Addendum to "The Blume-Emery-Griffiths Model at an Infinitely Many Ground States Interface and Exponential Decay of Correlations at all Nonzero Temperatures"

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Abstract. We present a corrected proof of Theorem 1.1 of Ref. [1] and extend it to a subset of parameter values in the disordered region.

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Due to the wrong definition of b in Eq. (9) of Ref. [1], the proof of Theorem 1.1 presented in Section 3 of this reference is incorrect. The definition of b has no parenthesis on it, i.e., $b \equiv e^{\beta y}\cosh\beta - 1$, and therefore b is not positive if $y < -1$, a statement that the authors have used in the course of the proof. The aim of this Note is to present the correct proof of Theorem 1.1. Its statement remains the same but convergence condition (2) is replaced by another one which will be determined below. At the end of this Note we extend the proof to a subset of the parameter values x and y in the disordered region $\{(x,y) : x < 0\,, 1 + 2x + y < 0\}$.

Proof of Theorem 1.1**.** As in Ref. [1], we consider the interface $x = 0$ and $y < -1$. We replace (15) of Ref. [1] by

$$\langle \sigma_i\, \sigma_j \rangle^0_\Lambda = \sum_{\Gamma_1 \supset \{i,j\}} e^{\beta y |\Gamma_1^*|} \left(\sum_{\{\sigma\} \in \{-1,1\}^{|\Gamma_1|}} \sigma_i\, \sigma_j \prod_{(k,l)\in\Gamma_1^*} e^{\beta \sigma_k \sigma_l} \right) \frac{Z^0_{\Lambda - (\Gamma_1 \cup \partial\Gamma_1)}}{Z^0_\Lambda}, \quad (1)$$

where Γ_1 is a one-connected subset of Λ containing $\{i,j\}$ and the superscript 0 means 0-boundary condition (which is equivalent to the free-boundary condition in the considered interface, see Ref. [1]). Identity (1) holds true because of the spin flip symmetry of the the model, namely, for any configuration giving a non-zero contribution to the numerator of the expected value, the spin variables σ_i and σ_j must be connected to each other by a path where the spin values are $+1$ or -1.

We first analyze (1) at high temperature, uniformly in y. Eq. (1) can be written as

$$\langle \sigma_i\, \sigma_j \rangle^0_\Lambda = \sum_{\Gamma_1 \supset \{i,j\}} \langle \sigma_i\, \sigma_j \rangle^I_{\Gamma_1} e^{\beta y |\Gamma_1^*|} Z^I_{\Gamma_1} \frac{Z^0_{\Lambda - (\Gamma_1 \cup \partial\Gamma_1)}}{Z^0_\Lambda}, \quad (2)$$

where the index I stands for d-dimensional Ising model with free boundary condition. Since $\Gamma_1 \subset \Lambda$, then Griffiths inequalities (see Section 4.1 of Ref. [2]) imply that $\langle \sigma_i\, \sigma_j \rangle^I_{\Gamma_1} \le \langle \sigma_i\, \sigma_j \rangle^I_\Lambda$ and from (2) we get that

$$\begin{aligned} \langle \sigma_i\, \sigma_j \rangle^0_\Lambda &\le \langle \sigma_i\, \sigma_j \rangle^I_\Lambda \; \frac{\sum_{\Gamma_1 \supset \{i,j\}} e^{\beta y |\Gamma_1^*|}\, Z^I_{\Gamma_1}\, Z^0_{\Lambda - (\Gamma_1 \cup \partial\Gamma_1)}}{Z^0_\Lambda} \\ &\equiv \langle \sigma_i\, \sigma_j \rangle^I_\Lambda \frac{\tilde{Z}_\Lambda}{Z^0_\Lambda} \le \langle \sigma_i\, \sigma_j \rangle^I_\Lambda, \end{aligned} \quad (3)$$

where we have used that $\tilde{Z}_\Lambda Z_\Lambda^{-1} \le 1$ because $\tilde{Z}_\Lambda$ is obtained from Z^0_Λ by restricting it to those configurations σ, for which there is a one-connected subset $\Gamma_1 \subset \Lambda$ containing i and j and $\sigma_{\Gamma_1} \in \{-1,1\}^{|\Gamma_1|}$. The Ising spin-spin correlation function decays exponentially if $\beta < \beta^I_d$, where β^I_d is given by the relation $4d^2(\exp\beta^I_d - 1) = 1$ (this condition is obtained using, for instance, the argument given in Section 3 of Ref. [1]), implying that $\langle \sigma_i\, \sigma_j \rangle^0_\Lambda$ decays exponentially if $\beta < \beta^I_d$ and $y \in \mathbb{R}$.

Now we analyze (1) in the regime $\beta \geq \beta_d^I$. Using that the product $(Z_\Lambda^0)^{-1} Z^0_{\Lambda-(\Gamma_1\cup\partial\Gamma_1)}$ is bounded above by 1, that $|\Gamma_1| \leq 2|\Gamma_1^*|$ and taking $y < -1$, we get from (1)

$$\langle \sigma_i\, \sigma_j \rangle_\Lambda^0 \leq \sum_{\Gamma_1 \supset \{i,j\}} 2^{|\Gamma_1|}\; e^{\beta(y+1)|\Gamma_1^*|} \leq \sum_{\Gamma_1 \supset \{i,j\}} 2^{2|\Gamma_1^*|}\, e^{(y+1)\,\beta\,|\Gamma_1^*|}. \tag{4}$$

As at the end of Section 3 of Ref. [1], the last sum in (4) is bounded above by

$$\sum_{n \geq \|i-j\|} \left(16 d^2\; e^{(y+1)\,\beta_d^I} \right)^n , \tag{5}$$

which is convergent (and decays exponentially with the distance $\|i-j\|$) as long as $y < y_d$, where y_d is defined by the relation $16d^2\, \exp[(y_d+1)\,\beta_d^I] = 1$, completing the proof. $\square$

Using now the Hamiltonian $H(\sigma) = -\sum_{(i,j)}[\sigma_i\sigma_j + y\sigma_i^2\sigma_j^2 + x(\sigma_i^2+\sigma_j^2)]$, the above analysis can be extended to a subset of the disordered region $\{(x,y) : x < 0\,, 1 + 2x + y < 0\}$. We observe that (1), (2), (3) and the first inequality in (4) still hold if the exponential factor $\exp(\beta y |\Gamma_1^*|)$ is replaced by $\exp[\beta(2x+y)|\Gamma_1^*| + \beta x |\partial\Gamma_1|]$. The last inequality in (4) and inequality (5) hold if $x \leq 0$ and if we replace y by $2x + y$. Therefore, we have extended Theorem 1.1 of Ref. [1]:

Theorem 1 *The spin-spin correlation function of the d-dimensional BEG model with Hamiltonian* $H(\sigma) = -\sum_{(i,j)}[\sigma_i\sigma_j + y\sigma_i^2\sigma_j^2 + x(\sigma_i^2 + \sigma_j^2)]$ *decays exponentially if (i)* x, $y \in \mathbb{R}$ *and* $\beta < \beta_d^I$ *or (ii)* $x \leq 0$, $\beta \geq \beta_d^I$ *and* $2x + y < y_d$, *where* β_d^I *and* y_d *are defined as above.*

[1] Gastão A. Braga and Paulo C. Lima, 2003, J. Phys. A: Math. Gen. **36** 9609.

[2] J. Glimm and A. Jaffe, 1987, *Quantum Physics, A functional integral point of view*, Second Edition, Springer, New York.