) \otimes R_T & \xrightarrow{i_{T*}} & A_*^T(X) \\ || & & \uparrow \ \iota_* \\ A_*^T(X^T = |D|^T) & \longrightarrow & A_*^T(|D|) \end{array}$$

we see that the map $\iota_* : A_*^T(|D|) \to A_*^T(X)$ becomes injective after inverting χ_{ij}. Therefore injectivity follows using again noetherian induction. $\square$

Recall that the equivariant ring of a point, $R_T = \mathbb{Z}[t]$, is a ring of polynomials. Let R_T^+ denote the multiplicative system of homogeneous elements of positive degree. We define the ring of fractions

$$\boxed{\mathcal{R}_T = (R_T^+)^{-1} \cdot R_T.}$$

Thus, in $\mathcal{R}_T$ (the image of) all nontrivial characters are invertible elements.

We get the following consequence.

2.4.4 Corollary: $i_* : A_*(X^T) \otimes \mathcal{R}_T \widetilde{\rightarrow} A_*^T(X) \otimes \mathcal{R}_T$

The map $i_* : A_*(X^T) \otimes \mathcal{R}_T \rightarrow A_*^T(X) \otimes \mathcal{R}_T$ *is an isomorphism.*

□

2.4.5 Theorem (explicit localization).

Let X be a smooth T-variety. Let $\alpha \in A_T^*(X) \otimes \mathcal{R}_T$. Then

$$\alpha = \sum_F i_{F*}\left(\frac{i_F^*\alpha}{c_{d_F}^T(\mathcal{N}_{F/X})}\right),$$

where the sum is taken over all components of X^T and d_F denotes the codimension of F in X.

Proof. From the surjectivity ensured by the Localization Theorem, we may write $\alpha = \sum_F i_{F*}(\beta_F)$. Since the irreducible components F of X^T are disjoint, it follows that $i_F^*\alpha = i_F^* i_{F*}(\beta_F)$ because the remaining components of X^T do not contribute for cycles in F.

The formula of self intersection 2 yields

$$i_F^* i_{F*}(\beta_F) = c_{d_F}^T(\mathcal{N}_{F/X}) \cdot \beta_F,$$

and so, by 2.3.3, we get $\beta_F = i_F^*\alpha/c_{d_F}^T(\mathcal{N}_{F/X})$ as desired. □

2.4.6 Homomorphism of integration

When X is a complete variety, the projection $\pi_X : X \to pt$ induces a pushforward $\pi_{X*} : A_*^T(X) \to R_T$ which is zero in A_i^T for $i > 0$ and is given by the degree of zero cycles for $i = 0$. Tensorizing by $\mathcal{R}_T$, we get the homomorphism of integration

$$\begin{array}{rccc} \pi_{X*}: & A_*^T(X)\otimes\mathcal{R}_T & \longrightarrow & \mathcal{R}_T \\ & \alpha & \longmapsto & \int_X \alpha. \end{array}$$

Replacing X by F, a component of X^T, we have a similar map π_{F*}.

Let us apply π_{X*} to both sides in the theorem of explicit localization.

Using the fact of that $\pi_{F*} = \pi_{X*} \circ i_{F*}$, we get the following.

2.4.7 Corollary. (Formula of integration)

Let X be a smooth and complete T-variety and let $\alpha \in A_T^*(X) \otimes \mathcal{Q}$. Then

$$\int_X \alpha = \sum_{F \subset X^T} \pi_{F*} \left(\frac{i_F^* \alpha}{c_{d_F}^T(\mathcal{N}_{F/X})} \right),$$

as an element of $\mathcal{R}_T$.

The previous corollary yields a formula of integration which is particularly useful for elements of the usual Chow group $A_0(X)$ of the form pullback of an element of $A_0^T(X)$. More precisely, consider the commutative diagram

$$\begin{array}{ccccc} X & \overset{i}{\hookrightarrow} & X_T & \leftarrow & X \times U \\ \pi_X \downarrow & \square & \downarrow \pi_X^T & & \downarrow \\ pt & \overset{j}{\hookrightarrow} & U/T & \leftarrow & U \end{array} \tag{7}$$

where the horizontal maps on the right are the quotient maps. Note that, by construction of the T-principal bundle, the inverse image of the point $pt \in U/T$ in U is the orbit $T \cdot u \cong T$ for some $u \in U$.

On the other hand, the inverse image in $X \times U$ is $X \times (T \cdot u)$.

The image of the latter subvariety in $X_T = (X \times U)/T$ is isomorphic to X.

Since X is smooth, we have that i is a regular embedding of codimension $d = \dim(U/T)$. Recalling the definition, we see that i induces the homomorphism,

$$i^* : A_0^T(X) = A_d(X_T) \longrightarrow A_0(X).$$

2.4.8 Proposition: case $a = i^*\alpha$, with $\alpha \in A_0^T(X)$.

With the hypotheses as in the previous corollary, suppose

$$a = i^*\alpha, \quad \textit{with} \ \ \alpha \in A_0^T(X).$$

Then

$$\int_X a = \sum_{F \subset X^T} \pi_{F*}\left(\frac{i_F^*\alpha}{c_{d_F}^T(\mathcal{N}_{F/X})}\right).$$

2.5 Bott's residues formula

Let $E_1, \ldots, E_s$ be T-equivariant vector bundles on a smooth and complete variety X of dimension n.

Let $p(x_1^1, \ldots, x_s^1, \ldots, x_1^n, \ldots, x_s^n)$ be a weighted homogeneous polynomial of degree n in the variables x_j^i, where x_j^i has degree i.

Denote by $p(E_1, \ldots, E_s)$ the polynomial in the Chern classes of $E_1, \ldots, E_s$, obtained by substituting $x_j^i = c_i(E_j)$.

The integration formula computes the degree of the zero cycle $p(E_1, \ldots, E_s) \cap [X]$ in terms of the restrictions of the bundles E_i's to the locus $X^T \subset X$ of fixed points.

Set for short

$$p(E_.) = p(E_1, \dots, E_s) \;\; \text{and} \;\; p^T(E_.) = p(E_{1\,T}, \dots, E_{s\,T})$$

the corresponding polynomial for the T-equivariant Chern classes of the bundles $E_1, \dots, E_s$. Note that $p(E) \cap [X] = i^*(p^T(E) \cap [X]_T)$. Employing the proposition 2.4.8, we get the following.

2.6 (Bott's residues formula)

Let X be a smooth, complete variety and let $E_1, \dots, E_s$ be T-equivariant vector bundles over X. Then

$$\int_X (p(E) \cap [X]) = \sum_{F \subset X^T} \pi_{F*} \left(\frac{p^T(E_{|F}) \cap [F]_T}{c_{d_F}(\mathcal{N}_{F/X})} \right), \tag{8}$$

where d_F denotes the codimension of the component F in X.

In spite of the possibly awe-inspiring appearance of the formula at first (and perhaps even a few subsequent) sight, we hope to convince the reader that it is in fact very efficient and rather simple to apply in practice.

2.7 Contribution of fixed points

Let X be a nonsingular T-variety and $F \subseteq X^T$ a connected (=irreducible) component of the fixed points locus. Write $d_F = \mathrm{codim}(F)$.

The T-equivariant Chern classes $c_k^T(E_{|F})$ and $c_{d_F}^T(\mathcal{N}_{F/X})$ can be computed in the equivariant Chow ring $A_*^T(F)$ in terms of the characters that occur in the decomposition of $E_{|F}$ and $\mathcal{N}_{F/X}$ into eigensubbundles and of the Chern classes of the latter.

2.7.1 Isolated fixed points

When X^T is a finite set of points, the classes

$$c_k^T(E_{|\,F}) \quad \text{and} \quad c_{d_F}^T(\mathcal{N}_{F/X}) = c_{\dim(X)}^T(\mathcal{T}X)$$

can be described purely in terms of characters associated to eigenbundles. More precisely, once we have the decomposition

$$E_{|\,F} = \bigoplus_{\chi} E^{\chi}_{|\,F}$$

into eigenspaces, the equivariant Chern classes of $E_{|\,F}$ can be determined. Indeed 2.2.2 yields the classes of each summand,

$$c_k^T(E^{\chi}_{|\,F}) = \binom{r}{k}\chi^k, \quad r = \text{rank of } E^{\chi}_{|\,F}. \tag{9}$$

Note that, in the above expression, χ^k means the kth.-iteration of the operator first Chern class introduced in (3). In particular, we deduce that $c^T_{\max}(E_{|F})$ is represented in the equivariant Chow ring of the fixed point F by the product of all characters that occur in the decomposition of the fiber $E_{|F}$ into eigenspaces, with their respective multiplicities. Here each character is already being considered as acting on the equivariant ring, according to (3). See the example 2.7.4.

Let us have a closer look at the replacement of $R_T = \mathbb{Z}[t]$ by

$$\boxed{\mathcal{R}_T = (R_T^+)^{-1}(R_T) = \mathbb{Q}[t, t^{-1}].}$$

On the right hand side of Bott's formula (8), the numerator $p^T(E_{|\,F})$ is a homogeneous polynomial of degree $n = \dim X$ in as many variables as there are characters appearing in the decomposition into eigensubbundles.

Typically, suppose that the original polynomial contains a term equal to, say $\xi = c_1^{n-2} \cdot c_2$ and (abusing) $E_{|\,F} = 2\chi_1 + \chi_2$.

Then we have $c_1^T(E_{|\,F}) = 2\chi_1 + \chi_2$, the right side now with the meaning of the operator in (3).

Similarly, $c_2^T(E_{|\,F}) = \chi_1^2 + 2\chi_1 \cdot \chi_2$.

Thus ξ yields at last the operator of degree n, to wit,
$(2\chi_1 + \chi_2)^{n-2} \cdot (\chi_1^2 + 2\chi_1 \cdot \chi_2)$.

Each character of $T = \mathbb{C}^*$ is of the form $\chi_i = t^{a_i}$, $a_i \in \mathbb{Z}$. The operator induced on R_T is $a_i \cdot t$, sorry for the abuse, cf. (4) where this time t means a hyperplane class!

Hence ξ appears in the final form

$$(2a_1 + a_2)^{n-2} \cdot (a_1^2 + 2a_1 \cdot a_2) \cdot t^n \in R_T. \tag{10}$$

That is, the numerator and the denominator on the right hand side of (8) are integer multiples of t^n. Cancelling t^n, we get in this way a rational number. So, the right hand side of (8) is just a finite sum of rational numbers obtained from the weights as described in (9).

More precisely, denote by $\tau_1(E,F),\ldots,\tau_r(E,F)$ the weights that occur in the decomposition of $E_{|F}$ into eigensubbundles, and for each integer $k \geq 0$, let $\sigma_k(E,F)$ denote the k-th elementary symmetric function on these weights.

2.7.2 Corollary: $c_k^T(E_{\,|\,F}) = \sigma_k(E,F)$

With notation as just above, we have that each equivariant Chern class $c_k^T(E_{\,|\,F})$ is represented in the equivariant Chow ring of the fixed point F by $\sigma_k(E,F)$. □

2.7.3 Corollary: $c_n^T(\mathcal{T}X)$

The equivariant top Chern class of the tangent bundle of X is given in the equivariant Chow ring of a fixed point F by the product of the weights that occur in the decomposition of the respective tangent space. □

In the next chapter we explain systematically how to apply Bott's formula to a few situations of interest in enumerative geometry. We can't resist however the compulsion to exhibit right away how the above result can be used to retrieve the number of zeros of a vector field in $\mathbb{P}^n$.

2.7.4 Zeros of vector fields in $\mathbb{P}^n$

Write $\mathcal{F} = \langle x_0, \ldots, x_n \rangle$ for the vector space of linear forms on these variables, a choice of homogeneous coordinates for $\mathbb{P}^n$. Consider the action of $T = \mathbb{C}^*$ given by $t \circ x_i = t^i x_i$. One sees at once that the set of fixed points in $\mathbb{P}^n$ is formed by the $n+1$ unitary points

$$P_0 = [1, 0, \ldots, 0], \ldots, P_n = [0, \ldots, 0, 1].$$

Let

$$\mathcal{A} = \{(P,\varphi) \in \mathbb{P}^n \times \mathcal{F} \mid \varphi(P) = 0\} \cdot$$

The tangent bundle admits the expression (cf. [6], p.200)

$$\mathcal{T}\mathbb{P}^n = \mathcal{A}^\vee \otimes \left(\mathcal{F}/\mathcal{A}\right). \tag{11}$$

The fiber over, say P_0, is given by

$$\langle x_1, \ldots, x_n \rangle^\vee \otimes \langle \overline{x_0} \rangle.$$

Hence, the decomposition of the tangent space into eigenspaces is

$$\mathcal{T}_{P_0}\mathbb{P}^n = (t^{-1} + \cdots + t^{-n}) \cdot t^0 = t^{-1} + \cdots + t^{-n}.$$

(Here we have used the property that the weight of the dual (resp. of a tensor product) is. . .)

The product of characters that occur in the decomposition gives

$$c_n^T(\mathcal{T}_{P_0}\mathbb{P}^n) = (-1)^n n!\, t^n,$$

where the term t^n now means n-iteration of hyperplane classes.

Recall that the cycle of zeros of a section of a vector bundle represents, under suitable conditions of regularity, the top Chern class of the bundle.

In particular, if the zeros of a vector field in $\mathbb{P}^n$ are isolated, then

$$\int c_n(\mathcal{T}\mathbb{P}^n) \cap [\mathbb{P}^n]$$

yields the number of zeros (counted with multiplicities).

Now Bott's formula (8) applied to the present situation displays, on its right hand side, $n+1$ terms, each equal to 1!

Indeed, each component F is just a point P_i, so that the homomorphism of integration $\pi_{F*} : \mathcal{R}_T \to \mathcal{R}_T$ is the identity. The numerator and denominator of the fractions that occur in the formula are both equal to $c_n^T(\mathcal{T}\mathbb{P}^n{}_{|P_i})$. We retrieve in this manner the well known fact that

the number of zeros of a (general) vector field in $\mathbb{P}^n$ is $n+1$.

3 Applications to enumerative geometry, I

Our goal henceforth is to give an idea of "practical applications" and usefulness of Bott's formula for the computation of some characteristic numbers.

3.1 Two lines in $\mathbb{P}^2$

Probably one of the simplest and yet instructive problems is the counting of the number of points of intersection of two general lines in the projective plane $\mathbb{P}^2$. The reader will quickly realize that the answer to our question is *one*...

Now that we know from the start the size of the answer, we may try and mess up the discussion a little bit and go on to perform the calculation using the usual Chow ring,

$$A^*(\mathbb{P}^2) = \mathbb{Z}[h]/\langle h^3 \rangle$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ represents the class of a line in $\mathbb{P}^2$.

Similarly, h^2 is the class of a point. Recalling that the product is induced by intersection, one sees at once that we want to compute the degree

$$\int c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2.$$

Our familiarity with the Chow ring of $\mathbb{P}^2$ is certainly enough to proclaim that this degree is 1.

However, the point here is to illustrate the use of Bott's formula. For this end, what really matters is realizing that, the cycle which solves the present geometric question,

$$c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 \cap [\mathbb{P}^2],$$

can be expressed as a polynomial function on the Chern classes of equivariant vector bundles for a suitable action of our torus $T = \mathbb{C}^*$.

3.1.1 Choice of the action

In practice, it suffices to consider diagonal actions of $T = \mathbb{C}^*$.

This amounts to picking a point (w_0, w_1, w_2) in the free group of weights $\mathbb{Z}^3$. The characters associated to the diagonal action of $\mathbb{C}^*$ are given by $\lambda_i = t^{w_i}$.

The homogeneous coordinates x_0, x_1, x_2 are eigenvectors with weights w_0, w_1, w_2.

That is, the action of T on $\mathbb{P}^2$ is given by

$$t \circ x_i = t^{w_i} \cdot x_i \quad \text{for all} \quad t \in \mathbb{C}^*, \ 0 \leq i \leq 2.$$

The fixed points locus of this $\mathbb{C}^*$-action is given by the system of equations

$$x_0 = t^{w_0} \cdot x_0, \ \ x_1 = t^{w_1} \cdot x_1, \ \ x_2 = t^{w_2} \cdot x_2 \quad \forall t \in \mathbb{C}^*.$$

The set of solutions is simply

$$F = \{[1,0,0],\ [0,1,0],\ [0,0,1]\} \subset \mathbb{P}^2,$$

provided that

the weights w_i's be chosen *all distinct* from each other.

This will be assumed till further notice.

We shall apply at first the version of Bott's formula for the case where the fixed points locus is as above, a finite set F.

The normal space at a fixed point, $\mathcal{N}_P$, is the same as $\mathcal{T}_P\mathbb{P}^2$.

The localization of the equivariant Chow ring of a point (1) is $\mathcal{R}_T = \mathbb{Q}[t, t^{-1}]$. Bott's formula (8) now reads

$$\int_{\mathbb{P}^2} c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 \cap [\mathbb{P}^2] = \sum_{P\in F} \pi_{P*}\left(\frac{\left(c_1^T(\mathcal{O}_{\mathbb{P}^2}(1)_{|P})\right)^2) \cap [P]_T}{c_2(\mathcal{N}_P)} \right).$$

3.1.2 Decomposition into eigensubbundles

The calculation of the right hand side requires knowledge of the explicit decomposition of E restricted to the locus of fixed points as a direct sum of eigensubbundles $\bigoplus_\lambda E^\lambda$.

This must be done for $E = \mathcal{O}(1)$ and $E = \mathcal{T}\mathbb{P}^2$, after restriction to a fixed point.

$\mathcal{O}(1)$ is the line bundle obtained by the quotient of the trivial bundle $\mathcal{F} = \langle x_0, x_1, x_2 \rangle$ of linear forms of $\mathbb{P}^2$, by the subbundle, $\mathcal{A}$, with fiber over a point $P \in \mathbb{P}^2$ given by the space of linear forms that vanish at P. In symbols,

$$\mathcal{O}(1) = \mathcal{F}/\mathcal{A}.$$

Recalling (11), we have

$$\mathcal{T}\mathbb{P}^2 = \mathrm{Hom}(\mathcal{A}, \mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{A}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(1).$$

3.1.3 The Chern classes

Continuing, we must study the weights of the representations induced on the fibers E_P for $E = \mathcal{O}_{\mathbb{P}^2}(1)$ and $E = \mathcal{T}\mathbb{P}^2$, at each fixed point P.

Once this is done, each Chern class $c_k^T(E_P^\lambda)$ will be represented in the $\mathbb{C}^*$-equivariant Chow ring of the point P, by

$$\binom{r}{k}\lambda^k, \text{ for } k \leq r = rk(E^\lambda), \text{ cf. } (2.7.2).$$

At the fixed point $P = [1, 0, 0]$, we have $\mathcal{A}_P = \langle x_1, x_2 \rangle$. Thus,

$$\mathcal{O}_{\mathbb{P}^2}(1)_P = \langle x_0, x_1, x_2 \rangle / \langle x_1, x_2 \rangle = \langle \overline{x_0} \rangle.$$

Here the weight is w_0. Hence, in $A_T^*(P) = R_T = \mathbb{Z}[t]$ we have

$$c_1^T(\mathcal{O}_{\mathbb{P}^2}(1)_P) = w_0 \cdot t.$$

Recalling the trick (10), we forget t and just keep the coefficient w_0.

Meanwhile,

$$\begin{aligned}\mathcal{T}_P\mathbb{P}^2 = \mathcal{A}_P^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(1)_P \quad &= \langle x_1, x_2\rangle^\vee \otimes \langle \overline{x_0}\rangle \\ &= (t^{-w_1} + t^{-w_2}) \cdot t^{w_0} \\ &= t^{w_0 - w_1} + t^{w_0 - w_2},\end{aligned}$$

the decomposition as a direct sum of eigenspaces.

The respective weights are $w_0 - w_1$ and $w_0 - w_2$.

Hence, the class $c_2^T(\mathcal{T}\mathbb{P}^2_P)$ is represented by the weight $(w_0 - w_1)\cdot(w_0 - w_2)$ in the Chow ring $A_*^T(P)$ of the fixed point $P = [1, 0, 0]$.

Similarly, we see that $c_2^T(\mathcal{T}_P\mathbb{P}^2)$ is represented by the weights

$(w_1-w_0)\cdot(w_1-w_2)$ at $[0,1,0]$ and $(w_2-w_0)\cdot(w_2-w_1)$ at $[0,0,1]$.

The weights of $\mathcal{O}_{\mathbb{P}^2}(1)$ at the fixed points $[0,1,0]$ and $[0,0,1]$ are w_1 and w_2.

Finally, we apply Bott's formula and get the incredible identity

$$\int_{\mathbb{P}^2} c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = \sum_{P\in F} \int_{[P]} \frac{c_1^T(\mathcal{O}_{\mathbb{P}^2}(1)_P)^2 \cap [P]_T}{c_2^T(\mathcal{T}\mathbb{P}^2_P)}$$

$$= \frac{w_0^2}{(w_0-w_1).(w_0-w_2)} + \frac{w_1^2}{(w_1-w_0).(w_1-w_2)} + \frac{w_2^2}{(w_2-w_0).(w_2-w_1)} \equiv 1!!!$$

$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$

$$[1,0,0] \qquad\qquad [0,1,0] \qquad\qquad [0,0,1]$$

3.2 Two lines in $\mathbb{P}^2$, bis

We now give an example of a simple application of the formula in the case where the set of fixed points is *infinite*. This actually occurs in our treatment of canonical curves in $\mathbb{P}^3$, as well as in the celebrated work of M. Kontsevich [9]. This is why we think it may be helpful to see how it works in a geometrically easier case, for which the answer is God-given.

Once again we let $T = \mathbb{C}^*$ act diagonally on $\mathbb{P}^2$, now with weights

$$w_0 = w_1 = a,\ w_2 = b \neq a.$$

That is,

$$t \circ x_0 = t^a \cdot x_0,\ t \circ x_1 = t^a \cdot x_1,\ t \circ x_2 = t^b \cdot x_2,\ \text{ for all } t \in \mathbb{C}^*.$$

Hence, the fixed points locus $X^T \subset X$ consists of two components:

- the line ℓ given by $x_2 = 0$, and
- the point $P = [0, 0, 1]$.

In the previous example, we have computed $\int_{\mathbb{P}^2} c_1(\mathcal{O}(1))^2$ in the case where X^T is a finite set. Now, Bott's formula's (8) yields

$$\int_{\mathbb{P}^2} c_1(\mathcal{O}(1))^2 = \int_\ell \frac{c_1^T(\mathcal{O}(1)_\ell)^2 \cap [\ell]_T}{c_1^T(\mathcal{N}_{\ell/\mathbb{P}^2})} + \int_P \frac{c_1^T(\mathcal{O}(1)_P)^2 \cap [P]_T}{c_2^T(\mathcal{T}_P(\mathbb{P}^2))}.$$

For the second term in the sum, no surprises:

$$w_2^2/\big((w_2 - w_0)\cdot(w_2 - w_1)\big) = b^2/(b-a)^2.$$

To find the contribution of the positive dimensional component $\ell \cong \mathbb{P}^1$, we need the values of $c_1^T\left(\mathcal{O}(1)_{|\,\ell}\right)$ and $c_1^T(\mathcal{N}_{\ell/\mathbb{P}^2})$ in the equivariant Chow ring of ℓ. We have $\mathcal{O}_{\mathbb{P}^2}(1)_{|\,\ell} = \mathcal{O}_\ell(1)$, eigenbundle on ℓ of rank 1 with weight a. In view of the lemma 2.2.2, it follows that

$$c_1^T(\mathcal{O}_\ell(1)) = h + at \ \in A_*^T(\ell),$$

where $A_*^T(\ell) = A_*(\ell)\otimes R_T$ with $R_T = \mathbb{Z}[t]$ and $A_*(\ell) = A_*(\mathbb{P}^1) = \mathbb{Z}[h]/\langle h^2\rangle$.

Warning! As to the normal bundle, even though

$$\mathcal{N}_{\ell/\mathbb{P}^2} \text{ and } \mathcal{O}_\ell(1) \textit{ are } \text{isomorphic,}$$

they *are not so as* $T-$ *bundles*: the weight this time for $\mathcal{N}_{\ell/\mathbb{P}^2}$ is $a-b$, not a, as we had in the previous case!

In fact, let us study the natural sequence,

$$\mathcal{T}\ell = \mathcal{T}\mathbb{P}^1 \hookrightarrow \mathcal{T}\mathbb{P}^2_{|\ell} \twoheadrightarrow \mathcal{N}_{\ell/\mathbb{P}^2}.$$

We know that $\mathcal{T}\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2)$. But here the weight is trivial, given that the action on ℓ is trivial.

To determine the weights of the central term, $\mathcal{T}\mathbb{P}^2_{|\ell}$, at each point $Q = [\alpha, \beta, 0] \in \ell$, look at

$$\mathcal{A}_Q = \langle \underbrace{x_2}_{t^b}, \underbrace{\beta x_0 - \alpha x_1}_{t^a} \rangle,$$

the corresponding space of linear forms. Its decomposition is $t^b + t^a$.

The fiber $\mathcal{T}_Q\mathbb{P}^2$ is given by $\mathcal{A}_Q^\vee \otimes \mathcal{F}/\mathcal{A}_Q$.

The decomposition into eigenspaces can now be written as

$$(t^{-b} + t^{-a}) \cdot (2t^a + t^b - t^b - t^a) = t^{a-b} + 1.$$

Discounting the trivial character, 1, which comes from $\mathcal{T}\mathbb{P}^1$, we may conclude that the weight in $\mathcal{N}_{\ell/\mathbb{P}^2}$ is $a - b$. Hence, again by (2.2.2) we have

$$c_1^T(\mathcal{N}_{\ell/\mathbb{P}^2}) = h + (a-b)t.$$

It remains to compute

$$\int_\ell \frac{(h+at)^2}{h+(a-b)t}\,.$$

Recalling that $(a-b)t$ is invertible in $A_*^T(\ell)\otimes\mathcal{R}_T$ and using the fact that $h^2=0$, we may write

$$\big(h+(a-b)t\big)^{-1}=\frac{\big(h-(a-b)t\big)}{\big(-(a-b)^2t^2\big)}\,.$$

This implies

$$\frac{(h+at)^2}{(h+(a-b)t)}=-\,\frac{(h+at)^2(h-(a-b)t)}{(a-b)^2t^2}\,.$$

Collecting the coefficient of h we get

$$\int_\ell\frac{(h+at)^2}{h+(a-b)t}=-\frac{2a(b-a)+a^2}{(b-a)^2}=\frac{a^2-2ab}{(b-a)^2}\,.$$

Finally, enjoy another bizarre manner to find the number 1:

$$\int_{\mathbb{P}^2}c_1(\mathcal{O}(1))^2=\frac{a^2-2ab}{(b-a)^2}+\frac{b^2}{(b-a)^2}=\frac{(b-a)^2}{(b-a)^2}=1.$$

3.3 2 lines meet 4 general other lines in $\mathbb{P}^3$

Fix x_1, x_2, x_3, x_4, homogeneous coordinates in $\mathbb{P}^3$.

Once again $T = \mathbb{C}^*$ acts with weights (w_1, w_2, w_3, w_4), $t \circ x_i = t^{w_i} \cdot x_i$, for all $t \in \mathbb{C}^*$.

Let $\mathcal{F} = \langle x_1, x_2, x_3, x_4 \rangle$ be the trivial vector bundle of linear forms on $\mathbb{P}^3$.

The action now moves on to the Grassmann variety, $Gr(2,4)$, that parameterizes the family of lines in $\mathbb{P}^3$. It carries the tautological sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where the fiber of $\mathcal{A}$ over $\ell \in Gr(2,4)$ is the bidimensional subspace of linear forms that vanish on the line ℓ.

3.3.1 Lines meeting a given line in $\mathbb{P}^3$

Say $\ell_0 := x_1 = x_2 = 0$. A line ℓ meets ℓ_0 if and only if the two subspaces $\mathcal{A}_{\ell_0} = \langle x_1, x_2 \rangle, \mathcal{A}_{\ell_0} \subset \mathcal{F}$ have a non zero intersection. This is the same as asking that the slant arrow below

$$\begin{array}{ccc} \mathcal{A} & \hookrightarrow & \mathcal{F} \\ & s \searrow & \downarrow \\ & & \mathcal{F}/\langle x_1, x_2 \rangle \end{array}$$

be not injective.

Thus, the set of lines incident to ℓ_0 is the zeros of the induced section,

$$\overset{2}{\wedge} s : \overset{2}{\wedge} \mathcal{A} \to \overset{2}{\wedge} \left(\mathcal{F}/\langle x_1, x_2 \rangle \right) = \mathcal{O} \text{ or, equivalently, } \mathcal{O} \to \overset{2}{\wedge} \mathcal{A}^\vee.$$

By general principles, setting $X = Gr(2,4)$, it follows that the solution is given by the top self-intersection,

$$\int_X (c_1(\mathcal{A}^\vee))^4 \cap [X].$$

This can be computed of course working directly in the not so complicated Chow ring of the grassmannian. We choose nevertheless to go on and apply mechanically Bott's formula.

3.3.2 Act on the grassmannian $Gr(2,4)$

We study the induced action on $Gr(2,4)$.

The maps $\mathcal{A} \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{Q}$ are equivariant.

Each point $\ell \in Gr(2,4)$ corresponds to a 2–dimensional subspace $\mathcal{A}_\ell \subset \mathcal{F}$. Explicitly,

$$\begin{aligned} A_\ell = \quad & \langle a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4, \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \rangle . \end{aligned}$$

The induced action, $t \circ \ell$, is computed by simply plugging in

$$x_i = t^{w_i} x_i .$$

3.3.3 Detect fixpoints

A fixed point $\ell \in Gr(2,4)$ is detected by the requirement that the subspace remain the same. That is, the matrix below must be of rank 2,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11}t^{w_1} & a_{12}t^{w_2} & a_{13}t^{w_3} & a_{14}t^{w_4} \\ a_{21}t^{w_1} & a_{22}t^{w_2} & a_{23}t^{w_3} & a_{24}t^{w_4} \end{pmatrix}.$$

Choosing the weights w_i's "sufficiently" distinct (so that $(\forall\, i<j, i'<j')$ if $(i,j) \neq (i',j')$ then $w_i + w_j \neq w_{i'} + w_{j'}$, e.g., 1,2,3,5),
the set of fixed points $F \subset Gr(2,4)$ is found to be the finite set

$$F = \{\langle x_1, x_2\rangle, \langle x_1, x_3\rangle, \langle x_1, x_4\rangle, \langle x_2, x_3\rangle, \langle x_2, x_4\rangle, \langle x_3, x_4\rangle\}$$

where $\langle x_i, x_j\rangle$ represents the line given by $x_i = x_j = 0$. (You may look at Plücker coordinates of ℓ.)

3.3.4 Find the weights

Let us study the weights of the induced representations on the fibers $\mathcal{A}$ and $\mathcal{T}_\ell Gr(2,4)$ for each fixed point $\ell \in F$.

We recall the identification of the tangent space cf. [6],

$$\mathcal{T}Gr(2,4) = \mathcal{A}^\vee \otimes \mathcal{Q}.$$

On the fiber over the fixed point $\ell = \langle x_1, x_2 \rangle = t^{w_1} + t^{w_2}$, we have

$$\begin{aligned} \mathcal{A} &= \langle x_1, x_2 \rangle, \\ \mathcal{T}_\ell Gr(2,4) &= \langle x_1, x_2 \rangle^\vee \otimes (\mathcal{F}/\langle x_1, x_2 \rangle) \\ &= (t^{-w_1} + t^{-w_2})(t^{w_3} + t^{w_4}). \end{aligned}$$

The latter space is generated by the eigenvectors

$$x_3 \otimes x_1^\vee, x_3 \otimes x_2^\vee, x_4 \otimes x_1^\vee, x_4 \otimes x_2^\vee$$

whose weights are equal to

$$w_3 - w_1, w_3 - w_2, w_4 - w_1, w_4 - w_2$$

respectively (abusing notation). The equivariant Chern class $c_4^T(\mathcal{T}_\ell Gr(2,4))$ is therefore represented by the weight

$$(w_3 - w_1) \cdot (w_3 - w_2) \cdot (w_4 - w_1) \cdot (w_4 - w_2)$$

in the $\mathbb{C}^*$-equivariant Chow ring of the fixed point $\ell = \langle x_1, x_2 \rangle \in Gr(2,4)$. We also have

$$c_1^T(\mathcal{A}_{|\langle x_1, x_2 \rangle}) = (w_1 + w_2)t \in A_T^*(\cdot) = \mathbb{Z}[t].$$

Putting it all together, we get in the right hand side of (8),

$$
\frac{(w_1+w_2)^4}{(w_3-w_1)(w_3-w_2)(w_4-w_1)(w_4-w_2)} + \frac{(w_1+w_3)^4}{(w_2-w_1)(w_2-w_3)(w_4-w_1)(w_4-w_3)}
$$

$$
+ \frac{(w_1+w_4)^4}{(w_2-w_1)(w_2-w_4)(w_3-w_1)(w_3-w_4)} + \frac{(w_2+w_3)^4}{(w_1-w_2)(w_1-w_3)(w_4-w_2)(w_4-w_3)}
$$

$$
+ \frac{(w_2+w_4)^4}{(w_1-w_2)(w_1-w_4)(w_3-w_2)(w_3-w_4)} + \frac{(w_3+w_4)^4}{(w_1-w_3)(w_1-w_4)(w_2-w_3)(w_2-w_4)}
$$

$$
\equiv 2!!!
$$

3.4 The 27 lines on a cubic surface

The next educational example will be the calculation of the number of lines contained in a general cubic surface $S \subset \mathbb{P}^3$.

Each cubic surface $S \subset \mathbb{P}^3$ is given as zeros of a section of the trivial 3rd symmetric power bundle, $\mathcal{S}_3\mathcal{F}$.

Composing with the quotient map $\mathcal{S}_3\mathcal{F} \twoheadrightarrow \mathcal{S}_3\mathcal{Q}$, we get a section $s : \mathcal{O} \to \mathcal{S}_3\mathcal{Q}$ over $Gr(2, \mathcal{F})$.

It can be easily checked that, for each line $\ell \in Gr(2, \mathcal{F})$, the section s vanishes in the fiber $\mathcal{S}_3\mathcal{Q}_\ell$ if and only if the surface S contains the line ℓ. We see that the cycle of the sought for locus in $Gr(2, 4)$ is given by the top Chern class of the bundle $\mathcal{S}_3\mathcal{Q}$. That is, the number we are after is the degree

$$\int_{Gr(2,4)} c_4(\mathcal{S}_3\mathcal{Q}).$$

We proceed to compute it using Bott's formula.

3.4.1 Find more weights

Let us study the weights of the induced representations on the fibers $\mathcal{S}_3\mathcal{Q}_\ell$ for each fixed point $\ell \in F$ (done already for $\mathcal{T}_\ell Gr(2,4)$).

The fiber of $\mathcal{S}_3\mathcal{Q}$ say over the the fixed point $\ell = \langle x_1, x_2 \rangle$ is the quotient space of $\mathcal{S}_3\mathcal{F}$ by the subspace of forms generated by x_1, x_2. It is generated by the classes

$$\bar{x}_3^3,\ \bar{x}_3^2 \cdot \bar{x}_4,\ \bar{x}_3 \cdot \bar{x}_4^2,\ \bar{x}_4^3$$

with respective weights $3w_3$, $2w_3 + w_4$, $w_3 + 2w_4$, $3w_4$.

Hence, the top Chern class $c_4^T(\mathcal{S}_3\mathcal{Q}_\ell)$ is represented by the weight

$$3w_3 \cdot (2w_3 + w_4) \cdot (w_3 + 2w_4) \cdot 3w_4$$

in the $\mathbb{C}^*$-equivariant Chow ring of the fixed point $\ell \in Gr(2,4)$.

3.4.2 Twenty seven

$$\int_{Gr(2,4)} c_4(\mathcal{S}_3\mathcal{Q}) = \sum_{\ell\in F}\int_{[l]} \frac{c_4^T(\mathcal{S}_3\mathcal{Q}_{|\,\ell})}{c_4^T(\mathcal{T}_\ell Gr(2,4))}$$

$$= 9\frac{w_0\cdot(2w_0+w_1)\cdot(w_0+2w_1)\cdot w_1}{(w_0-w_2)\cdot(w_0-w_3)\cdot(w_1-w_2)\cdot(w_1-w_3)}$$

$$+9\frac{w_0\cdot(2w_0+w_2)\cdot(w_0+2w_2)\cdot w_2}{(w_0-w_1)\cdot(w_0-w_3)\cdot(w_2-w_1)\cdot(w_2-w_3)}$$

$$+9\frac{w_0\cdot(2w_0+w_3)\cdot(w_0+2w_3)\cdot w_3}{(w_0-w_1)\cdot(w_0-w_2)\cdot(w_3-w_0)\cdot(w_3-w_2)}$$

$$+9\frac{w_1\cdot(2w_1+w_2)\cdot(w_1+2w_2)\cdot w_2}{(w_1-w_0)\cdot(w_1-w_3)\cdot(w_2-w_0)\cdot(w_2-w_3)}$$

$$+9\frac{w_1\cdot(2w_1+w_3)\cdot(w_1+2w_3)\cdot w_3}{(w_1-w_0)\cdot(w_1-w_2)\cdot(w_3-w_0)\cdot(w_3-w_2)}$$

$$+9\frac{w_2\cdot(2w_2+w_3)\cdot(w_2+2w_3)\cdot w_3}{(w_2-w_0)\cdot(w_2-w_1)\cdot(w_3-w_0)\cdot(w_3-w_1)} = 27.(!!!\,{}^{(\star}_{(\star}-)$$

4 Enumerative applications, II

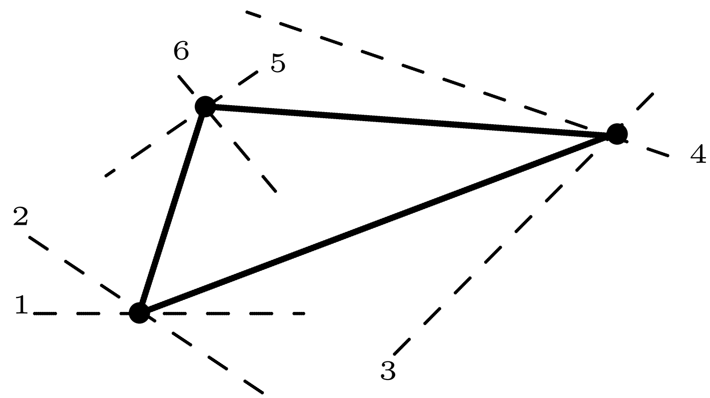
We move on to a slightly harder situation, inasmuch as the parameter space requires a little more care.

It will serve us as a guiding tool to more sophisticated cases.

4.1 Triangles

Question: how many triangles are there with its set of vertices meeting six general lines?

A triangle is determined by its set of vertices, hence it depends on ∞^6 parameters, two for each vertex.



Of course there is an easy combinatorial answer: pick 2 of the lines to meet on a vertex, 2 more on a second, hence the 3rd is determined by the remaining pair of lines: find **15**$= \binom{6}{2} \cdot \binom{4}{2}$ divided by 3!. (If you pick first lines 1,2, then 3,4, you get the same triangle as if had first picked 3,4, then 1,2. . .)

4.1.1 Choice of parameter space

Our interest here is to test the answer against the use of Bott's machinery on an appropriate parameter space.

There are several many ways to produce a parameter space for the family of triangles. The most naïve is to start with $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and take care of the diagonals in order to handle the unavoidable degenerate configurations. Of course with this approach, each honest triangle appears 3! times.

A (lot) more educated way is to work with $\text{Hilb}^3\mathbb{P}^2$, the Hilbert scheme. Its points correspond to subschemes of $\mathbb{P}^2$ of degree 3 and dimension zero. Such a subscheme is essentially the same as a homogeneous ideal $I \subset \mathbb{C}[x_0, x_1, x_2]$ such that for all sufficiently high degree d the homogeneous piece $I_d \subset \mathcal{F}_d$ is a subspace of codimension 3. Among such are the honest triangles, e.g.,

$$\langle x_0x_1, x_0x_2, x_2x_1\rangle,$$

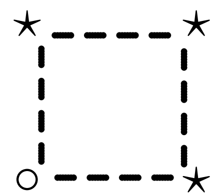
the slightly spurious $\langle x_0^2, x_0x_1, (x_1 - x_2)x_1\rangle$, or even $\langle x_0, x_1^3\rangle$ as well as

the truly outrageous $\langle x_0^2, x_0x_1, x_1^2\rangle$.

4.1.2 Sidetrack

We take a sidetrack. Namely, use the parameter space suggested by Fernando Xavier's thesis [15],[14]; see also [13].

Start by completing a general triangle to a 4–gon: simply add a point $0 := [0,0,1] \in \mathbb{P}^2$, fixed once and for all:



$\star$'s triangle with an extra vertex

Now the 4–gon can be described as the base locus of a pencil of conics $\ni 0$, e.g.,

$$\begin{cases} q_1 := x_0^2 + a_1 x_0x_2 + a_2x_1^2 + a_3x_1x_2, \\ q_2 := x_0x_1 + a_4x_0x_2 + a_5x_1^2 + a_6x_1x_2. \end{cases}$$

4.1.3 Other grass

More precisely, let

$$X = Gr(2, \mathcal{F}_2^0)$$

be the grassmannian of pencils of conics passing through 0.

A general point of X corresponds to a 4-gon with a vertex on 0, hence to a general triangle. The 6 coefficients $a_1, \ldots, a_6$ are local coordinates on X around $\langle x_0^2, x_0x_1 \rangle$.

Of course this pencil has too big a base locus: no 4-gon, nor triangle in sight. We proceed to modify X so that the open set corresponding to honest triangles (plus the attached vertex 0) will fit in a new compactification with a more interesting boundary.

4.1.4 triangle = 4–gon minus 1–gon

The first step is to understand how to get rid of the auxiliary vertex. This done, at least generically, by defining a new quadric q_3 containing the vertices but, in general, not 0. For this, write

$$\begin{aligned} q_1 &:= \alpha_{11}x_0 + \alpha_{12}x_1, \\ q_2 &:= \alpha_{21}x_0 + \alpha_{22}x_1 \end{aligned}$$

for suitable linear forms a_{ij}. By Cramer's rule, a zero of q_1, q_2 with nonzero x_0 or x_1 requires

$$q_3 := \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = 0. \tag{12}$$

4.1.5 First blowup: X'

We have defined a rational map,

$$\begin{array}{ccc} X & \dashrightarrow & Gr(3,\mathcal{F}_2) \\ \langle q_1, q_2\rangle & \mapsto & \langle q_1, q_2, q_3\rangle. \end{array}$$

4.1 Proposition. *The closure $X' \subset X \times Gr(3,\mathcal{F}_2)$ of the graph of the map above is isomorphic to the blowup of X along the subvariety*

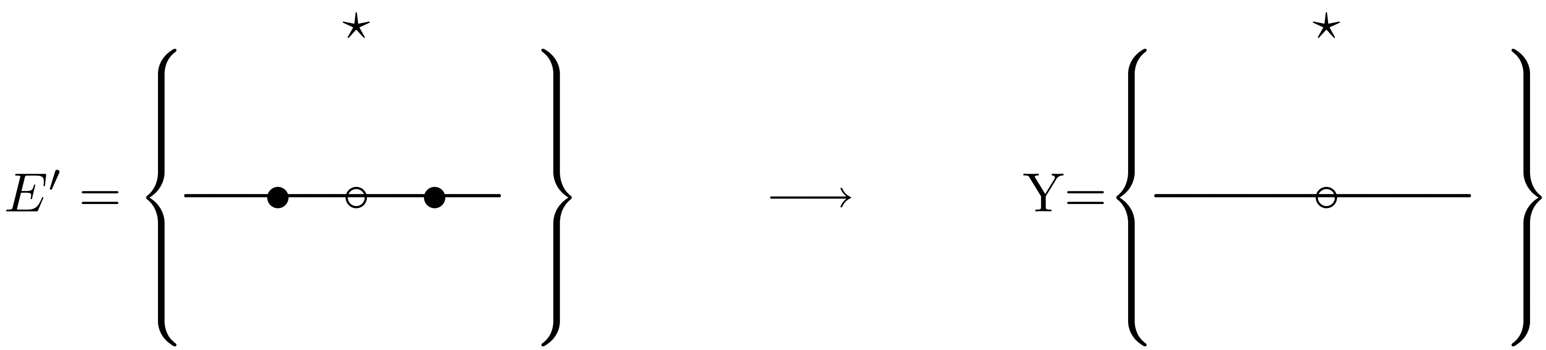
$$\begin{array}{cccc} Y = & \check{\mathbb{P}}^1_0 \times \mathbb{P}^2 & \hookrightarrow & X \\ & (\lambda_0, \langle \lambda_1, \lambda_2\rangle) & \mapsto & \langle \lambda_0\lambda_1, \lambda_0\lambda_2\rangle \end{array}$$

where $\check{\mathbb{P}}^1_0$ stands for the pencil of lines through 0.

Moreover, a general point on the exceptional divisor E' corresponds to triangles two vertices of which are aligned with the special point 0. The fiber of $E' \to Y$ over $(\lambda_0, \langle\lambda_1, \lambda_2\rangle) \in Y$ is

$$\mathbb{P}\left(\mathcal{F}_2^{\langle\lambda_1,\lambda_2\rangle}/(\lambda_0 \cdot \langle\lambda_1, \lambda_2\rangle)\right). \tag{13}$$

Here $\mathcal{F}_2^{\langle\lambda_1,\lambda_2\rangle}$ is the space of conics through $\star = \langle\lambda_1, \lambda_2\rangle$.



The map $E' \to Gr(3, \mathcal{F}_2)$ is given by

$$\mathbb{P}\left(\mathcal{F}_2^{\langle\lambda_1,\lambda_2\rangle}/(\lambda_0 \cdot \langle\lambda_1, \lambda_2\rangle)\right) \ni \bar{q}_3 \mapsto \langle\lambda_0\lambda_1, \lambda_0\lambda_2, q_3\rangle.$$

4.1.6 Produce cubics

Our next task is to produce cubic equations, certainly needed whenever the vertices become aligned.

We do this stepwise.

At first, we ignore the 3rd quadric and just look at the cubics arising from $\langle q_1, q_2 \rangle \cdot \mathcal{F}$.

Precisely, denoting by $\mathcal{A}$ the tautological rank 2 subbundle of $\mathcal{F}_2^0$ over X, form the multiplication map,

$$\mu : \mathcal{A} \otimes \mathcal{F} \longrightarrow \mathcal{F}_3^0. \tag{14}$$

4.1.7 2nd blowup center Y'

4.2 Proposition. *The generic rank of the pull back* $\mu_{X'}$ *is 6. Its rank drops to 5 on the scheme union of the exceptional divisor* E' *and a subscheme* $Y' \subset X'$*, such that* $Y' \cong \check{\mathbb{P}}^2$*. The embedding*

$$\check{\mathbb{P}}^2 \cong Y' \subset X \times Gr(3, \mathcal{F}_2)$$

is defined by

$$\ell \mapsto (\ell \cdot 0^\vee, \ell \cdot \mathcal{F}),$$

where $0^\vee = \langle x_0, x_1 \rangle$ *denotes the pencil of lines through* 0*. The intersection* $E' \cap Y'$ *is transversal, equal to the image of* $\check{\mathbb{P}}^1_0$*.*

In coordinates, the above means the following. Represent $\mu_{X'}$ (14) by a 6×9–matrix; perform elementary row operations. Eventually we get a new matrix,

$$\mu' = \begin{bmatrix} I_5 & \star \\ 0 & \rho' \end{bmatrix}$$

with 5 pivots and a row, say the bottom one, which is divisible by the local equation of E'.

Dividing, we get a new row. This row represents a cubic in $\mathcal{F}_3^0$. It vanishes identically along Y'. The description of Y' is easily uncovered by substituting the local equations (given by the entries of ρ') into the conics q_1, q_2, q_3.

4.1.8 The exceptional divisor $E'' \to Y'$

4.3 Proposition. *Let $X'' \subset X' \times Gr(6, \mathcal{F}_3^0)$ be the closure of the graph of the rational map $X' \dashrightarrow Gr(6, \mathcal{F}_3^0)$ induced by μ. Then X'' is isomorphic to the blowup of X' along Y' (4.2). The exceptional divisor $E'' \to Y'$ is the projectivization of the vector bundle with fiber over $\ell \in \check{\mathbb{P}}^2 \cong Y'$ the vector space $\mathcal{F}_3^0 / \left(\ell \cdot \mathcal{F}_2^0\right)$ of cubics vanishing at 0 modulo that of conics.*

$$E''_\ell = \mathbb{P}\left(\mathcal{F}_3^0 / \left(\ell \cdot \mathcal{F}_2^0\right)\right) \tag{15}$$

A general point of E'' corresponds to a choice of three points on a line.

The final step produces one additional cubic equation.

Precisely, we have the following.

4.1.9 Last blowup center Y''

4.4 Proposition. *Let $\mathcal{B}$ denote the rank 3 bundle/X', pullback of the tautological subbundle of $Gr(3,6)$ via (4.1).*

Consider the multiplication map,

$$\nu : \mathcal{B} \otimes \mathcal{F} \longrightarrow \mathcal{F}_3. \tag{16}$$

Let $\mathcal{M} \subset (\mathcal{F}_3)_{|X''}$ be the saturation of the image of the pullback $\nu_{|X''}$. Then the following holds:

1. *the generic rank of $\mathcal{M}$ is 7;*
2. *the Fitting subscheme $Y'' \subset X''$ defining the locus where $\mathcal{F}_3/\mathcal{M}$ is of rank ≥ 4 is isomorphic to the blowup of $\mathbb{P}^2$ at 0.*

Any point of Y'' is given by a a pair (ℓ, P) such that $P \in \mathbb{P}^2$ lies on the line $\ell \ni 0$. The embedding

$$Y'' \subset X'' \subset X' \times Gr(6, \mathcal{F}_3^0) \subset X \times Gr(3, \mathcal{F}_2) \times Gr(6, \mathcal{F}_3^0) \qquad (17)$$

sends

$$(\ell, P) \in Y'' \subset \check{\mathbb{P}}_0^1 \times \mathbb{P}^2 \qquad \text{to} \qquad (\ell \cdot P^\vee, \ell \cdot \mathcal{F}, \ell \cdot \mathcal{F}_2).$$

4.1.10 The parameter space of triangles X'''

4.5 Proposition. *Let $X''' \subset X'' \times Gr(7, \mathcal{F}_3)$ be the closure of the graph of the rational map $X'' \to Gr(7, \mathcal{F}_3)$ induced by ν. Then we have that X''' is isomorphic to the blowup of X'' along Y''. A point x''' on the exceptional divisor E''' corresponds to a choice of a point P on a line $\ell \ni 0$ together with a degree 3 divisor thereon. The degenerate triangle assigned to such $\mathrm{x}''' \in E'''$ is defined by that degree 3 divisor on ℓ.*

Summarizing, our parameter space for the family of triangles is a subvariety

$$X''' \subset Gr(2, \mathcal{F}_2^0) \times Gr(3, \mathcal{F}_2) \times Gr(6, \mathcal{F}_3^0) \times Gr(7, \mathcal{F}_3).$$

4.1.11 Points of X''' are...

A point of X''' is given by a 4-tuple

$$(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) \in Gr(2, \mathcal{F}_2^0) \times Gr(3, \mathcal{F}_2) \times Gr(6, \mathcal{F}_3^0) \times Gr(7, \mathcal{F}_3)$$

such that

$$\begin{cases} \mathrm{x}_1 \text{ is a pencil of conics through } 0, \\ \mathrm{x}_2 \text{ is a net of conics through a triangle,} \\ \mathrm{x}_3 \text{ is a codimension 4 system of cubics through a 4-gon} \\ \quad \text{containing the triangle,} \\ \mathrm{x}_4 \text{ is a codimension 3 system of cubics through the triangle.} \end{cases}$$

4.1.12 Incidence to a line: D'''_ℓ

For each line $\ell \not\ni 0$, the variety X''' contains a hypersurface D'''_ℓ defined by the condition that the base locus of each of the 4 linear systems meet ℓ.

The hypersurface D'''_ℓ is the strict transform in X''' of a hypersurface $D_\ell \subset X$ defined by the condition $\ell \cap q_1 \cap q_2 \neq \emptyset$.

D_ℓ is in turn the image of $\widetilde{D}_\ell \subset \ell \times X$,

$$\widetilde{D}_\ell = \{(P, \pi) \in \ell \times X \mid P \in \text{base locus of pencil } \pi\}.$$

4.1.13 Class of D_ℓ

Studying the diagram of natural maps of vector bundles over $\ell \times X$,

$$\begin{array}{ccc} \mathcal{A} & \hookrightarrow & \mathcal{F}_2^0 \\ & \searrow & \downarrow \\ & & \mathcal{O}(2) \end{array}$$

we see that

the slant arrow vanish at $(P, \pi) \in \ell \times X$
if and only if $\widetilde{D}_\ell \ni (P, \pi)$.

Hence we have

$$\widetilde{D}_\ell = c_2(\mathcal{A}(-2)) = c_2(\mathcal{A}) - 2hc_1(\mathcal{A}),$$

where $h = c_1 \mathcal{O}_\ell(1)$. It follows that

$$D_\ell = -2c_1(\mathcal{A}).$$

A local coordinates check shows that D_ℓ contains the blowup center Y (4.1) with multiplicity one.

Hence the formula for its strict transform in X',

$$D'_\ell = -2c_1(\mathcal{A}) - E'.$$

Similarly, (abusing notation, omitting pullbacks) we get for the succeeding strict transforms,

$$D''_\ell = -2c_1(\mathcal{A}) - E' - E'',$$

$$D'''_\ell = -2c_1(\mathcal{A}) - E' - E'' - E'''.$$

Now we are asked to compute the self intersection $(D'''_\ell)^6$.

This is what we intend to do via Bott's formula, anticipating <u>*finite*</u> fixpts

$$\int (D'''_\ell)^6 = \sum_F \int_{[F]} \frac{\left(2c_1^T(\mathcal{A}_F) + E' + E'' + E'''\right)^6 \cap [F]_T}{c_6^T(\mathcal{T}X'''_F)}. \qquad (18)$$

4.1.14 Fixpoints on $X, \ldots, X'''$

We have to find the contributions of fixed points for a suitable torus action, starting at X and following all the way up to X'''.

- at $X = Gr(2, \mathcal{F}_2^0)$. If the weights are sufficiently generic, the Plücker embedding shows that the fixed points are the 10 monomial pencils,

$$\begin{gathered}\langle x_0^2, x_0x_2\rangle, \langle x_0^2, x_0x_1\rangle, \langle x_0x_1, x_0x_2\rangle,\\ \langle x_0x_1, x_1x_2\rangle, \langle x_0x_1, x_1^2\rangle, \langle x_1^2, x_1x_2\rangle,\\ \langle x_0x_2, x_1x_2\rangle,\\ \langle x_0x_2, x_1^2\rangle, \langle x_0^2, x_1^2\rangle, \langle x_0^2, x_1x_2\rangle\end{gathered} \tag{19}$$

The first 6, $\pi_1 = \langle x_0^2, x_0x_2\rangle, \ldots, \pi_6 = \langle x_1^2, x_1x_2\rangle$ have a fixed line through 0 and therefore lie on Y cf. (4.1).

The seventh, $\langle x_0x_2, x_1x_2\rangle$, lies in the (isomorphic) image of Y' (4.2) down in X.

The last three lie off Y (as well as off the images of the other blowup centers, Y' (4.2), Y'' (4.1.9)). They lift (isomorphically) all the way up to X'''

So their contribution can be obtained at once, down on X. The exceptional divisors give no contribution here. On the numerator of (18) we have for $2c_1^T(\mathcal{A}_{|F})$,

$$\begin{aligned}
\pi_8 := \langle x_0x_2, x_1^2\rangle &\quad\leftrightarrow\quad 2(w_0 + w_2 + 2w_1),\\
\pi_9 := \langle x_0^2, x_1^2\rangle &\quad\leftrightarrow\quad 2(2w_0 + 2w_1),\\
\pi_{10} := \langle x_0^2, x_1x_2\rangle &\quad\leftrightarrow\quad 2(2w_0 + w_1 + w_2).
\end{aligned}$$

On the denominator of (18) we get

$$\mathcal{N}_{F/X} = \mathcal{T}X'''_F = \mathcal{T}X_F = \left(\mathcal{F}_2^0/\mathcal{A}_{|F}\right) \otimes \mathcal{A}_{|F}^\vee.$$

Eigen-decomposing as in 2.7.1, we have

$$\begin{aligned} \mathcal{F}_2^0 &= \left(\sum_{0\leq i\leq j\leq 2} t^{w_i+w_j}\right) - t^{w_1+w_2} \\ &= t^{w_0+w_1} + t^{w_0+w_2} + t^{w_1+w_2} + t^{2w_0} + t^{2w_1}. \end{aligned}$$

Hence,

$$F = \langle x_0x_2, x_1^2\rangle \rightsquigarrow c_6^T(\mathcal{T}X_F) =$$

$$\left(t^{w_0+w_1} + t^{w_1+w_2} + t^{2w_0}\right)\left(t^{-(w_0+w_2)} + t^{-2w_1}\right) \rightsquigarrow$$

$$(w_1 - w_2)(w_1 - w_0)(w_0 - w_2)(w_0 - w_1)(w_2 - w_1)(2w_0 - 2w_1),$$

$$F = \langle x_0^2, x_1^2\rangle \rightsquigarrow c_6^T(\mathcal{T}X_F) =$$

$$\left(t^{w_0+w_1} + t^{w_0+w_2} + t^{w_1+w_2}\right)\left(t^{-2w_0} + t^{-2w_1}\right) \rightsquigarrow$$

$$(w_1 - w_0)(w_0 - w_1)(w_2 - w_0)(w_0 + w_2 - 2w_1)(w_2 + w_1 - 2w_0)(w_2 - w_1),$$

$$F = \langle x_0^2, x_1x_2\rangle \rightsquigarrow c_6^T(\mathcal{T}X_F) =$$

$$\left(t^{w_0+w_1} + t^{w_0+w_2} + t^{2w_1}\right)\left(t^{-2w_0} + t^{-w_1-w_2}\right) \rightsquigarrow$$

$$(w_1 - w_0)(w_0 - w_2)(w_2 - w_0)(w_0 - w_1)(2w_1 - 2w_0)(w_1 - w_2).$$

We collect the first three contributions:

$$\frac{(2(w_0+w_2+2w_1))^6}{(w_1-w_2)(w_1-w_0)(w_0-w_2)(w_0-w_1)(w_2-w_1)(2w_0-2w_1)}+$$

$$\frac{(2(2w_0+2w_1))^6}{(w_1-w_0)(w_0-w_1)(w_2-w_0)(w_0+w_2-2w_1)(w_2+w_1-2w_0)(w_2-w_1)}+$$

$$\frac{(2(2w_0+w_1+w_2))^6}{(w_1-w_0)(w_0-w_2)(w_2-w_0)(w_0-w_1)(2w_1-2w_0)(w_1-w_2)}+$$

more to come... (**??**)

Consider now the fiber of E' (4.1),(13) over, say $\pi_1 = \langle x_0^2, x_0x_2 \rangle$.

According to (13), it is just

$$\mathbb{P}^2 \cong \mathbb{P}\left(\langle [x_0x_1], [x_1x_2], [x_2^2] \rangle\right), \tag{20}$$

where [ ] indicates classes mod. $\langle x_0^2, x_0x_2 \rangle$. A quick verification reveals that the induced action has the obvious "unit" fixed points again, namely,

$$\pi'_{11} = [x_0x_1], \pi'_{12} = [x_1x_2], \pi'_{13} = [x_2^2]. \tag{21}$$

None lies on Y'(4.2), because $\langle x_0^2, x_0x_2 \rangle$ is not of the form $\ell \cdot 0^\vee$. Likewise, π'_{12}, π'_{13} are both outside the image of Y'' (4.1.9) (since the corresponding nets, $\langle x_0^2, x_0x_2, x_1x_2 \rangle$ and $\langle x_0^2, x_0x_2, x_2^2 \rangle$, are not of the form $\ell \cdot \mathcal{F}$). Only π'_{11}, with net $\langle x_0^2, x_0x_2, x_0x_1 \rangle$, sits in the image of Y''.

Hence π'_{12}, π'_{13} give immediate contribution for (18).

As to π'_{11}, we postpone its examination to (**??**).

Let us handle the contributions of π'_{12}, π'_{13} explicitly. Recall that the exceptional divisor $E' \to Y$ is the projective bundle $\mathbb{P}\left(\mathcal{N}_{Y/X}\right)$.

Here we point out a subtlety. We have in one hand the geometric description (13). It yields the normal bundle $\mathcal{N}_{Y/X}$ only up to a twist by a line bundle on Y. In particular, it does not reveal the induced action neither on $\mathcal{T}E'$ nor on $\mathcal{T}X'_{E'}$.

The exact sequence of representations

$$0 \to \mathcal{T}Y_{\pi_1} \longrightarrow \mathcal{T}X_{\pi_1} \longrightarrow (\mathcal{N}_{Y/X})_{\pi_1} \to 0$$

splits. Recalling the formulas for the tangent space to a grassmannian, we may write the decomposition into eigenspaces,

$$\begin{aligned} (\mathcal{N}_{Y/X})_{\pi_1} &= \mathcal{T}X_{\pi_1} - \mathcal{T}Y_{\pi_1} \\ &= (x_0^2 + x_0x_2)^\vee \otimes \left(\mathcal{F}_2^0 - (x_0^2 + x_0x_2)\right) - \mathcal{T}Y_{\pi_1} \\ &= \tfrac{x_1^2}{x_0^2} + \tfrac{x_1^2}{x_2x_0} + \tfrac{x_2x_1}{x_0^2}. \end{aligned}$$

Our task now is to reconcile the two descriptions,

$$\mathbb{P}\left((\mathcal{N}_{Y/X})_{\pi_1}\right) = \mathbb{P}\left(\langle[x_0x_1],[x_1x_2],[x_2^2]\rangle\right), \text{ cf. } (20)$$

and

$$(\mathcal{N}_{Y/X})_{\pi_1} = \langle\frac{x_1^2}{x_0^2} + \frac{x_1^2}{x_2x_0} + \frac{x_2x_1}{x_0^2}\rangle.$$

For this, we recall how a normal vector such as that represented by, say, $\eta = x_1^2/x_0^2$, gives rise to a point on the exceptional divisor.

First lift η to a tangent vector, still written η, in $\mathcal{T}X_{\pi_1}$. Find that η can be expressed as

$$(x_0^2)^\vee \otimes x_1^2.$$

By this we mean

$$\eta \in \mathcal{T}X_{\pi_1} \;=\; \mathrm{Hom}(\langle x_0^2, x_0x_2\rangle, \mathcal{F}_2^0/\langle x_0^2, x_0x_2\rangle)$$

such that

$$\eta(x_0^2) = x_1^2,\;\; \eta(x_0x_2) = 0.$$

We use this tangent vector to construct the curve with tangent η at $t = 0$, given by

$$\gamma_t = \langle x_0^2 + tx_1^2, x_0x_2\rangle.$$

Since $\gamma_t \in X \setminus Y$ for $t \neq 0$, it lifts to a curve in X', namely, using (12),

$$\gamma_t' = \langle x_0^2 + tx_1^2, x_0x_2, tx_1x_2\rangle \in Gr(3, \mathcal{F}_2).$$

Letting $t \to 0$ we find the limit point $\langle x_0^2, x_0x_2, x_1x_2\rangle \in Gr(3, \mathcal{F}_2)$.

Thus the normal direction corresponding to $\eta = x_1^2/x_0^2$ gives $\pi'_{12} = [x_1 x_2]$:

$$x_1^2/x_0^2 \leftrightarrow \langle x_0^2, x_0x_2, x_1x_2\rangle = [x_1x_2] = \pi'_{12}. \qquad (22)$$

Similarly, we have

$$x_1^2/(x_2x_0) \leftrightarrow \langle x_0^2, x_0x_2, x_0x_1\rangle = [x_0x_1] = \pi'_{11} \qquad (23)$$

and

$$x_1x_2/x_0^2 \leftrightarrow \langle x_0^2, x_0x_2, x_2^2\rangle = [x_2^2] = \pi'_{13}. \qquad (24)$$

Also

$$\mathcal{O}(E')_{E'} = \mathcal{O}(-1)_{\mathbb{P}(\mathcal{N}_{Y/X})}.$$

The fiber over $\pi'_{12} = [x_1 x_2] \leftrightarrow x_1^2/x_0^2$ (22) is of course, (tautologically!)

$$(\mathcal{O}(E')_{E'})_{\langle x_1 x_2 \rangle} = x_1^2/x_0^2 = t^{2w_1 - 2w_0}.$$

As to the tangent space at any $\mathrm{x}' \in E'$ lying over $\mathrm{x} \in Y = \check{\mathbb{P}}_0^1 \times \mathbb{P}^2$, we know that

$$\begin{aligned} \mathcal{T}_{\mathrm{x}'} X' &= \mathcal{T}_{\mathrm{x}'} E' + (\mathcal{N}_{E'/X'})_{\mathrm{x}'} \\ &= \mathcal{T}_{\mathrm{x}'}(E'/Y) + \mathcal{T}_{\mathrm{x}} Y + \langle \mathrm{x}' \rangle \\ &= \mathcal{T}_{\mathrm{x}'} \mathbb{P}\left((\mathcal{N}_{Y/X})_{\mathrm{x}}\right) + \mathcal{T}_{\mathrm{x}}(\check{\mathbb{P}}_0^1 \times \mathbb{P}^2) + \langle \mathrm{x}' \rangle. \end{aligned}$$

At $\pi'_{11} = [x_0x_1] \leftrightarrow \frac{x_1^2}{x_0x_2}$, (22) which lies over

$$\pi_1 = \langle x_0^2, x_0x_2\rangle = (x_0, \langle x_0, x_2\rangle) \in \check{\mathbb{P}}_0^1 \times \mathbb{P}^2,$$

we find,

$$\mathcal{T}_{\mathrm{x}'}X' = \langle \mathrm{x}'\rangle^\vee \otimes \left(\langle \tfrac{x_1^2}{x_0^2} + \tfrac{x_1^2}{x_2x_0} + \tfrac{x_2x_1}{x_0^2}\rangle / \langle \mathrm{x}'\rangle \right) + \mathcal{T}_{\mathrm{x}}(\check{\mathbb{P}}_0^1 \times \mathbb{P}^2) + \langle \mathrm{x}'\rangle$$

$$=$$

$$\underbrace{(\frac{x_1^2}{x_0x_2})^\vee \otimes \langle \frac{x_1^2}{x_0^2} + \frac{x_2x_1}{x_0^2}\rangle}_{\left(t^{2w_1-w_0-w_2}(t^{2w_1-2w_0}+t^{w_2+w_1-2w_0})\right)} + \underbrace{\frac{x_1}{x_0}}_{\left(t^{w_1-w_0}\right)} + \underbrace{\frac{x_1}{x_0} + \frac{x_1}{x_2}}_{\left(t^{w_1-w_0}+t^{w_1-w_2}\right)}$$

$$+ \underbrace{\frac{x_1^2}{x_0x_2}}_{t^{(2w_1-w_0-w_2)}} .$$

This will be needed later on for the calculation of $\mathcal{T}_{\mathrm{x}''}X''$ for points x'' lying over π'_{11}.

At $\pi'_{12} = [x_1 x_2] \leftrightarrow \frac{x_1^2}{x_0^2}$, (22) we find,

$$\mathcal{T}_{\mathrm{x}'} X' = \langle \mathrm{x}' \rangle^\vee \otimes \left(\langle \frac{x_1^2}{x_0^2} + \frac{x_1^2}{x_2 x_0} + \frac{x_2 x_1}{x_0^2} \rangle / \langle \mathrm{x}' \rangle \right) + \mathcal{T}_{\mathrm{x}} (\check{\mathbb{P}}_0^1 \times \mathbb{P}^2) + \langle \mathrm{x}' \rangle$$

$$=$$

$$\underbrace{(\frac{x_1^2}{x_0^2})^\vee \otimes \langle \frac{x_1^2}{x_2 x_0} + \frac{x_2 x_1}{x_0^2} \rangle}_{\left(t^{2w_0 - 2w_1}(t^{2w_1 - w_2 - w_0} + t^{w_2 + w_1 - 2w_0})\right)} + \underbrace{\frac{x_1}{x_0}}_{\left(t^{w_1 - w_0}\right)} + \underbrace{\frac{x_1}{x_0} + \frac{x_1}{x_2}}_{\left(t^{w_1 - w_0} + t^{w_1 - w_2}\right)} + \underbrace{\frac{x_1^2}{x_0^2}}_{t^{(2w_1 - 2w_0)}} .$$

$$\mathcal{T}_{\pi'_{12}} X''' = \mathcal{T}_{\pi'_{12}} X'' = \mathcal{T}_{\pi'_{12}} X' =$$

$$\underbrace{\langle x_1 x_2 \rangle^\vee \otimes \langle x_0 x_1, x_2^2 \rangle}_{\left(t^{-w_1 - w_2}(t^{(w_0 + w_1)} + t^{(2w_2)})\right)} + \underbrace{\mathcal{T}_{\langle x_0 \rangle} \check{\mathbb{P}}_0^1}_{(t^{-w_0}(t^{w_1}))} + \underbrace{\mathcal{T}_{\langle x_0, x_2 \rangle} \mathbb{P}^2}_{(t^{w_1}(t^{-w_0} + t^{-w_2}))} + \underbrace{\langle x_1 x_2 \rangle}_{t^{w_1 + w_2}} .$$

Hence the expression for $c_6^T(\mathcal{T}_{\pi'_{12}} X')$,

$$(w_0 - w_2)(w_2 - w_1)(w_1 - w_0)(w_1 - w_0)(w_1 - w_2)(w_1 + w_2).$$

At $\pi'_{13} = [x_2^2]$, we find for $\mathcal{T}_{\pi'_{13}} X''' = \mathcal{T}_{\pi'_{13}} X'' = \mathcal{T}_{\pi'_{13}} X'$,

$$\underbrace{\langle x_2^2 \rangle^\vee \otimes \langle x_0 x_1, x_1 x_2 \rangle}_{\left(t^{-2w_2}(t^{(w_0+w_1)}+t^{(w_1+w_2)})\right)} \;+\; \underbrace{\mathcal{T}_{\langle x_0 \rangle} \check{\mathbb{P}}^1_0}_{(t^{-w_0}(t^{w_1}))} \;+\; \underbrace{\mathcal{T}_{\langle x_0, x_2 \rangle} \mathbb{P}^2}_{(t^{w_1}(t^{-w_0}+t^{-w_2}))} \;+ \underbrace{\langle x_2^2 \rangle}_{t^{2w_2}}.$$

Hence the expression for $c_6^T(\mathcal{T}_{\pi'_{13}} X')$,

$$(w_0 + w_1 - 2w_2)(w_1 - w_2)(w_1 - w_0)(w_1 - w_0)(w_1 - w_2)(2w_2).$$

We may collect the total contribution of π'_{12}, π'_{13} (21) to (18), to wit,

$$\pi'_{11} = [x_0 x_1] \;\to\; \text{wait till } (\textbf{??})$$

$$\pi'_{12} = [x_1 x_2] \;\to\; \frac{(2(2w_0+w_0+w_2)+(w_1+w_2))^6}{(w_0-w_2)(w_2-w_1)(w_1-w_0)(w_1-w_0)(w_1-w_2)(w_1+w_2)},$$

$$\pi'_{13} = [x_2^2] \;\to\; \frac{(2(2w_0+w_0+w_2)+(2w_2))^6}{(w_0+w_1-2w_2)(w_1-w_2)(w_1-w_0)(w_1-w_0)(w_1-w_2)(2w_2)}.$$

Move on next to the fibers of E' (4.1),(13) over the remaining points $\pi_2, \ldots, \pi_6$ in (19).

$$\begin{aligned}
x_0\langle x_0, x_1\rangle &\leftarrow \mathbb{P}\left(\langle [x_0x_2], [x_1x_2], [x_1^2]\rangle\right),\\
x_0\langle x_1, x_2\rangle &\leftarrow \mathbb{P}\left(\langle [x_0^2], [x_1^2], [x_1x_2]\rangle\right),\\
x_1\langle x_0, x_2\rangle &\leftarrow \mathbb{P}\left(\langle [x_0^2], [x_0x_2], [x_1^2]\rangle\right),\\
x_1\langle x_0, x_1\rangle &\leftarrow \mathbb{P}\left(\langle [x_1x_2], [x_0x_2], [x_0^2]\rangle\right),\\
x_1\langle x_1, x_2\rangle &\leftarrow \mathbb{P}\left(\langle [x_0^2], [x_0x_1], [x_0x_2]\rangle\right)
\end{aligned}$$

We have again just the obvious "unit" fixed points, totaling $3{\times}5{=}15$.

On the first fiber, we detect $\pi'_{21} = [x_0x_2]_{\langle x_0^2, x_0x_1\rangle} \in Y'$(4.2), because $\langle x_0^2, x_0x_1\rangle$ is of the form $\ell \cdot 0^\vee$ and the 3rd quadric is divisible by the fixed line $\ell = x_0$ too. It will be accounted for on due time (25).

Here we wish to focus on the fixpts with immediate contribution to (18), namely,

$$\pi'_{22} = [x_1 x_2]_{\langle x_0^2, x_0 x_1 \rangle} \text{ and } \pi'_{23} = [x_1^2]_{\langle x_0^2, x_0 x_1 \rangle}.$$

Both lie off the image of Y'' as well, cf. (17). The total contribution to (18) of each of these 2 points is,

$$\pi'_{22} = [x_1 x_2]_{\langle x_0^2, x_0 x_1 \rangle} \longrightarrow \frac{(2(2w_0+w_0+w_1)+(w_1+w_2))^6}{(w_2-w_0)(2w_2-w_0-w_1)(w_1-w_0)(w_1-w_0)(w_1-w_2)(w_0+w_1)},$$

$$\pi'_{23} = [x_1^2]_{\langle x_0^2, x_0 x_1 \rangle} \longrightarrow \frac{(2(2w_0+w_0+w_2)+(w_1+w_2))^6}{(w_0-w_2)(w_2-w_1)(w_1-w_0)(w_1-w_0)(w_1-w_2)(w_1+w_2)}$$

The fiber of E'' over $\pi'_{21} = [x_0x_2]_{\langle x_0^2, x_0x_1\rangle}$ is given by (15), namely cubics vanishing at 0 mod. $\ell\cdot$conics vanishing at 0.

Omitting from the 9 cubic monomials vanishing at 0 those of the form $x_0\cdot$(conic vanishing at 0), we find the fiber

$$E''_{\pi'_{21}} = \mathbb{P}\left(x_2^2x_0, x_1^3, x_2x_1^2, x_1x_2^2\right). \tag{25}$$

Only the unit points are fixed, totaling 4. Among these, just $x_2^2x_0$ fits the prescription for Y'' in (4.1.9); to be handled below (**??**). Thus, the remaining 3 points contribute immediately to (18).

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