

DEGENERATE SINGULARITIES OF ONE DIMENSIONAL FOLIATIONS

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ABSTRACT. We give formulas for the degrees of the spaces of holomorphic foliations in the complex projective plane with a dicritical singularity of prescribed order. Blowing up such singularity induces, generically, a foliation with only finitely many leaves tangent to the exceptional line; we find the degree of the locus defined by imposing a leaf of total contact with the exceptional line.

INTRODUCTION

Holomorphic foliations of degree d on the complex projective plane \mathbb{P}^2 are defined by nonzero twisted 1-forms, $\omega = \sum a_i dz_i$, with homogeneous polynomials $a_i(z_0, z_1, z_2)$ of degree $d+1$, up to scalar multiples, satisfying $\sum a_i z_i = 0$. The parameter space of foliations of degree d is a projective space \mathbb{P}^N (cf. 2).

The purpose of this work is to compute the dimensions and degrees of the subvarieties of \mathbb{P}^N corresponding to foliations displaying certain degenerate singularities. Given an integer $k \geq 2$ we study the locus, $M_k \subset \mathbb{P}^N$, of foliations with a singularity of order $\geq k$. These are foliations defined in local coordinates by a holomorphic 1-form that can be written as $\omega = a_k dx + b_k dy + \text{higher order terms}$, with $a_k(x, y), b_k(x, y)$ homogeneous polynomials of degree k . It turns out that M_k is the birational image of an explicit projective bundle over \mathbb{P}^2 . This enables us to find a formula for the degree of M_k .

Another interesting type of non-generic foliation presents a so called *dicritical* singularity of order k : require $a_k x + b_k y$ to vanish. This defines a closed subset $D_k \subset M_k$.

A characteristic feature of a foliation with a dicritical singularity is best phrased in terms of the foliation induced on the blowup of \mathbb{P}^2 at the singular point: through all but finitely many points on the exceptional line, the leaf is transversal. We say a foliation with such a singularity has the property of *maximal contact* (*MC* for short) if some leaf of the induced foliation has a contact of order k with the exceptional line. Thus we may consider the subvariety $C_k \subset D_k$ consisting of dicritical foliations with *MC*.

Requiring a leaf of a foliation to be tangent to a line at a given point defines a hyperplane in the parameter space \mathbb{P}^N . Therefore, the degree of each of the loci $C_k \subset D_k \subset M_k \subset \mathbb{P}^N$ can be rephrased loosely as the number of foliations with a singularity of the specified type and further tangent to the appropriate number of flags (point, line) in \mathbb{P}^2 . It turns out that the degrees of C_k, D_k, M_k are expressed as explicit polynomials in k, d .

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This fits into the tradition of classical enumerative geometry: answers to questions such as determining the number of plane algebraic curves that have singularities of prescribed orders, besides passing through an appropriate number of points in general position, are often given by “node” polynomials. There is also a wealth of results and conjectures on generating functions for counting suitably singular members of linear systems of curves on surfaces, cf. Göttsche [5], Kleiman and Piene [11]. We hope similar results can be formulated in the setting of foliations.

1. THE SPACE OF FOLIATIONS

The main reference for this material is Jouanolou [8]. A projective 1-form of degree d in \mathbb{P}^2 is a global section of $\Omega_{\mathbb{P}^2}^1(d+2)$, for some $d \geq 0$.

We denote by S_d the space $\text{Sym}_d(\mathbb{C}^3)^\vee$ of homogeneous polynomials of degree d in the variables z_0, z_1, z_2 . We write $\partial_i = \partial/\partial z_i$, thought of as a vector field basis for \mathbb{C}^3 . The dual basis will also be written as dz_0, dz_1, dz_2 whenever we think of differential forms. Recalling Euler sequence

$$(1) \quad 0 \rightarrow \Omega_{\mathbb{P}^2}^1(d+2) \rightarrow \mathcal{O}_{\mathbb{P}^2}(d+1) \otimes S_1 \rightarrow \mathcal{O}_{\mathbb{P}^2}(d+2) \rightarrow 0$$

and taking global sections we get the exact sequence

$$0 \rightarrow H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(d+2)) \rightarrow S_{d+1} \otimes S_1 \xrightarrow{\iota_R} S_{d+2} \rightarrow 0$$

where $\iota_R(\sum a_i dz_i) = \sum a_i z_i$ is the contraction by the radial vector field. Thus a 1-form $\omega \in H^0(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(d+2))$ can be written in homogeneous coordinates as

$$\omega = a_0 dz_0 + a_1 dz_1 + a_2 dz_2$$

where the a_i 's are homogeneous polynomials of degree $d+1$ satisfying

$$a_0 z_0 + a_1 z_1 + a_2 z_2 = 0.$$

The space of foliations of degree d in \mathbb{P}^2 is the projective space

$$(2) \quad \mathbb{P}^N = \mathbb{P}(H^0(\Omega_{\mathbb{P}^2}^1(d+2)))$$

of dimension

$$N = 3 \binom{d+2}{2} - \binom{d+1}{2} - 1 = d^2 + 4d + 2.$$

We have $\Omega_{\mathbb{P}^2}^1(d+2) = \text{Hom}(\mathcal{T}\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d+2))$. Any nonzero

$$(3) \quad \omega : \mathcal{T}\mathbb{P}^2 \rightarrow \mathcal{O}_{\mathbb{P}^2}(d+2)$$

induces a (singular, integrable) distribution of dimension one subspaces given by $p \mapsto \ker \omega_p$. A nonzero multiple of ω yields the same distribution.

1.1. Singularities. Twisting (3) by $\mathcal{O}_{\mathbb{P}^2}(-d-2)$ we get a map

$$\omega \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^2}(-d-2)} : \mathcal{T}\mathbb{P}^2(-d-2) \rightarrow \mathcal{O}_{\mathbb{P}^2}.$$

The singular scheme of ω is defined by the ideal sheaf image of the above map. If finite, it consists of

$$\int_{\mathbb{P}^2} c_2(\Omega_{\mathbb{P}^2}^1(d+2)) = d^2 + d + 1$$

points counted with multiplicity. In local coordinates, say around $0 = [0, 0, 1] \in \mathbb{P}^2$, writing $\omega = adx + bdy$ the singular scheme of ω is given by the ideal $\langle a, b \rangle$. We say 0 is a nondegenerate singularity if the jacobian determinant $|\partial(a, b)/\partial(x, y)|$ is nonzero. A generic 1-form has only nondegenerate (hence isolated) singularities; see [8, p. 87]. The order of the singularity 0 is $\nu_0(\omega) = \min\{\text{order}_0(a), \text{order}_0(b)\}$. It can easily be checked that this is independent of the choice of coordinates.

In fact, if \mathcal{I} is the ideal sheaf of the singular scheme, then for each $p \in \mathbb{P}^2$ there is a unique nonnegative integer k such that the stalk at p satisfies $\mathcal{I}_p \subset \mathfrak{m}_p^k$ and $\mathcal{I}_p \not\subset \mathfrak{m}_p^{k+1}$, where \mathfrak{m} denotes the ideal sheaf of p . Thus, setting $\mathcal{E} = \Omega_{\mathbb{P}^2}^1(d+2)$, we see that the order of the singularity p is at least k if and only if the image of ω in the quotient $\mathcal{E}/\mathfrak{m}_p^k \mathcal{E}$ is zero.

1.2. Jet bundles. The preceeding discussion entices us to recall the notion of jet bundles associated to a vector bundle, cf. [6, 16.7], [13]. Let \mathcal{E} be a vector bundle over a smooth projective variety X . For $k \geq 0$ the k th-jet bundle associated to \mathcal{E} , denoted $\mathcal{P}^k(\mathcal{E})$, is a fiber bundle over X with fiber over $x \in X$ given by

$$\mathcal{P}^k(\mathcal{E})_x = (\mathcal{O}_X/m_x^{k+1}) \otimes \mathcal{E}_x$$

where m_x is the maximal ideal of the point x .

For each $k \geq 0$ we have exact sequences

$$(4) \quad 0 \rightarrow \text{Sym}_{k+1} \Omega_X^1 \otimes \mathcal{E} \rightarrow \mathcal{P}^{k+1}(\mathcal{E}) \rightarrow \mathcal{P}^k(\mathcal{E}) \rightarrow 0.$$

Consider the evaluation map

$$ev : X \times H^0(X, \mathcal{E}) \rightarrow \mathcal{E}$$

given by $ev(x, s) = (x, s(x))$. The map ev lifts to natural maps fitting into the following commutative diagram:

$$(5) \quad \begin{array}{ccc} X \times H^0(X, \mathcal{E}) & \xrightarrow{ev_k} & \mathcal{P}^k(\mathcal{E}) \\ & \searrow ev_{k-1} & \downarrow \\ & & \mathcal{P}^{k-1}(\mathcal{E}) \end{array}$$

We think of $ev_k(s)$ as the Taylor expansion of s truncated at order $k+1$. We include for the reader's convenience the following

1.3. Lemma of global generation. *Notation as in (5) above, given k , replacing \mathcal{E} by a sufficiently high twist $\mathcal{E} \otimes \mathcal{L}^m$ by an ample line bundle \mathcal{L} , we have that*

- (i) *the map ev_k is surjective.*
- (ii) *Set $W_x = \{s \in H^0(X, \mathcal{E}) \mid (ev_{k-1})_x(s) = 0\}$. Then*

$$(ev_k)_x(W_x) = (\text{Sym}_k \Omega_X^1 \otimes \mathcal{E})_x.$$

Proof. Let \mathcal{J} be the ideal of the diagonal of $X \times X$. Consider the projection maps $p_i : X \times X \rightarrow X$, $i = 1, 2$. We have the exact sequence of sheaves over $X \times X$,

$$(6) \quad \begin{array}{ccccc} \mathcal{J}^{k+1} & \xlongequal{\quad} & \mathcal{J}^{k+1} & & \\ \downarrow & & \downarrow & & \\ \mathcal{J}^k & \hookrightarrow & \mathcal{O} & \twoheadrightarrow & \mathcal{O}/\mathcal{J}^k \\ \downarrow & & \downarrow & & \parallel \\ \mathcal{J}^k/\mathcal{J}^{k+1} & \hookrightarrow & \mathcal{O}/\mathcal{J}^{k+1} & \twoheadrightarrow & \mathcal{O}/\mathcal{J}^k \end{array}$$

Tensoring by $p_2^* \mathcal{E} \otimes \mathcal{L}^m$ and taking $(p_1)_*$ yields the diagram

$$\begin{array}{ccccc}
(p_1)_*(\mathcal{J}^{k+1} p_2^* \mathcal{E} \otimes \mathcal{L}^m) & = & (p_1)_*(\mathcal{J}^{k+1} p_2^* \mathcal{E} \otimes \mathcal{L}^m) \\
\downarrow & & \downarrow & & \\
(p_1)_*(\mathcal{J}^k p_2^* \mathcal{E} \otimes \mathcal{L}^m) & \hookrightarrow & (p_1)_*(p_2^* \mathcal{E} \otimes \mathcal{L}^m) & \xrightarrow{ev_{k-1}} & (p_1)_*((\mathcal{O}/\mathcal{J}^k) \otimes p_2^* \mathcal{E} \otimes \mathcal{L}^m) \\
\downarrow & & \downarrow ev_k & & \parallel \\
\mathrm{Sym}_k \Omega_X^1 \otimes \mathcal{E} \otimes \mathcal{L}^m & \twoheadrightarrow & \mathcal{P}^k(\mathcal{E} \otimes \mathcal{L}^m) & \twoheadrightarrow & \mathcal{P}^{k-1}(\mathcal{E} \otimes \mathcal{L}^m).
\end{array}$$

The vertical central arrow above fits into the exact sequence

$$\begin{array}{ccccc}
(p_1)_*(p_2^* \mathcal{E} \otimes \mathcal{L}^m) & \xrightarrow{ev_k} & \mathcal{P}^k(\mathcal{E} \otimes \mathcal{L}^m) & \rightarrow & (R^1 p_1)_*(\mathcal{J}^{k+1} p_2^* \mathcal{E} \otimes \mathcal{L}^m) \\
\parallel & & & & \\
X \times H^0(X, \mathcal{E} \otimes \mathcal{L}^m) & & & &
\end{array}$$

Surjectivity of ev_k now follows upon killing $(R^1 p_1)_*(\mathcal{J}^{k+1} p_2^* \mathcal{E} \otimes \mathcal{L}^m)$. This is possible for $m \gg 0$ because \mathcal{L} is ample (cf. [7, ch. III, Thm. 8.8, p. 252]). Similarly, (ii) follows from the identification $W_x = (p_1)_*(\mathcal{J}^k p_2^* \mathcal{E} \otimes \mathcal{L}^m)_x$. The right hand side is the fiber of $\ker ev_{k-1}$ at x due to exactness of the middle row together with cohomology and base change ([7, Thm. 12.11, p. 290]). \square

1.4. Singularities of order k . We apply the previous lemma to $\mathcal{E} = \Omega_{\mathbb{P}^2}^1, \mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$. In order to simplify the notation we set for short in the sequel

$$\Omega = \Omega_{\mathbb{P}^2}^1 \quad \text{and} \quad V := H^0(\Omega(d+2)).$$

1.5. Remark. Fix $k \leq d+1$. It follows from the explicit calculation of $H^i(\mathbb{P}^2, \Omega(d+2))$ (Bott's formula [1], [12, p. 8]) that the conclusions of the previous lemma hold for $m = d+2$. Hence

- (i) ev_k is surjective for $k \leq d+1$ and
- (ii) $ev_k(\ker(ev_{k-1})) = \mathrm{Sym}_k \Omega \otimes \Omega(d+2)$.

The lemma below is included for lack of a convenient reference.

1.6. Lemma. *Let X be a projective variety of dimension m . Let \mathcal{E} be a vector subbundle of the trivial bundle $X \times \mathbb{C}^{n+1}$. Let $Z \subseteq \mathbb{P}^n$ be the image of the projectivization $\mathbb{P}(\mathcal{E}) \subset X \times \mathbb{P}^n$ under the projection map $q : X \times \mathbb{P}^n \rightarrow \mathbb{P}^n$. Assume $\mathbb{P}(\mathcal{E}) \xrightarrow{q} Z$ generically injective. Then the degree of Z is equal to the degree of the Segre class $s_m \mathcal{E}$.*

Proof. Write $e := \mathrm{rank} \mathcal{E}$; $u := \dim Z = m + e - 1$. Set $h =$ hyperplane class of \mathbb{P}^n and likewise $H = c_1 \mathcal{O}_{\mathcal{E}}(1)$. We have $\deg Z = \int h^u \cap [Z] = \int h^u \cap q_*[\mathbb{P}(\mathcal{E})]$. By the projection formula, we get $\deg Z = \int H^u \cap [\mathbb{P}(\mathcal{E})]$. Pushing forward to X the assertion follows from [4, §3.1, p. 47]. \square

We describe now the locus $M_k \subset \mathbb{P}^N$ of foliations of given degree d that have some singularity of order $\geq k$.

1.7. Proposition. *For $1 \leq k \leq d+1$, denote by*

$$M_k = \{[\omega] \in \mathbb{P}^N \mid [\omega] \text{ has a singularity of order at least } k\}.$$

Then we have

$$\mathrm{cod}_{\mathbb{P}^N} M_k = k(k+1) - 2$$

and

$$\deg(M_k) = \int_{\mathbb{P}^2} c_2(\mathcal{P}^{k-1}(\Omega(d+2))).$$

Proof. Define

$$\mathcal{M}_k = \ker (ev_{k-1} : \mathbb{P}^2 \times V \rightarrow \mathcal{P}^{k-1}(\Omega(d+2))).$$

In view of the previous remark, we see that \mathcal{M}_k is a vector subbundle of V of co-rank equal to $\text{rank } \mathcal{P}^{k-1}(\Omega(d+2))$. By construction, the projective bundle associated to \mathcal{M}_k is the incidence variety,

$$\mathbb{P}(\mathcal{M}_k) = \{(p, [\omega]) \in \mathbb{P}^2 \times \mathbb{P}^N \mid p \text{ is a singularity of } [\omega] \text{ and } \nu_p(\omega) \geq k\}.$$

Let $q : \mathbb{P}(\mathcal{M}_k) \rightarrow \mathbb{P}^N$ denote the projection on the second factor. We have $M_k = q(\mathbb{P}(\mathcal{M}_k))$. It is easy to check that q is generically injective. It follows from the lemma just above that $\deg(M_k) = \int s_2(\mathcal{M}_k) \cap [\mathbb{P}^2]$. Since $s_2(\mathcal{M}_k) = c_2(\mathcal{P}^{k-1}(\Omega(d+2)))$, the assertions now follow from (4). \square

Using the proposition we may now derive an explicit formula for the degree of $M_k \subset \mathbb{P}^N$. See also the script in §5. We find

1.8. Corollary. *The degree of M_k is given by*

$$\frac{1}{2}k(k+1) \left[(k^2 + k - 1)(d^2 - (2k - 3)d) + \frac{1}{4}(4k^4 - 8k^3 - 7k^2 + 21k - 6) \right].$$

\square

2. DICRITICAL SINGULARITIES

If $\omega \in H^0(\mathbb{P}^2, \Omega(d+2))$ and p is a singularity of ω , we say that p is dicritical if the local expression of ω is

$$\omega_p = a_k dx + b_k dy + h.o.t$$

with $a_k x + b_k y = 0$. In the case $k = 1$, we say that p is a *radial singularity*.

Observe that this condition is equivalent to

$$\omega_p = f(x, y)(ydx - xdy) + h.o.t$$

for some homogeneous polynomial f of degree $k - 1$.

The main result of this section is the following.

2.1. Proposition. *For all $1 \leq k \leq d$ there exists a subbundle \mathcal{D}_k of $\mathcal{M}_k \rightarrow \mathbb{P}^2$ such that*

- (i) $\mathbb{P}(\mathcal{D}_k) = \{(p, [\omega]) \in \mathbb{P}^2 \times \mathbb{P}^N \mid p \text{ is a dicritical singularity of } [\omega] \text{ with } \nu_p(\omega) \geq k\}$.
- (ii) Set $D_k = q(\mathbb{P}(\mathcal{D}_k))$. Then the codimension of D_k in \mathbb{P}^N is $k(k+2)$ and
- (iii) the degree of D_k is the coefficient of the degree two part of $c(\mathcal{P}^{k-1}(\Omega(d+2)))c(\text{Sym}_{k+1} \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d+2))$.

2.2. Remark. Before proceeding to the proof of the proposition, we explain an invariant way of expressing the condition that a singularity be dicritical. Suppose that \mathcal{E} is a vector bundle of rank 2. Then for all $k \geq 1$ we have the following exact sequence (e.g., see [3, Appendix 2, A2.6.1.]),

$$0 \rightarrow \bigwedge^2 \mathcal{E} \otimes \text{Sym}_{k-1} \mathcal{E} \rightarrow \text{Sym}_k \mathcal{E} \otimes \mathcal{E} \xrightarrow{P_k} \text{Sym}_{k+1} \mathcal{E} \rightarrow 0,$$

where the first map is given by

$$(a \wedge b \otimes c) \mapsto (ac \otimes b) - (bc \otimes a)$$

and the second by multiplication, i.e.,

$$a \otimes b \mapsto ab.$$

Say x, y form a local basis for \mathcal{E} . Then for $a_k, b_k \in \text{Sym}_k \mathcal{E}$, we have that $a_k x + b_k y = 0$ in $\text{Sym}_{k+1} \mathcal{E}$ if and only if there is some $c \in \text{Sym}_{k-1} \mathcal{E}$ such that $a_k \otimes x + b_k \otimes y$ is equal to the image of $x \wedge y \otimes c$, to wit, $xc \otimes y - yc \otimes x$.

2.3. Construction of \mathcal{D}_k . We have the following diagram,

$$\begin{array}{ccccc}
 & & \text{Sym}_k \Omega \otimes \Omega(d+2) & & \\
 & \nearrow J_k & \downarrow & & \\
 \mathcal{M}_k & & \mathcal{P}^k(\Omega(d+2)) & & \\
 & \nearrow ev_k & \downarrow & & \\
 \mathcal{M}_k \twoheadrightarrow V & \longrightarrow & \mathcal{P}^{k-1}(\Omega(d+2)) & &
 \end{array}$$

The map J_k defined in the previous diagram is surjective in view of remark 1.5. We obtain the surjective map

$$(7) \quad \mathcal{M}_k \xrightarrow{J_k} \text{Sym}_k \Omega \otimes \Omega(d+2) \xrightarrow{P_k} \text{Sym}_{k+1} \Omega(d+2).$$

T_k

Explicitly, on the fiber over $p \in \mathbb{P}^2$ the map is as follows:

$$T_k(p, \omega) = (p, a_k x + b_k y)$$

where

$$\omega_p = a_k dx + b_k dy + h.o.t.$$

is the local expression of ω in a neighborhood of p . Set

$$(8) \quad \mathcal{D}_k := \ker \left(\mathcal{M}_k \xrightarrow{T_k} \text{Sym}_{k+1} \Omega(d+2) \right).$$

Thus \mathcal{D}_k is a vector bundle of rank $= \text{rank}(\mathcal{M}_k) - (k+2)$. Recalling 2.2, we see that the projective bundle associated to \mathcal{D}_k is the incidence variety, $\mathbb{P}(\mathcal{D}_k) = \{(p, [\omega]) \in \mathbb{P}^2 \times \mathbb{P}^N \mid p \text{ is a dicritical singularity of } [\omega] \text{ with } \nu_p(\omega) \geq k\}$. It can be shown that q is generically injective. Using 1.6, we see that the degree of \mathcal{D}_k is given by $\int s_2(\mathcal{D}_k) \cap [\mathbb{P}^2]$. This finishes the proof of the proposition 2.1. \square

A formula for the degree of \mathcal{D}_k can be made explicit.

2.4. Corollary. *The degree of \mathcal{D}_k is given by*

$$(k+1)^2 \left[\frac{1}{2}(k^4 + k^2 - 2k + 2) - (k^3 + k^2 + k - 1)d + \frac{1}{2}(k^2 + 2k + 2)d^2 \right].$$

\square

2.5. Remarks. (i) We have by construction the following diagram:

$$\begin{array}{ccccc}
 & & & \text{Sym}_{k-1} \Omega \otimes \overset{2}{\wedge} \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d+2) & \\
 & \nearrow d_k & & \downarrow & \\
 \mathcal{D}_k & \longrightarrow \mathcal{M}_k & \xrightarrow{J_k} & \text{Sym}_k \Omega \otimes \Omega(d+2) & \\
 & \searrow 0 & \searrow T_k & \downarrow P_k & \\
 & & & \text{Sym}_{k+1} \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d+2). &
 \end{array}$$

By definition of \mathcal{D}_k we obtain a map

$$d_k : \mathcal{D}_k \rightarrow \text{Sym}_{k-1} \Omega \otimes \overset{2}{\wedge} \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$$

given in the fibers by $d_k(p, \omega) = f(x, y)dx \wedge dy$ where f is a polynomial of degree $k-1$.

(ii) In the case $k=1$ we have

$$\omega = \lambda(ydx - xdy) + h.o.t.$$

with $\lambda \in \mathbb{C}$ i.e., a *radial* singularity. Thus 2.1 and 2.4 give formulas for the codimension and degree of the space of foliations with a radial singularity:

$$\begin{cases} \text{cod}_{\mathbb{P}^N} D_1 = 3; \\ \deg D_1 = 10d^2 - 8d + 4. \end{cases}$$

For $d=1$ we find **6**. Looking at Jouanolou's classification, [8, p. 14] presently we have $N=7$ and D_1 must coincide with the (projection of a) Segre image of reducible foliations, $\mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^7$. In fact, up to a projective change of coordinates we have $w = z(xdy - ydx)$.

(iii) In the case $k=d+1$ the map $J_{d+1} : \mathcal{M}_{d+1} \rightarrow \text{Sym}_{d+1} \Omega_{\mathbb{P}^2}^1 \otimes \Omega(d+2)$ is no longer surjective: its image is $\text{Sym}_d \Omega \otimes \overset{2}{\wedge} \Omega_{\mathbb{P}^2}^1 \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$. Indeed, suppose that ω is a form of degree $d+1$ which has p as singularity of order $d+1$. Then a local expression of ω is

$$\omega_p = a_{d+1}dx + b_{d+1}dy.$$

However, this form defines a projective form of degree $d+1$ in \mathbb{P}^2 if and only if

$$a_{d+1}x + b_{d+1}y = 0,$$

i.e., if p is a dicritical singularity. Therefore we can write $\omega_p = f(x, y)(ydx - xdy)$ for some homogeneous polynomial f of degree d . That is, $\omega_p \in \text{Sym}_d \Omega_p^1 \otimes \overset{2}{\wedge} \Omega_p^1$. Hence T_{d+1} (see (7)) is the zero map. This shows that $\mathcal{M}_{d+1} = \mathcal{D}_{d+1}$. Summarizing, for a foliation of degree d , any singularity of order $d+1$ is automatically dicritical.

3. MAXIMAL CONTACT

Consider a degree d form $\omega \in H^0(\mathbb{P}^2, \Omega(d+2))$ with a dicritical singularity at $p = [0 : 0 : 1]$ of order $k \geq 2$. Denote by

$$\pi : \tilde{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$$

the blowup of \mathbb{C}^2 at p . Write ω in local coordinates (x, y) around $0 \in \mathbb{C}^2$ as

$$\omega = \sum_{j=k}^{d+1} a_j dx + b_j dy$$

where a_j, b_j are homogeneous polynomials of degree j . The blowup of \mathbb{C}^2 at $(0, 0)$

$$\tilde{\mathbb{C}}^2 = \{(x, y), [s : t] \mid tx = sy\} \subset \mathbb{C}^2 \times \mathbb{P}^1$$

is covered by the usual two charts

$$V_0 = \{((x, y), [1 : t]) \mid tx = y\} \simeq \{(x, t) \mid t, x \in \mathbb{C}\},$$

$$V_1 = \{((x, y), [s : 1]) \mid x = sy\} \simeq \{(s, y) \mid s, y \in \mathbb{C}\}.$$

Over V_0 we have $dy = tdx + xdt$. Thus

$$\begin{aligned} \pi^*(\omega)(x, t) &= \sum_{j=k}^{d+1} a_j(x, tx)dx + b_j(x, tx)(tdx + xdt) = \\ &= \sum_{j=k}^{d+1} (a_j(x, tx) + tb_j(x, tx))dx + xb_j(x, tx)dt = \\ &= x^k \sum_{j=k}^{d+1} x^{j-k} [(a_j(1, t) + tb_j(1, t))dx + xb_j(1, t)dt]. \end{aligned}$$

Since $p = (0, 0)$ is a dicritical singularity we have

$$\omega = f(x, y)(ydx - xdy) + h.o.t$$

where f is a polynomial of degree $k-1$; so $a_k = yf(x, y)$, $b_k = -xf(x, y)$.

Hence we may write

$$\pi^*(\omega)(x, t) = x^k(xb_k(1, t)dt + x\alpha) = x^{k+1}(-f(1, t)dt + \alpha)$$

where α is a 1-form. The strict transform of ω is

$$(9) \quad \tilde{\omega} = -f(1, t)dt + (a_{k+1} + tb_{k+1})dx + x\alpha_1$$

for some 1-form α_1 . Over V_0 the exceptional divisor is given by $x = 0$, and by (9) we have

$$\tilde{\omega} \wedge dx = -f(1, t)dt \wedge dx + x\alpha_1 \wedge dx.$$

The leaves of $\tilde{\omega}$ passing through each point $(0, t_0)$ with $f(1, t_0) \neq 0$ are transverse to the exceptional divisor. On the other hand, the points $(0, t_0)$ such that $f(1, t_0) = 0$ but aren't singularities of $\tilde{\omega}$ are exactly the points of tangency of leaves of $\tilde{\omega}$ with the exceptional divisor.

Next, we study the relationship between the multiplicity of t_0 as a zero of $f(1, t)$ and the order of tangency of the leaf of $\tilde{\omega}$ with the exceptional divisor at $(0, t_0)$.

3.1. Lemma. *The intersection multiplicity of a leaf of $\tilde{\omega}$ with the exceptional divisor at a point $(0, t_0)$ is the multiplicity of t_0 as zero of $f(1, t)$ plus one.*

Proof. We may assume $t_0 = 0$. By (9) we have that $\tilde{\omega}$ has the following form

$$\tilde{\omega} = (-f(1, t) + xF(x, t))dt + (g(t) + xG(x, t))dx$$

with $f(1, 0) = 0$. Observe that $g(0) \neq 0$ because we are assuming that $p := (0, 0)$ is a nonsingular point of $\tilde{\omega}$. Let $h(x, y) = 0$ be a local equation for a leaf of $\tilde{\omega}$ through $(0, 0)$, where h is a non constant holomorphic function. We have

$$(\tilde{\omega} \wedge dh)(p) = -g(0) \frac{\partial h}{\partial t}(p) dt \wedge dx = 0.$$

Hence $\frac{\partial h}{\partial t}(p) = 0 \neq \frac{\partial h}{\partial x}(p)$. Therefore, we can find a local analytic parameterization of $h = 0$ of the form $y = t, x = \gamma(t)$ defined in a neighborhood of $t = 0$ such that

$$\begin{cases} \gamma(0) = 0, \\ \gamma'(0) = \frac{\partial h}{\partial t}(p) / \frac{\partial h}{\partial x}(p) = 0. \end{cases}$$

Since $(\gamma(t), t)$ parameterizes a leaf of $\tilde{\omega}$ we find that

$$f(1, t) + \gamma(t)F(\gamma(t), t) + \gamma'(t)(g(t) + \gamma(t)G(\gamma(t), t)) \equiv 0.$$

Hence, repeatedly differentiating with respect to t yields

$$\gamma^{(j)}(0) = 0 \ \forall j \leq r \implies \gamma^{(r+1)}(0) = \frac{\frac{\partial^r f}{\partial t^r}(1, 0)}{g(0)}.$$

Now, γ has intersection multiplicity n with $x = 0$ at $(0, 0)$ if the first non-vanishing derivative of $x(\gamma(t), t) = \gamma(t)$ at 0 is precisely n . Thus the intersection multiplicity of $h = 0$ with $x = 0$ is n if and only if $t = 0$ is a zero of order $n - 1$ of $f(1, t)$. \square

From the above Lemma we have that if $\omega \in D_k$, then the order of tangency of the leaves of ω with the exceptional divisor is $\leq k$, and is equal to k precisely in the case that $f = l^{k-1}$ where l is a polynomial of degree one.

3.2. Degree of the MC locus. Recall that we say that a form ω has the MC property if it has a dicritical singularity p of order k such that the strict transform of ω under the blowup of p has a leaf with maximal order of contact with the exceptional divisor of the blowup.

Consider a form with a dicritical singularity of order k ,

$$\omega = f(x, y)(ydx - xdy) + h.o.t.$$

(i.e. f is a polynomial of degree $k - 1$). Then ω has the MC property if and only if $f(1, t) = (t - t_0)^{k-1}$ or $f(s, 1) = (s - s_0)^{k-1}$, i.e.,

$$f(x, y) = (ax + by)^{k-1} \text{ for some } a, b \in \mathbb{C}.$$

Therefore we can parameterize the set of forms that has the MC property as follows. The Veronese-type map $\Omega \rightarrow \text{Sym}_{k-1} \Omega$ induces an embedding

$$v_k : \mathbb{P}(\Omega \otimes \wedge^2 \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)) \rightarrow \mathbb{P}(\text{Sym}_{k-1} \Omega \otimes \wedge^2 \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d+2))$$

which is locally given by $v_k(p, l \otimes (ydx - xdy)) = (p, l^{k-1} \otimes (ydx - xdy))$. In order to simplify the notation set

$$\mathcal{E} := \Omega \otimes \wedge^2 \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d+2)$$

and

$$\mathcal{E}_k := \text{Sym}_{k-1} \Omega \otimes \wedge^2 \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d+2).$$

Define

$$\mathbb{V}_k := v_k(\mathbb{P}(\mathcal{E})) \subset \mathbb{P}(\mathcal{E}_k).$$

3.3. Lemma. *The codimension of \mathbb{V}_k in $\mathbb{P}(\mathcal{E}_k)$ is $k-2$ and its cycle class is*

$$[\mathbb{V}_k] = uH_k^{k-2} + vhH_k^{k-3} + wh^2H_k^{k-4} \cap \mathbb{P}(\mathcal{E}_k)$$

where H_k (resp. h) denotes the relative hyperplane class of $\mathbb{P}(\mathcal{E}_k)$ (resp. \mathbb{P}^2) and

$$\begin{aligned} u &= (k-1), \\ v &= -\frac{1}{2}(k-1)(k-2)(3k+2d-5), \\ w &= \frac{1}{8}(k-2)(k-1)^2(9k^2-47k+72+(12k-60)d+12d^2). \end{aligned}$$

Proof. It is clear that $\text{cod}(\mathbb{V}_k) = \dim \mathbb{P}(\mathcal{E}_k) - \dim \mathbb{P}(\mathcal{E}) = k-2$. Recalling that the Chow ring $A_*(\mathbb{P}(\mathcal{E}_k))$ is generated by H_k and h (see [4, Thm. 3.3., p. 64]) we can express

$$(10) \quad [\mathbb{V}_k] = uH_k^{k-2} + vhH_k^{k-3} + wh^2H_k^{k-4} \cap \mathbb{P}(\mathcal{E}_k).$$

With this notation, the relative hyperplane class of $\mathbb{P}(\mathcal{E})$ is H_2 , and we have $v_k^*(H_k) = (k-1)H_2$. Consider the following diagram:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{v_k} & \mathbb{P}(\mathcal{E}_k) \\ & \searrow \rho & \downarrow \pi \\ & & \mathbb{P}^2 \end{array}$$

To find the coefficient u we multiply by h^2H_k both sides of (10) to obtain:

$$h^2H_k \cap v_{k*}(\mathbb{P}(\mathcal{E})) = uh^2H_k^{k-1} \cap \mathbb{P}(\mathcal{E}_k).$$

By the projection formula we have:

$$h^2v_{k*}((k-1)H_2 \cap \mathbb{P}(\mathcal{E})) = uh^2H_k^{k-1} \cap \mathbb{P}(\mathcal{E}_k).$$

Applying π_* to this last equation we find

$$\begin{aligned} (k-1)h^2 \cap \rho_*(H_2 \cap \rho^*\mathbb{P}^2) &= uh^2\pi_*(H_k^{k-1} \cap \pi^*\mathbb{P}^2), \\ (k-1)h^2 \cap s_0(\mathcal{E}) \cap \mathbb{P}^2 &= uh^2 \cap s_0(\mathcal{E}_k) \cap \mathbb{P}^2. \end{aligned}$$

Hence $u = k-1$. Next, multiplying (10) by hH_k^2 we get

$$hH_k^2 \cap v_{k*}(\mathbb{P}(\mathcal{E})) = uhH_k^k \cap \mathbb{P}(\mathcal{E}_k) + vh^2H_k^{k-1} \cap \mathbb{P}(\mathcal{E}_k).$$

Again the projection formula yields

$$h(k-1)^2 \cap v_{k*}(H_2^2 \cap \mathbb{P}(\mathcal{E})) = uhH_k^k \cap \mathbb{P}(\mathcal{E}_k) + vh^2H_k^{k-1} \cap \mathbb{P}(\mathcal{E}_k).$$

Applying π_* we obtain

$$\begin{aligned} h(k-1)^2\rho_*(H_2^2 \cap \mathbb{P}(\mathcal{E})) &= uh\pi_*(H_k^k \cap \pi^*\mathbb{P}^2) + vh^2\pi_*(H_k^{k-1} \cap \pi^*\mathbb{P}^2), \\ (k-1)^2hs_1(\mathcal{E}) \cap \mathbb{P}^2 &= ((k-1)hs_1(\mathcal{E}_k) + vh^2s_0(\mathcal{E}_k)) \cap \mathbb{P}^2, \end{aligned}$$

hence $v = (k-1)^2s_1(\mathcal{E}) - (k-1)s_1(\mathcal{E}_k)$.

Similarly we obtain $w = [(k-1)^3s_2(\mathcal{E}) - us_2(\mathcal{E}_k) - vhs_1(\mathcal{E}_k)] \cap [\mathbb{P}^2]$.

The lemma follows from the calculation of the Segre classes of \mathcal{E} and \mathcal{E}_k . Observe that $\bigwedge^2 \Omega = \mathcal{O}_{\mathbb{P}^2}(-3)$, so that $\mathcal{E} = \Omega(d-1)$ and $\mathcal{E}_k = \text{Sym}_{k-1} \Omega \otimes \mathcal{O}_{\mathbb{P}^2}(d-1)$. These classes can be computed with SCHUBERT [9]. \square

By Remark (2.5) we have a rational map ψ_k as in the diagram

$$(11) \quad \begin{array}{ccc} \psi_k : \mathbb{P}(\mathcal{D}_k) & \dashrightarrow & \mathbb{P}(\mathcal{E}_k) \\ \uparrow & & \uparrow \\ \psi_k^{-1}(\mathbb{V}_k) & & \mathbb{V}_k \end{array}$$

Set $\Gamma_k := \overline{\psi_k^{-1}(\mathbb{V}_k)} \subset \mathbb{P}(\mathcal{D}_k)$. Thus

$$\Gamma_k = \{(p, [\omega]) \mid \omega_p = l^{k-1}(ydx - xdy) + h.o.t \text{ for some } l \in \Omega_p\}.$$

The image

$$C_k := q(\Gamma_k) \subset \mathbb{P}^N$$

parameterizes the space of foliations with the *MC* property.

3.4. Lemma. *We have*

- (i) $\text{cod}_{\mathbb{P}(\mathcal{D}_k)} \Gamma_k = \text{cod}_{\mathbb{P}(\mathcal{E}_k)} \mathbb{V}_k = k - 2$.
- (ii) *Let $Z \subset \mathbb{P}(\mathcal{D}_k)$ denote the indeterminacy locus of ψ_k . Then*

$$\text{cod}_{\mathbb{P}(\mathcal{D}_k)}(Z) = k.$$

Proof. Since ψ_k (cf. diagram 11) is induced by a surjective map of vector bundles, its fibers have the same dimension n . Therefore $\dim \Gamma_k = \dim \mathbb{V}_k + n$, and $\dim \mathbb{P}(\mathcal{D}_k) = \dim \mathbb{P}(\mathcal{E}_k) + n$. Hence the equality for the codimension follows:

$$\text{cod}_{\mathbb{P}(\mathcal{D}_k)}(Z) = \text{rank}(\text{Sym}_{k-1} \Omega \otimes \wedge^2 \Omega(d+2)) = k.$$

□

We may now find the degree of the locus of dicritical foliations with maximal contact with the exceptional line.

3.5. Proposition. (i) *The codimension of C_k in \mathbb{P}^N is*

$$\text{cod}_{\mathbb{P}^N} C_k = k^2 + 3k - 2.$$

(ii) *The degree of C_k is given by the formula*

$$(k-1) \frac{1}{2} \left[\frac{1}{4}(4k^6 + 20k^5 - 15k^4 - 66k^3 + 211k^2 - 218k + 112) \right. \\ \left. - (2k^5 + 7k^4 + 2k^3 + 24k^2 - 49k + 44)d + (k^4 + 2k^3 + 10k^2 + k + 16)d^2 \right].$$

Proof. First of all, the restriction $q|_{\Gamma_k}$ is generically injective. For instance, it can be checked that the 1-form

$$\omega = (z_2^{d-(k-1)}(z_0 + z_1)^{k-1} + z_0^d + z_1^d)(z_1 dz_0 - z_0 dz_1)$$

has $p = [0 : 0 : 1]$ as its unique singularity with order k and p is a reduced point of the fiber $(q|_{\Gamma_k})^{-1}([\omega])$. To compute the codimension observe that

$$\text{cod}_{\mathbb{P}^N} C_k = \text{cod}_{\mathbb{P}^N} D_k + \text{cod}_{\mathbb{P}(\mathcal{D}_k)} C_k = k(k+2) + k - 2 = k^2 + 3k - 2.$$

Put $n = \dim \Gamma_k$. By lemma 3.4 (ii) we have that $\dim Z < n$, hence $A_n(Z) = 0$. Using the excision exact sequence (cf. [4, Prop. 1.8, p. 21])

$$A_n(Z) \rightarrow A_n(\mathbb{P}(\mathcal{D}_k)) \rightarrow A_n(\mathbb{P}(\mathcal{D}_k) \setminus Z) \rightarrow 0,$$

we deduce that

$$A_n(\mathbb{P}(\mathcal{D}_k)) \simeq A_n(\mathbb{P}(\mathcal{D}_k) \setminus Z).$$

Therefore, using that the class $\psi_k^*[\mathbb{V}_k]$ is known in $\mathbb{P}(\mathcal{D}_k) \setminus Z$, we can do the computations in $A_n(\mathbb{P}(\mathcal{D}_k))$. Recalling (11) ψ_k is a linear projection, we have that $\psi_k^*H_k = H := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{D}_k)}(1))$. Therefore

$$\begin{aligned} \deg C_k &= \deg q_* \Gamma_k = \int H^n \cap [\Gamma_k] = \int H^n \cap \psi_k^*[\mathbb{V}_k] = \\ (\text{Lemma 3.3}) \quad & \int (uH^{n+k-2} + vH^{n+k-3} + wh^2H^{n+k-4}) \cap [\mathbb{P}(\mathcal{D}_k)] = \\ & \int (uH^{r+1} + vH^r + wh^2H^{r-1}) \cap [\mathbb{P}(\mathcal{D}_k)], \end{aligned}$$

where $r = \text{rank } \mathcal{D}_k$. Applying p_{1*} and using the definition of Segre class [4, §3.1, p. 47], we see that what we are calculating is:

$$\int (us_2(\mathcal{D}_k) + vs_1(\mathcal{D}_k) + ws_0(\mathcal{D}_k)) \cap [\mathbb{P}^2].$$

From Lemma 3.3 we know the values of u, v, w . The classes $s_1(\mathcal{D}_k)$ and $s_2(\mathcal{D}_k)$ are known from (8) and the beginning of the proof of 1.7. We finish using SCHUBERT [9]. \square

4. CONCLUDING REMARKS

It is worth mentioning that for foliations of degree $d \geq 2$, the scheme of singularities completely determines the foliation. Moreover, the schemes of $d^2 + d + 1$ points that can occur as singular scheme of a foliation are known, cf. [2]. It would be nice to work out the enumerative geometry of the loci of foliations with scheme of singularities subject to collisions in the spirit of [10].

The reader is invited to check that formulas similar to 1.8, 2.4 and 3.5 can be written down for an arbitrary surface. Precisely, given a smooth, projective surface X , we may fix an ample divisor class h and look at the space of foliations $\mathbb{P}^N = \mathbb{P}(H^0(\Omega_X^1 \otimes \mathcal{O}((d+2)h)))$ for $d \gg 0$. The degree of M_k can be written as $\frac{1}{72}k(k+1)[(k-1)(k+2)(4k^2+4k+3)c_1^2 + 12(2k+1)(k^2+k-1)(d+2)hc_1 + 6(k^2+k+4)c_2 + 36(k^2+k-1)(d+2)^2h^2]$

where we set for short $c_i = c_i\Omega_X^1$. Substituting in the Chern numbers for \mathbb{P}^2 , ($h^2 = 1, c_1h = -3, c_2 = 3$) reproduces 1.8. We include a script below.

5. SCHUBERT/MAPLE SCRIPT

```
with(schubert): DIM:=2; omega:=bundle(2,c); f:=expand(Symm(k,omega));
g:=convert(%,list); s0:=sum(1, 'j'=0..k-1): s0:=factor(%);
s1:=sum('j', 'j'=0..k-1): s1:=factor(%); s2:=sum('j^2', 'j'=0..k-1):
s2:=factor(%); s3:=sum('j^3', 'j'=0..k-1): s3:=factor(%);
G:=g;l:=[]: for i to nops(g) do if has(g[i],k^3)then print(i):
l:=[op(1),i]: g[i]:=subs(k^3=s3,g[i]) fi od;g;
G:=g;l:=[]: for i to nops(g) do if not i in l then
if has(g[i],k^2)then print(i):l:=[op(1),i]: g[i]:=subs(k^2=s2,g[i])
fi fi od;g; G:=g;l:=[]: for i to nops(g) do
if not i in l then if has(g[i],k) then print(i):
l:=[op(1),i]: g[i]:=subs(k=s1,g[i]) fi fi od; g;l; g[2]:=s0;
collect(convert(g,'+'),t); omega*o((d+2)*h); mtaylor(%%*,t,3);
chern(2,%); factor(%); #P2:c1^2=9*h^2,c2=3*h^2,c1=-3*h2
subs(c1^2=9*h^2,%); subs(c2=3*h^2,%); subs(c1=-3*h,%); print(indets(%));
factor(%); subs(h=1,%); collect(%,d);
```

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REFERENCES

- [1] R. Bott. *Homogeneous vector bundles*. Ann. of Math. **66**, 203-248 (1957).
- [2] A. Campillo and J. Olivares. *Polarity with respect to a foliation and Cayley-Bacharach Theorems*. J. reine angew. Math. **534**, 95-118 (2001).
- [3] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, 150. New York etc.: Springer-Verlag, (1994).
- [4] W. Fulton. *Intersection Theory*. Springer-Verlag. New York. (1985).
- [5] L. Göttsche. *A conjectural generating function for numbers of curves on surfaces*, Comm. Math. Phys. **196**, 523-533, eprint alg-geom/9711012 (1998).
- [6] A. Grothendieck et J. Dieudonné. *Éléments de Géométrie Algébrique IV-4*. Publ. Math. IHES, **32**, 5-361 (1967).
- [7] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. **52**. New York - Heidelberg - Berlin: Springer-Verlag. (1977).
- [8] J.P. Jouanolou. *Equations de Pfaff algébriques*. Lectures Notes in Mathematics, **708**. Springer-Verlag, (1979).
- [9] S. Katz and S.A. Stromme. *Schubert: a maple package for intersection theory*, <http://www.mi.uib.no/schubert/>
- [10] D. Kerner. *On the collisions of singular points of complex algebraic plane curves*. Singularities II, 89-110, Contemp. Math., 475, Amer. Math. Soc., Providence, RI, (2008).
- [11] S. L. Kleiman and R. Piene. *Node polynomials for families: methods and applications*. Math. Nachr. **271**, 69-90, (2004).
- [12] C. Okonek, M. Schneider and H. Spindler. *Vector bundles on complex projective spaces*, Progress in Math. **3**, Birkhäuser, (1980)
- [13] R. Piene. *Numerical characters of a curve in projective n-space*. In: Holm, P. (ed.) Real and complex singularities. Proceedings, Oslo 1976, pp. 475- 496. Sijthoff & Noordhoff (1977).

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