ENUMERATION OF TROPES

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Abstract. Singularities of the plane sections of a *general surface* in three-space are well known, and counted; in particular, *all* plane sections are reduced. Fixing integers d, k, we give formulas for the degree of the locus of surfaces of degree d admitting a plane which is tangent along some curve of degree k.

Dedicated to Prof. Heisuke Hironaka
on his 80th birthday.

1. introduction

We look for surfaces special in the sense that some plane section is tangent along a curve. We say a surface of degree d has a $k-$*trope*, if a plane is tangent to S along a curve of degree k. The curve or its supporting plane will be referred to as a $k-$*trope*.

Tropes have entered my vocabulary upon reading D. Eklund, [1]. He calls a plane in $\mathbb{P}^3$ a *trope* of a quartic surface if their intersection is an irreducible conic counted with multiplicity two $(d = 4, k = 2)$.

Fixing integers d, k, we show how to get formulas for the degree of the locus of surfaces of degree d admitting a $k-$trope. We also indicate how to handle the similar question in higher dimension. For historical accounts and further motivation on the subject,the reader may consult W. Fulton's tome [2], S. Kleiman and R. Piene [4] and the references therein.

2. parameter space

As customary we apply the toolbox of intersection theory to a suitable parameter space. Let us start by finding the dimension of the family of surfaces of degree d with a $k-$trope. Such a surface has an equation of the form

$$S := xF(x, y, z, w) + G(y, z, w)^2 H(y, z, w) = 0, \tag{1}$$

for some polynomial F of degree $d-1$ and polynomials G, H such that $\deg G = k$, $\deg H = d - 2k$. Count constants: We have fixed the plane $x = 0$; now the freedom of G, H in (1) is

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$$\tbinom{k+2}{2} - 1 + \tbinom{d-2k+2}{2} - 1.$$

Thus we expect and do obtain a family of dimension

$$(2)\qquad \begin{aligned} N_{d,k} &= 3 + \tbinom{d+2}{3} + \tbinom{k+2}{2} - 1 + \tbinom{d-2k+2}{2} - 1 \\ &= 1 + \tbinom{d+2}{3} + \tbinom{k+2}{2} + \tbinom{d-2k+2}{2} \end{aligned}$$

of surfaces of degree d with a $k-$trope. The previous discussion globalizes to yield a suitable projective bundle.

Write $\mathcal{F}_i = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(i))$ for the space of homogeneous polynomials of degree i. There are exact sequences of vector bundles over the dual projective space $\check{\mathbb{P}}^3 = \mathbb{P}(\mathcal{F}_1)$:

$$\mathcal{O}_{\check{\mathbb{P}}^3}(-1) \rightarrowtail \check{\mathbb{P}}^3 \times \mathcal{F}_1 \twoheadrightarrow \overline{\mathcal{F}}_1$$

and

$$\mathcal{O}_{\check{\mathbb{P}}^3}(-1) \otimes \mathcal{F}_{k-1} \rightarrowtail \check{\mathbb{P}}^3 \times \mathcal{F}_k \xrightarrow{\rho} \overline{\mathcal{F}}_k.$$

Here $\overline{\mathcal{F}}_k$ denotes the vector bundle with fiber the space of equations of plane curves of degree k. The homomomorphism ρ is defined by restriction of elements of $\mathcal{F}_k$ to a (varying) plane in $\mathbb{P}^3$. The kernel consists of multiples of the equation of a plane.

Set

$$k' = d - 2k,\ X = \mathbb{P}(\overline{\mathcal{F}}_k)\ .$$

Note that X parameterizes the family of plane curves of degree k in $\mathbb{P}^3$. Form the diagram of vector bundles over X,

$$(3)\qquad \begin{tikzcd} \mathcal{O}_{\check{\mathbb{P}}^3}(-1) \otimes \mathcal{F}_{d-1} \arrow[r, equal] \arrow[d, tail] & \mathcal{O}_{\check{\mathbb{P}}^3}(-1) \otimes \mathcal{F}_{d-1} \arrow[d, tail] \\ \widetilde{\mathcal{F}}_{d,k} \arrow[r, tail] \arrow[d] & X \times \mathcal{F}_d \arrow[d, two heads] \\ \mathcal{O}_{\overline{\mathcal{F}}_k}(-2) \otimes \overline{\mathcal{F}}_{k'} \arrow[r, tail] & \overline{\mathcal{F}}_d \end{tikzcd}$$

where the bottom row amounts to squaring the equation of a plane curve of degree k and multiplying by all equations of complementary degree $k' = d - 2k$. We have just defined a vector subbundle $\widetilde{\mathcal{F}}_{d,k}$ of the trivial bundle $X \times \mathcal{F}_d$, with rank

$$\operatorname{rk} \widetilde{\mathcal{F}}_{d,k} = \tbinom{d+2}{3} + \tbinom{d-2k+2}{2},$$

thus obtaining

$$\dim \mathbb{P}(\widetilde{\mathcal{F}}_{d,k}) = 1 + \tbinom{k+2}{2} + \operatorname{rk} \widetilde{\mathcal{F}}_{d,k} = N_{d,k}.$$

Since $\mathbb{P}(\widetilde{\mathcal{F}}_{d,k})$ is a projective subbundle of the trivial bundle $X \times \mathbb{P}(\mathcal{F}_d)$, we get the diagram of maps

$$(4)\qquad \begin{tikzcd} \mathbb{P}(\widetilde{\mathcal{F}}_{d,k}) \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, "\pi"] & X \times \mathbb{P}(\mathcal{F}_d) \arrow[d] \\ X & Y_{d,k} \subset \mathbb{P}(\mathcal{F}_d) \end{tikzcd}$$

The image $Y_{d,k}$ of π is the desired locus of surfaces of degree d with a $k-$trope. We wish to determine $\deg Y_{d,k}$.

3. SURFACES WITH A TROPE ARE SINGULAR

Calculating the gradient of S in (1), we find

$$\nabla S = x\nabla F + F\nabla x + G(G\nabla H + 2H\nabla G).$$

We see that ∇S vanishes along

$$x = F = G = 0, \tag{5}$$

usually $k(d-1)$ singular points.

3.1. **test** $d = 2, k = 1$. Clearly, if a quadric surface has a 1–trope, it is a cone. Hence it carries infinitely many 1–tropes. In this case, the map π in (4) shrinks dimension.

3.2. **test** $d = 3, k = 1$. An apparently sad thing is that the image of $\mathbb{P}(\widetilde{\mathcal{F}}_{3,1})$ in $\mathbb{P}(\mathcal{F}_3) \simeq \mathbb{P}^{19}$ must be contained in the locus of cubic surfaces with at least 2 singular points, which is 17-dimensional. As the map $\mathbb{P}(\widetilde{\mathcal{F}}_3) \to \mathbb{P}(\mathcal{F}_3)$ can be shown to be generically injective we should realize that for the general cubic surface with two singularities, there is a plane section which cuts a double line. Is this correct?

Let $\Sigma_d^2 \subset \mathbb{P}(\mathcal{F}_d)$ be the locus of surfaces with 2 double points. The degree of Σ_d^2 is given by

$$2(d-1)^2(d-2)(4d^3 - 8d^2 + 8d - 25)$$

cf. [5]. For $d = 3$, we get **280**. As observed before, $Y_{3,1}$ sits in Σ_3^2 and is of the same dimension. It turns out that the calculation of $\deg Y_{d,k}$ done below gives the same answer for $d = 3$. Hence $Y_{3,1} = \Sigma_3^2$. For $d \geq 4$ however, $Y_{d,1}$ has dimension strictly smaller than the locus of surfaces with $d-1$ singularities.

4. CALCULATION OF THE DEGREES

4.1. **Lemma.** *For fixed $k \geq 1$ and all sufficiently big d, the map*
$$\pi : \mathbb{P}(\widetilde{\mathcal{F}}_{d,k}) \to Y_{d,k}$$
is generically injective.

Proof. We show first that the tangent map of π is injective at a general point, hence $\dim Y_{d,k} = N_{d,k}$. Set as before $k' = d - 2k$ and
$$X' = \mathbb{P}(\overline{\mathcal{F}}_k) \times_{\check{\mathbb{P}}^3} \mathbb{P}(\overline{\mathcal{F}}_{k'})$$
There is a diagram over X' similar to (3), obtaining exact sequences

$$\begin{array}{ccccc}
\mathcal{O}_{\check{\mathbb{P}}^3}(-1) \otimes \mathcal{F}_{d-1} & \rightarrowtail & \mathcal{F}'_{d,k} & \twoheadrightarrow & \mathcal{O}_{\overline{\mathcal{F}}_k}(-2) \otimes \mathcal{O}_{\overline{\mathcal{F}}_{k'}}(-1) \\
\| & & \downarrow & & \downarrow \\
\mathcal{O}_{\check{\mathbb{P}}^3}(-1) \otimes \mathcal{F}_{d-1} & \rightarrowtail & \widetilde{\mathcal{F}}_{d,k} & \twoheadrightarrow & \mathcal{O}_{\overline{\mathcal{F}}_k}(-2) \otimes \overline{\mathcal{F}}_{k'}
\end{array}$$

Pick $(x, \overline{G}, \overline{H}) \in X'$ with $x \in \check{\mathbb{P}}^3$ and $\overline{G}, \overline{H}$ equations of curves of degrees k, k' in the plane x. The fiber of $\mathcal{F}'_{d,k}$ over $(x, \overline{G}, \overline{H})$ is the subspace of $\mathcal{F}_d$ consisting of all S as in (1). In particular, moding out by the equation of the plane x yields a multiple of $\overline{G}^2$. It follows that $\mathcal{F}'_{d,k}$ sits inside the pullback of $\widetilde{\mathcal{F}}_{d,k}$ to X'. We get a diagram of maps of projective bundles

$$\begin{tikzcd}
\mathbb{P}(\mathcal{F}'_{d,k}) \arrow[d] \arrow[rrd, "\psi"] & \subset & \mathbb{P}(\widetilde{\mathcal{F}}_{d,k}) \times_X X' \arrow[d] & \\
X' & & \mathbb{P}(\widetilde{\mathcal{F}}_{d,k}) & \subset \mathbb{P}(\mathcal{F}_d) \times X.
\end{tikzcd}$$

The map $\psi : \mathbb{P}(\mathcal{F}'_{d,k}) \longrightarrow \mathbb{P}(\widetilde{\mathcal{F}}_{d,k})$ is an isomorphism. Given a vector space V, recall the usual identification $T_{(v)}\mathbb{P}(V) = V/(v)$ for the tangent space of the associated projective space $\mathbb{P}(V)$ at a point (v); it comes from Euler exact sequence, to wit

$$\mathcal{O}_{\mathbb{P}(V)} \rightarrowtail V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \twoheadrightarrow T_{\mathbb{P}(V)}. \ .$$

Thus a tangent vector at $(x, \overline{G}, \overline{H}, F) \in \mathbb{P}(\mathcal{F}'_{d,k})$ can be written as (L', G', H', F') with

$$L' \in \mathcal{F}_1/\left(x\right), G' \in \mathcal{F}_k/\left(x, G\right), H' \in \mathcal{F}_{k'}/(x, H), F' \in \mathcal{F}_{d-1}/(F),$$

where we have set for short $(x, G) = x\mathcal{F}_{k-1} + (G)$, the subspace spanned by a representative G of $\overline{G}$ and multiples of x (and likewise for (x, H)). It gives the $\mathbb{C}[\epsilon]-$point $(\epsilon^2 = 0)$

$$\begin{aligned} S + \epsilon S' = &\ (x + \epsilon L')(F + \epsilon F') + (G + \epsilon G')^2(H + \epsilon H') \\ = &\ xF + G^2H + \epsilon(xF' + FL' + 2GHG' + G^2H') \end{aligned}$$

with $S' := xF' + FL' + 2GHG' + G^2H'$. Now suppose S' is zero in $T_S\mathbb{P}(\mathcal{F}_d) = \mathcal{F}_d/(S)$. Set $\overline{F} = F(0, y, z, w)$. We may assume x does not occur in G, H, L', G', H'. We deduce $\overline{F}L' + 2GHG' + G^2H' \in \langle G^2H \rangle$. Hence G divides $\overline{F}L'$. Since F, G, H are general, we conclude G divides L'. Now if $\deg G = k \geq 2$, this implies $L' = 0 = F' = G' = H'$ as desired. If $L' \neq 0$ then $G = L'$ up to scalar and $k = 1, k' = d - 2$. It follows that $\overline{F} + 2HG' + GH'$ lies in the ideal $\langle G, H \rangle$. This is avoided by genericity, provided $d > 2$. If $d = 2, k' = 0$, then $H = 1, H' = 0$ and we have the situation as in the case (3.1). To finish, observe that the dimension calculated in (2) can be retrieved as follows: subtract from the quantity $\dim(X \times \mathbb{P}(\mathcal{F}_d))(= \binom{k+2}{2} + \binom{d+3}{3} - 1)$, the Hilbert polynomial of a plane curve of degree $2k$, to wit, $p(d) = 2dk + 1 - (2k-1)(k-1)$. Likewise, if a surface S of degree d has two distinct $k-$tropes, it lies in the image of a variety of dimension

$$\dim(X \times X \times \mathbb{P}(\mathcal{F}_d)) - 2(2dk + 1 - (2k-1)(k-1)).$$

This is strictly smaller than $\dim Y_{d,k}$ for all big d. $\square$

We recall that the total Segre class of a vector bundle $\mathcal{E}$ is equal to the inverse of the total Chern class, cf. [2, p.50,§3.2]:

$$s(\mathcal{E}) = c(\mathcal{E})^{-1}.$$

There are polynomial formulas expressing the ith Segre class in terms of Chern classes and vice-versa. Also, if $\mathcal{E}' \subset \mathcal{E}$ is a vector subbundle, Whitney's formula holds: $s(\mathcal{E}) = s(\mathcal{E}')s(\mathcal{E}/\mathcal{E}')$.

4.2. **Proposition.** *Notation as above, we have the formula*

$$deg\, Y_{d,k} = \int s_{\dim X}(\widetilde{\mathcal{F}}_d) \cap X,$$

with the top dimensional Segre class calculated from

$$s(\widetilde{\mathcal{F}}_d) = s(\mathcal{O}_{\check{\mathbb{P}}^3}(-1))^{\binom{d+2}{3}} s(\mathcal{O}_{\overline{\mathcal{F}}_k}(-2) \otimes \overline{\mathcal{F}}_{d-2k}).$$

Proof. Let h denote the hyperplane class of $\mathbb{P}(\mathcal{F}_d)$. By general principles from intersection theory,

$$\deg Y_{d,k} = \int h^{N_{d,k}} \cap Y_{d,k}.$$

Since $\widetilde{\mathcal{F}}_d$ is a vector subbundle of $X \times \mathcal{F}_d$, the degree of $Y_{d,k}$ can be calculated upstairs. Indeed, using projection formula and generic injectivity ensured in Lemma 4.1,

$$\deg Y_{d,k} = \int \pi^\star h^{N_{d,k}} \cap \mathbb{P}(\widetilde{\mathcal{F}}_d).$$

Pushing forward via $\mathbb{P}(\widetilde{\mathcal{F}}_{d,k}) \to X$ and recalling (3), we get the assertion (cf. [2, 3.1]). $\square$

5. EXAMPLES

Pushing down via $X = \mathbb{P}(\overline{\mathcal{F}}_k) \to \check{\mathbb{P}}^3$, we get, for each k, a polynomial in d. The first few values seem to point towards a nice closed formula in terms of k, d, but so far we haven't managed to sort it out. Here is a sample:

$k = 1:\ \binom{d}{3}(d^2 - d + 4)(3d^4 - 6d^3 + 11d^2 - 8d + 12)/6;$

$k = 2:\ \binom{d-2}{2}(d^2 - 5d + 12)(d^2 - 5d + 14)\big(207d^{10} - 2259d^9 + 4278d^8 + 74486d^7 - 699689d^6 + 3156341d^5 - 8896820d^4 + 16501736d^3 - 19769000d^2 + 14106240d - 4838400\big)/(2^4 \cdot 3^4 \cdot 5 \cdot 7);$

$\dots$

$k = 6:\ \binom{d-10}{2}(d^2 - 21d + 116)(d^2 - 21d + 118) \cdots (d^2 - 21d + 162)(891d^{10} - 54158d^9 + 1376712d^8 - 18439700d^7 + 127907907d^6 - 231224361d^5 - 3266133060d^4 + 28228362487d^3 - 98166708726d^2 + 149105049696d - 52825449600)/(2^{22}3^{14}5^6 7^3 11^2 13^2 17 \cdot 19 \cdot 23 \cdot 29).$

See the appendix for a MAPLE code.

6. HIGHER DIMENSION

Could move up, say to $\mathbb{P}^4$. Now one may either keep cutting with hyperplanes, or stick to $\mathbb{P}^2$'s. In the former case, the hypersurfaces will presently be singular along a curve, cf. (5).

So we take the decision to stick to sections by $\mathbb{P}^2$'s. We are now looking at hypersurfaces in $\mathbb{P}^4$ of the form

$$S = x_0 F_0 + x_1 F_1 + G^2 H$$

where $G, H \in \mathbb{C}[x_2, x_3, x_4], \deg G = k$. Presently S needs no longer be singular.

Write $\mathbb{G}\mathrm{r} = \mathbb{G}\mathrm{r}(3,5)$ for the grassmannian of $\mathbb{P}^2$'s in $\mathbb{P}^4$, with tautological sequence

$$\mathcal{A} \rightarrowtail \mathbb{G}\mathrm{r} \times \mathcal{F}_1 \twoheadrightarrow \overline{\mathcal{F}}_1,$$

where $\mathrm{rk}\mathcal{A}{=}2$, $\mathrm{rk}\mathcal{F}_1{=}5$. The fibers of $\mathcal{A}$ give equations for $\mathbb{P}^2 \subset \mathbb{P}^4$. Take symmetric powers. Put $\mathcal{A}_d = \ker(\mathbb{G}\mathrm{r} \times \mathcal{F}_d \twoheadrightarrow \overline{\mathcal{F}}_d)$. Set

$$X = \mathbb{P}(\overline{\mathcal{F}}_k),$$

a projective bundle over $\mathbb{G}\mathrm{r}(3,5)$. Form the diagram of vector bundles over X (similar to (3) on p. 2),

$$\begin{tikzcd}
\mathcal{A}_d \arrow[d, tail] \arrow[r, equal] & \mathcal{A}_d \arrow[d, tail] \\
\widetilde{\mathcal{F}}_{d,k} \arrow[d, two heads] \arrow[r, tail] & X \times \mathcal{F}_d \arrow[d, two heads] \\
\mathcal{O}_{\overline{\mathcal{F}}_k}(-2) \otimes \overline{\mathcal{F}}_{d-2k} \arrow[r, tail] & \overline{\mathcal{F}}_d .
\end{tikzcd}$$

We find a projective subbundle $\mathbb{P}(\widetilde{\mathcal{F}}_{d,k}) \subset X \times \mathbb{P}(\mathcal{F}_d)$. It parameterizes the data (p, C, f) where

$$\begin{cases} p \simeq \mathbb{P}^2 \subset \mathbb{P}^4, \text{ plane;} \\ C \subset p: \text{ plane curve of deg } = k \\ f = \text{ threefold with deg } = d \text{ such that} \\ p \cap f \text{ is non-transverse all along } C. \end{cases}$$

The degree of the image of $\mathbb{P}(\widetilde{\mathcal{F}}_{d,k})$ in $\mathbb{P}(\mathcal{F}_d)$ can be found as before, using computer algebra. We list the first few values.

$k = 1: \ \frac{1}{6^3}\binom{d}{4}\big(18d^{10} - 162d^9 + 801d^8 - 2740d^7 + 6862d^6 - 13348d^5 + 20137d^4 - 24370d^3 + 23374d^2 - 14028d + 15120\big);$

$k = 2: \ \frac{10}{3^2 \cdot 11!}\binom{d-2}{2}\big(11916d^{20} - 307512d^{19} + 3384207d^{18} - 18547587d^{17} + \cdots + 973950299424000d^2 - 435442434716160d + 97440421478400\big);$

$k = 4, \ \frac{5}{3^2 19!}\binom{d-6}{2}\prod_0^7 (d^2 - 13d + 54 + 2i)\big(10499814d^{22} - 918982866d^{21} + \cdots - 2001589710277173359616000d + 62713770449105018880000\big).$

7. CONCLUDING REMARKS

Back in 1992, I was happily counting singular curves with up to six nodes varying in a family of smooth surfaces, partly just to amuse myself, partly trying to answer a question posed by Sheldon Katz on rational plane sections on a quintic threefold.

For instance, if we fix a smooth projective surface Y and let the curves vary in a 6–dimensional linear system with numerical invariants

$$d := c_1L \cdot c_1L,\ k := c_1L \cdot c_1K_Y,\ s := c_1K_Y \cdot c_1K_Y,\ x := c_2(Y)$$

we found (cf. [6]) a formula for the number of six-nodal curves, $N_6 = (81/80)d^6 + (81/40x - 567/8 + 81/20k)d^5 + (27/16x^2 + (27/4k - 1701/16)x - 81/8s + 8109/4 + 27/4k^2 - 4077/16k)d^4 + (3/4x^3 + (9/2k - 63)x^2 + \cdots$

Slightly later, according to Barbara Fantechi, Lothar Göttsche reportedly perceived in this maze of letters and numbers the first terms of an "obvious" expansion of the product of 4 power series depending solely on the four Chern numbers x, s, k, d, leading to his ground-breaking conjecture, [3].

It turns out that the case of intersections of a general quintic threefold with varying $\mathbb{P}^2$'s in $\mathbb{P}^4$ was of interest for string theory.

At about the same time, the cases of (singular) rational curves lying on K3-surfaces of low genera helped to confirm a generating series (physically) found by Yau and Zaslow,

$$\prod_{k>0} \frac{1}{(1-q^g)^{24}} = \sum_{d\geq 0} N_d q^d$$

$= 1+24q+324q^2+3200q^3+25650q^4+176256q^5+1073720q^6+5930496q^7+\cdots$ eventually proven au *goût des mathématiciens* by A. Beauville.

Recently, Yu-Jong Tzeng used Levine and Pandharipande's theory of algebraic cobordism to prove Göttsche's conjecture.

We may hope the formulas given here may also fit into some generating series. So, long live resolution of and/or counting singularities!

8. Appendix: Maple codes

```
with(schubert): #Thanks to Katz & Stromme!
DIM:=3; Q:=4-o(-h); answ:=[ ]:
for k to 3 do
  Sk:=Symm(k,Q); Sk1:=Symm(d-2*k,Q);
  F0:=o(-h)*Symm(d-1,4);mk:=binomial(k+2,2)-1;
  DIM:=3+mk; F0:=F0+o(-2*H)*Sk1;
  F0:=collect(mtaylor(chern(DIM,-F0),h,4),H);
  for i from degree(F0,H) by -1 to ldegree(F0,H) do
    F0:=subs(H^(i)=chern(i-mk,-Sk),F0)
  od:
  factor(%):F0:=%/h^3;answ:=[op(answ),[k,F0]]:
  print(k,degree(collect(F0,d),d),F0)
od:

with(schubert): #now P2's in P4
grass(2,5,q); Q:=dual(Qq);
sd:=Symm(d,Q);bfd:=collect(Symm(d,5)-sd,t);
for k to 6 do
#k:=1;
Sk:=Symm(k,Q); Sk1:=Symm(d-2*k,Q);
F0:=sd;mk:=binomial(k+2,2)-1;
DIM:=Gq[dimension_]+mk; F0:=F0+o(-2*H)*Sk1;
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```
F0:=collect(chern(DIM,-F0),H);
F0:=collect(F0-mtaylor(F0,H,mk),H):
for i from degree(F0,H) by -1 to ldegree(F0,H)  do
F0:=subs(H^(i)=chern(i-mk,-Sk),F0)
od:
F0:=subs(q1=t*q1,q2=t^2*q2,F0):
F0:=coeff(mtaylor(F0,t,7),t^6):integral(Gq,%);
print(k,factor(%));
od:
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