BEST CONSTANTS, OPTIMAL SOBOLEV INEQUALITIES ON RIEMANNIAN MANIFOLDS AND APPLICATIONS

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A thesis submitted to the
Graduate School—New Brunswick
Rutgers, The State University of New Jersey
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
Graduate Program in Mathematics

Written under the direction of
Prof. Yan Yan Li

and approved by

New Brunswick, New Jersey
October, 2003
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In Chapter 1, a sharp form for a pointwise estimate of blow-up solutions of a $p$-Laplacian type equation, which has been used among other things in the proof of the validity of optimal Sobolev inequalities on complete Riemannian manifolds, is obtained through the comparison principle and the careful choice of a test function.

Chapter 2 shows several ranges in which first order optimal Sobolev inequalities are not valid in Riemannian manifolds with positive scalar curvature somewhere, by showing that the associated Sobolev quotient becomes arbitrarily large for a family of minimizers of the Sobolev quotient localized at a point where the scalar curvature is positive. A comparison is made between those ranges where the optimal inequalities are known to be valid and those where we found they are not, showing that there remain a few ranges where it is not known if the optimal inequalities hold or not.

In Chapter 3, the best constants for Sobolev trace inequalities on Riemannian manifolds with boundary are established for any $1 < p < n$. A version of the concentration-compactness principle for manifolds with boundary and the almost everywhere convergence of the gradients of solutions of a $p$-Laplace type equation are used in the proof.

In Chapter 4, the best constant for certain second order Sobolev inequalities on compact Riemannian manifolds with or without boundary, and its application to the
resolution of fourth order nonlinear elliptic partial differential equations with critical exponent are studied. More precisely, if \((M, g)\) is a smooth compact Riemannian manifold, with or without boundary, of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\), the norm

\[
\|u\| = \left( \|\Delta_g u\|_{L^p(M)}^p + \|u\|_{L^p(M)}^p \right)^{1/p}
\]

is considered on each of the spaces \(H^{2,p}(M)\), \(H^{2,p}_0(M)\) and \(H^{2,p}(M) \cap H^{1,p}_0(M)\). The existence of an asymptotically sharp inequality associated to the critical Sobolev embedding of these spaces is shown. The non-validity of the associated optimal inequality for Riemannian manifolds with positive scalar curvature somewhere in the \(p = 2\) case is also proven. As an application of the asymptotically sharp inequality, the influence of the geometry in the existence of solutions for some fourth order problems involving critical exponents on manifolds is investigated. In particular, new phenomena arise in Brezis-Nirenberg type problems on manifolds with positive scalar curvature somewhere, in contrast with the Euclidean case. The existence of solutions is proven through a suitable version of the concentration-compactness principle and issues of regularity of solutions are addressed in some cases.
Preface

This work deals on matters related to best constants and optimal Sobolev inequalities on Riemannian manifolds and some of their applications to the resolution of nonlinear elliptic partial differential equations with critical exponent. It comprises four chapters.

In Chapter 1 we obtain a sharp form for a pointwise estimate of blow-up solutions of a $p$–Laplacian type equation, which has been used among other things in the proof of the validity of optimal Sobolev inequalities on complete Riemannian manifolds. It will be the subject of a joint paper with Prof. Yan Yan Li [19].

Chapter 2 concerns the ranges in which first order optimal Sobolev inequalities are not valid.

Chapter 3 deals with the problem of finding the best constant for Sobolev trace inequalities on Riemannian manifolds with boundary. Part of the results of this chapter were announced in the First Workshop in Nonlinear Analysis and PDEs, held in Belo Horizonte, Brazil, February 2002, and were published on Nonlinear Analysis: Theory, Methods and Applications [18].

In Chapter 4 we study the best constant for second order Sobolev inequalities on compact Riemannian manifolds with or without boundary, and its application to the resolution of fourth order nonlinear elliptic partial differential equations with critical exponent and under different boundary conditions, specially fourth order Brezis-Nirenberg type problems. The non-validity of the associated optimal inequality in the $p = 2$ case is also considered. The results of this chapter are a joint work with Prof. Marcos Montenegro, currently at Universidade Federal de Minas Gerais, Brazil, and were published on Journal de Mathématiques Pures et Appliquées [20].
Acknowledgements

I would like to thank my thesis advisor, Prof. Yan Yan Li, for his infinite patience as well as for his mathematical and financial support, and the members of the committee: Profs. Haim Brezis, Zheng-Chao Han, Xiaochun Rong and Thierry Aubin. I would like also to thank Prof. Marcos Montenegro for his mathematical help in this work and Alfredo Rios, Augusto Ponce and Prof. Antônio Zumpano for helping me in several ways, giving me peace of mind from many bureaucratic matters.

I also want to thank CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), for its financial support during the first four years of my studies at Rutgers, Universidade Federal of Minas Gerais (UFMG), for sustaining me during the whole period of my studies, and my Mathematics Department (Departamento de Matemática) at UFMG, for all the help they gave me, particularly during the final stages of writing my thesis.

My thanks go also to the secretaries, Carla Ortiz and Barbara Sirman, who helped me make the necessary arrangements and preparations for my defense while I was in Brazil.

Thanks to my relatives Alda, Didico, Evânio e Maria do Carmo, for backing me during the specially hard times at the very first year of my graduate studies.

My heartily thanks to Dona Zelita, for coming to the US and helping us during the difficult times we had (given our inexperience) with the birth of our daughter and thereafter helping baby-sit her.

My greatest thanks to my wife Valéria, without whose support I could not have finished this work.

And, above all, I thank and praise G-d. May this work be one of His infinite ways of leading me closer and closer to Him.
Dedication

I dedicate this work to my wife Valéria and my daughter Rachel for the inspiration and joy they bring into my life.
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Chapter 0

Introduction

In the following, we present the problem of best constants and optimal Sobolev inequalities in its historical context and some of the most important results in the field, as well as a detailed description of each chapter of this work.

0.1 Best Constants and the Yamabe Problem

The importance of best constants of Sobolev inequalities in the study of nonlinear partial differential equations was first recognized during the work done towards solving the Yamabe problem, which was the first nonlinear PDE with critical exponent successfully solved.

The Yamabe problem is simply stated as follows: “Given a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\), is there a metric \(\tilde{g}\) conformal to \(g\) such that the corresponding scalar curvature \(R_{\tilde{g}}\) is constant?” The existence of such a metric is equivalent to the existence of a positive smooth solution to the following semilinear elliptic equation with critical exponent:

\[-\frac{4(n-1)}{n-2} \Delta_g u + R_g u = R_{\tilde{g}} u^{2^* - 1} \quad \text{in } M, \quad (1)\]

where \(R_{\tilde{g}}\) is some real constant. Here, \(\Delta_g u = \text{div}_g(\nabla_g u)\) denotes the Laplace-Beltrami operator and \(2^* = 2n/(n-2)\) is the critical exponent for the Sobolev embedding \(H^{1,2}(M) \hookrightarrow L^q(M)\).

In attempting to solve this equation using variational techniques, Yamabe [90] considered, for \(2 < q \leq 2^*\), the functionals

\[ J_q(u) = \frac{4(n-1)}{n-2} \int_M |\nabla_g u|^2 \, dv_g + \int_M R_g(x) u^2 \, dv_g \bigg/ \left( \int_M |u|^q \, dv_g \right)^{2/q} \]
defined in $H^{1,2}(M) \setminus \{0\}$, and searched for positive minimizers of these functionals. For $2 < q < 2^*$, since the embedding $H^{1,2}(M) \hookrightarrow L^q(M)$ is compact, it is easy to find a sequence of non-negative functions $\{u_q\}_{2 < q < 2^*}$ such that $\|u_q\|_{L^q(M)} = 1$ and $J_q(u_q) = \mu_q$, where

$$\mu_q = \inf_{H^1(M) \setminus \{0\}} J_q.$$ 

Such minimizers are weak solutions of the corresponding subcritical elliptic equation

$$-\Delta_g u + R_g u = \mu_q u^{q-1} \quad \text{in } M,$$

and regularity theory for elliptic equations together with the maximum principle ensure that $u_q \in C^\infty(M)$ and $u_q > 0$. On the other hand, since the embedding $H^{1,2}(M) \hookrightarrow L^{2^*}(M)$ is only continuous, the direct variational method is unable to show the existence of a solution to (1). Instead, Yamabe assumed the functions $\{u_q\}_{2 < q < 2^*}$ to be uniformly bounded in $M$ and thereby showed that the subcritical solutions $u_q$ approach the solution $u$ to (1) when $q \to 2^*$.

However, Trudinger [85] showed that Yamabe’s assumption about the uniform boundedness of the subcritical solutions was not always correct. In certain situations, Yamabe’s assumption is indeed true: denoting $\mu = \mu_{2^*}$, it is well known that this is a conformal invariant, sometimes called the *Yamabe invariant*, and if $\mu \leq 0$ the functions $\{u_q\}_{2 < q < 2^*}$ are in fact uniformly bounded in $M$ (Yamabe’s equation has the right sign in this case; see [7]). However, if $\mu > 0$, this assertion is false (for a counterexample in the sphere see [7]).

In correcting Yamabe’s proof, Trudinger proved that there exists a positive constant $\alpha(M)$, which in principle depends on the Riemannian manifold $M$, such that if $\mu = \mu(M) < \alpha(M)$, then the Yamabe problem has a solution. On the other hand, Aubin showed that $\mu(M) \leq \mu(S^n)$, where $S^n$ denotes the $n$-dimensional sphere with the standard metric. In fact, Aubin proved that the constant $\alpha(M)$ does not depend on the manifold $M$ and, moreover, gave its explicit value in terms of the best constant $K(n,2)$ of the Sobolev inequality in $\mathbb{R}^n$, whose precise value he had already obtained.
(as well as, independently, Talenti [83]):

\[
\frac{1}{K(n, 2)} = \inf_{u \in L^2(\mathbb{R}^n) \setminus \{0\}, \nabla u \in L^2(\mathbb{R}^n)} \frac{\|\nabla u\|_{L^2(\mathbb{R}^n)}}{\|u\|_{L^2(\mathbb{R}^n)}} = \sqrt{\frac{n(n-2)\omega_n^{2/n}}{4}}.
\]

Aubin [9] showed that \(K(n, 2)\) is the best constant for the Sobolev inequality on any Riemannian manifold in the sense that, given \(\varepsilon > 0\), there exists \(A_\varepsilon > 0\) such that for all \(u \in H^{1,2}(M)\) we have

\[
\|u\|_{L^2(M)} \leq (K(n, 2) + \varepsilon) \|\nabla u\|_{L^2(M)} + A_\varepsilon\|u\|_{L^2(M)},
\]

and thereby proved [10] the following result:

**Theorem.** If

\[
\mu < 4(n-1) \frac{1}{n-2} \frac{1}{K^2(n, 2)} = n(n-1)\omega_n^{2/n},
\]

then (1) has a positive smooth solution with \(R_\tilde{g} = \mu\).

In particular, \(\alpha(M) = \mu(S^n)\).

Thus, Aubin’s result reduced solving Yamabe’s problem to finding appropriate test functions \(v\) that would satisfy

\[
J(v) := J_{2\cdot}(v) < n(n-1)\omega_n^{2/n}.
\]

Aubin himself was able to find local test functions for compact nonlocally conformally flat Riemannian manifolds of dimension \(n \geq 6\), and by a careful study of the asymptotic expansion of the element of volume in normal geodesic coordinates proved the Yamabe problem for such manifolds. Such a local approach cannot work for conformally flat manifolds, since for the sphere we have \(\mu(S^n) = n(n-1)\omega_n^{2/n}\). The solution of the Yamabe problem for the remaining cases was done by Schoen [81], who constructed a global test function \(v\) which was simply the Green function for the conformal Laplace operator \(-\Delta_g + [(n-2)/4(n-1)]R_g\) with the singularity smoothed out; for the analysis of the asymptotic expansion of \(J(v)\), he had to use the positive mass theorem, which had recently been proved by him and Yau. It must be remarked that Schoen’s proof works only in the cases not covered by Aubin. For full details on all this, as well as a
full account of the origins of this problem and several other solutions to it, see [7], [63] and the references therein.

Later, many other nonlinear problems in PDEs were solved using the best constant $K(n, 2)$. Specially, it is worthy mentioning the Brezis-Nirenberg problem (see [24]), which is also related to this work (see Chapter 4). For a survey of such problems and their solutions, the reader may consult [7] or [51].

### 0.2 Optimal Inequalities

By showing that the best constant for the Sobolev critical inequality does not depend on the manifold, Aubin also introduced the study of optimal Sobolev inequalities. Let $(M, g)$ be a $n$-dimensional smooth Riemannian manifold without boundary. We denote by $H^{1, p}(M) = H^1_0(M)$ the Sobolev space of order $p$, which by definition is the completion of $C_0^\infty(M)$ under the norm

$$
\|u\|_{H^{1, p}(M)} = \left( \|\nabla u\|_{L^p(M)}^p + \|u\|_{L^p(M)}^p \right)^{1/p}.
$$

(2)

Let $1 \leq p < n$. If $M$ is compact, the embedding $H^{1, p}(M) \hookrightarrow L^q(M)$ is compact for $1 \leq q < p^*$ and only continuous for $q = p^*$. If $M$ is complete, the validity of the Sobolev embedding $H^{1, p}(M) \hookrightarrow L^{p^*}(M)$ depends on assuming additional hypotheses on $M$. Indeed, there are complete manifolds for any dimension $n \geq 2$ for which the Sobolev embedding is not valid; for example, complete noncompact Riemannian manifolds of finite volume (see [51], Proposition 3.5, for details, and Proposition 3.4 for other examples). Aubin [8] and Cantor [29] proved, independently, that the Sobolev embedding holds for complete manifolds with positive injectivity radius and bounded sectional curvature. Almost a decade later, Varopoulos [88] showed that the Sobolev embedding holds for complete manifolds if only the Ricci curvature is bounded from below, as long as there is also a lower bound for the volume of small balls, uniform with respect to their center (Varopoulos used semigroup techniques to prove this result; a more natural proof is presented in [51]). By a geometrical result due to Croke [34], a lower bound on the injectivity radius implies a lower bound for the volume of small balls which is uniform with respect to their center. Therefore, the Sobolev imbedding is valid for
complete manifolds with positive injectivity radius and Ricci curvature bounded from below. On the other hand, there are complete manifolds whose Ricci curvatures are not bounded from below for which the Sobolev embedding is valid (see [51]).

Therefore, under the hypotheses discussed above, there exist numbers $A, B \in \mathbb{R}$ such that for all $u \in H^{1,p}(M)$ there holds

$$
\|u\|_{L^p(M)} \leq A \|\nabla u\|_{L^p(M)} + B \|u\|_{L^p(M)}. 
$$

Following the terminology and notation introduced by Hebey in [51], we call this inequality the generic Sobolev inequality of order $p$, and consider the best constants associated with it:

$$
\alpha_p(M, g) = \inf \left\{ A \in \mathbb{R} : \text{there exists } B \in \mathbb{R} \text{ such that } (I_{p,\text{gen}}^1) \text{ is satisfied} \right\},
$$

$$
\beta_p(M, g) = \inf \left\{ B \in \mathbb{R} : \text{there exists } A \in \mathbb{R} \text{ such that } (I_{p,\text{gen}}^1) \text{ is satisfied} \right\}.
$$

A priori, the first best constant $\alpha_p(M, g)$ depends on the metric $g$, but Aubin [9] proved that $\alpha_p(M, g)$ depends only on $n$ and $p$. In fact, $\alpha_p(M, g) = K(n, p)$, where $K(n, p)$ is the best constant for the Sobolev embedding for $M = \mathbb{R}^n$ under the Euclidean metric, i.e.:

$$
\frac{1}{K(n, p)} = \inf_{\nabla u \in L^p(\mathbb{R}^n) \setminus \{0\}} \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}}. 
$$

The value of $K(n, p)$ was explicitly computed independently by Aubin [8] himself and Talenti [83]:

$$
K(n, 1) = \frac{1}{n} n^{1/n} \omega_{n-1}^{1/n},
$$

$$
K(n, p) = \frac{p-1}{n-p} \left( \frac{n-p}{n(n-1)} \right)^{1/p} \left[ \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{p}\right) \Gamma\left(n + 1 - \frac{n}{p}\right) \omega_{n-1}} \right]^{1/n},
$$

where $\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$ and $\Gamma$ is the gamma function. Aubin proved that $\alpha_p(M, g) = K(n, p)$ for complete manifolds with positive injectivity radius and bounded sectional curvature, but later, Hebey [49] showed that Aubin’s conclusion still remains valid for complete manifolds with positive injectivity radius and Ricci curvature bounded from below.
Different from the first best constant, for compact Riemannian manifolds the second best constant $\beta_p(M, g)$ is simple to determine and depends very strongly on the metric $g$ of the manifold (see [51], Theorem 4.1; see also Section 3.5 here):

$$\beta_p(M, g) = \text{vol}_g(M)^{-1/n}.$$  

For complete Riemannian manifolds for which the Sobolev embedding holds (and so $\text{vol}_g(M)^{-1/n} = 0$), the value of the second best constant remains an open problem. All one can say is that for certain manifolds we have $\beta_p(M, g) \leq 0$ (those manifolds for which $(I^1_{1p})$ is valid with $B = 0$; see [51], chapter 8, for several examples and counterexamples, and the relation with the Faber-Krahn inequality), and even actually $\beta_p(M, g) < 0$ for some (for instance, the hyperbolic space; see Theorem 7.7 in [51]).

A natural question is if the best constants are achieved, i.e., if there exists a constant $B$ such that

$$\|u\|_{L^p^*} \leq K(n, p) \|\nabla u\|_{L^p(M)} + B \|u\|_{L^p(M)} \quad (I^1_p)$$

for all $u \in H^{1,p}(M)$, or if there exists a constant $A$ such that

$$\|u\|_{L^p^*} \leq A \|\nabla u\|_{L^p(M)} + \text{vol}_g(M)^{-1/n} \|u\|_{L^p(M)} \quad (J^1_p)$$

for all $u \in H^{1,p}(M)$. These inequalities are called optimal Sobolev inequalities.

By extension, one can consider the validity or not of the following optimal scaled inequalities

$$\|u\|_{L^{p^*}}^a \leq K(n, p)^a \|\nabla u\|_{L^p(M)}^a + B \|u\|_{L^p(M)}^a \quad (I^a_p)$$

$$\|u\|_{L^{p^*}}^a \leq A \|\nabla u\|_{L^p(M)}^a + \text{vol}_g(M)^{-a/n} \|u\|_{L^p(M)}^a \quad (J^a_p)$$

for $0 < a \leq p$, and of the optimal inequality with lower-order remainder term

$$\|u\|_{L^{p^*}}^p \leq K(n, p)^p \|\nabla u\|_{L^p(M)}^p + B \|u\|_{L^r(M)}^p \quad (I^p)$$

for $1 \leq r < p^*$. 

The first general result concerning these optimal inequalities was obtained by Aubin himself [9], who showed $(I^1_p)$ to be true for $1 \leq p < n$ for complete manifolds with positive injectivity radius and constant sectional curvature (if $n = 2$, this last hypothesis
can be replaced by bounded sectional curvature). Brezis and Lieb [23] proved the validity of (I\_2) for Euclidean bounded domains; in addition, they considered on balls the problem of the second best constant given the first best constant (for details and further results on this problem, see chapter 5 of [51]). Their work was later extended to arbitrary smooth Euclidean domains by Adimurthi and Yadava [3], who also considered optimal Sobolev trace inequalities (see the end of this section).

Aubin’s result was extended by Hebey and Vaugon [54], in the case p = 2, for complete, conformally flat manifolds, with positive injectivity radius and bounded sectional curvature. They were later able to refine this result in [55], demanding conformal flatness only outside some compact subset of M. Finally, Hebey and Vaugon [55] extended this result even further, proving (I\_2) for complete manifolds with positive injectivity radius and bounded Riemann curvature up to the first order. Surprisingly, (I\_2) is false if one only assumes that the Ricci curvature is bounded from below, instead of the bound on the Riemann curvature and its first covariant derivative, at least for Riemannian manifolds of dimension n ≥ 4 (for if (I\_2) is valid for such manifolds, their Ricci curvature is necessarily bounded, but there exist complete Riemannian manifolds with positive injectivity radius and Ricci curvature bounded from below but not from above; see [51], Theorem 7.3, for details).

Aubin, based on positive results he obtained for the standard spheres S^n, conjectured in [8] that (I\_p) is true for 1 < p ≤ 2, and that (I\_p) is true for 2 < p < n and a = 2. Druet [38] proved that (I\_p) is indeed false for 2 < p < \sqrt{n} for any Riemannian manifold with positive scalar curvature at some point; in particular, it follows from Aubin’s result quoted in the beginning of the previous paragraph, that (I\_1) is true for the standard spheres, but (I\_p) is not, for 2 < p < \sqrt{n}. Druet also showed that the positive scalar curvature hypothesis is necessary, by proving that (I\_p) is valid for any 1 < p < n in flat tori. Moreover, Aubin, Druet and Hebey [11] proved that (I\_p) is always valid for compact manifolds with non-positive sectional curvature of dimension n = 2, 3 or 4.

Recently, Aubin and Li [12] and, independently, Druet [39], obtained more general positive results for the inequalities above. Druet proved Aubin’s conjecture for compact
manifolds. Aubin and Li proved Aubin’s conjecture and \((I_p^r, I_p^a)\) for several ranges of \(p, r\) and \(a\), in complete manifolds with positive injectivity radius and bounded Riemann curvature up to the first order (for the specific values of these ranges, see Theorems 0.1 and 0.2 in Section 0.4 below). In particular, the confirmation of Aubin’s conjecture implies that \((I_p^1)\) is valid for \(n \geq 2\) and \(1 < p < n\) for any complete manifold with positive injectivity radius and bounded Riemann curvature up to the first order.

Concerning optimal inequalities in complete or compact Riemannian manifolds without boundary, it should be also mentioned the recent results by Druet, Hebey and Vaugon [41], and by Hebey [52], where further ranges for the validity of \((I_p^r)\) in lower dimensions are considered, and the work [64] by Li and Ricciardi on the determination of the sharp optimal Sobolev inequality for \(p = 2\).

With respect to the validity or not of the second optimal inequalities \((J_p^1)\) and \((J_p^p)\) on compact Riemannian manifolds, work done by Bakry [15] and by Druet and Hebey (presented in [51]; see also Section 3.5 here) showed that \((J_p^1)\) is always valid, while \((J_p^p)\) is valid if \(n = 2\) or if \(n \geq 3\) and \(1 < p \leq 2\), but is not valid if \(n \geq 3\) and \(p > 2\).

If \((M, g)\) is a compact Riemannian manifold with boundary, then \(H^{1,p}(M)\) denotes the completion of \(C^\infty(M)\) under the norm (2) and \(H^{1,p}(M) \neq H_0^{1,p}(M)\). The critical Sobolev embedding is valid and therefore the same questions concerning the best constants of \((I_{p,\text{gen}}^1)\) and its associated optimal inequalities apply, except that now we have to consider two distinct Sobolev spaces. When we consider \((I_{p,\text{gen}}^1)\) on \(H_0^{1,p}(M)\), the same results for best constants and optimal inequalities in compact manifolds without boundary described above remain true. On the other hand, if we consider \((I_{p,\text{gen}}^1)\) on \(H^{1,p}(M)\), Cherrier [33] has shown that the first best constant is \(2^{1/n}K(n, p)\). However, besides this result, little further work was done in this setting, specially concerning the study of optimal Sobolev inequalities. A different optimal Sobolev inequality on \(H^{1,2}(M)\) which includes a boundary term on the right-hand side of \((I_p^2)\) was considered by Adimurthi and Yadava [3], for bounded Euclidean domains of \(\mathbb{R}^n\) and \(n \geq 5\), and by Li and Zhu [66], for compact Riemannian manifolds with boundary and dimension \(n \geq 3\). In [91] and [92], Zhu establishes some general forms of sharp Sobolev inequalities on the half space and compact Riemannian manifolds with boundary for \(p = 2\), showing
that some previous results were special cases.

Sobolev-type inequalities are related to many other inequalities that appear in Analysis and the study of partial differential equations. For elementary and self-contained proofs of the equivalence of Sobolev inequalities and other classical inequalities, see the suggestively titled [16] by Bakry, Coulhon, Ledoux and Saloff-Coste. Specially, Gagliardo-Nirenberg type inequalities are exhaustively discussed. Following their ideas, Hebey proves in [51], chapter 8, the equivalence between the Euclidean-type generic Sobolev inequality and the Nash inequality, in the sense that one is valid if and only if the other is. The best constant in Nash’s inequality for the Euclidean space was obtained by Carlen and Loss [31], and optimal versions of it for Riemannian compact manifolds were studied by Druet, Hebey and Vaugon [42] and Humbert [59] (which applied techniques similar to the ones used by Druet in order to prove the validity of the optimal Sobolev inequality (\(I_p^*)\)); in addition, Humbert also studied the optimal trace version of Nash inequality on compact manifolds with boundary in [58] and [60]. Some works on the best constant, sharp inequalities and extremal functions for the Hardy inequality which are related and which we would like to mention are [26], [71], [27] and [80]; Hardy inequalities with remainder terms were studied in [28] and [2].

0.3 Chapter 1: A sharp pointwise estimate

Pointwise estimates of blow-up solutions are an important ingredient in the study of the asymptotic behavior of solutions of singularly perturbed problems and, in this role, are crucial in the determination of the validity of optimal inequalities on Riemannian manifolds.

For instance, let \( \Omega \subset \mathbb{R}^n \) be a bounded smooth domain, and for \( \varepsilon \geq 0 \) consider the problem

\[
\begin{aligned}
-\Delta u &= \varepsilon u + n(n-2)u^{2^*-1} & \quad \text{in } \Omega, \\
u &> 0 & \quad \text{in } \Omega, \\
u &= 0 & \quad \text{on } \partial \Omega,
\end{aligned}
\]

(4)
for $n \geq 4$, and
\[
\begin{align*}
-\Delta u &= n(n - 2)u^{2^*-1-\varepsilon} \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
for $n \geq 3$. When $\varepsilon = 0$, these problems become the same. Brezis and Nirenberg [24] proved that (4) has a solution $u_\varepsilon$ if $0 < \varepsilon < \lambda_1$, where $\lambda_1$ denotes the first eigenvalue of the Laplacian with homogeneous Dirichlet boundary condition, and it is well known that (5) has a solution $u_\varepsilon$ for any $\varepsilon > 0$, but if $\varepsilon = 0$ and $\Omega$ is starshaped, then Pohozaev’s identity shows that (4) has no solution. If $\varepsilon = 0$ but $\Omega$ has non-trivial topology (i.e., at least one of the homology groups of $\Omega$ over $\mathbb{Z}_2$ is nontrivial), Bahri and Coron [14] proved that (4) has a solution. Even when $\Omega$ is contractible, (4) can still have a solution for $\varepsilon = 0$, as Ding [36] has shown, if the geometry is non-trivial in a certain sense. The asymptotic behavior of the nearly critical solutions $u_\varepsilon$ of (4) and (5) when $\varepsilon \to 0$ was studied by many authors, beginning with Atkinson and Peletier [6], who investigated the asymptotic behavior of radially symmetric solutions of (5) in balls of $\mathbb{R}^n$. Later, Brezis and Peletier [25] gave different proofs for some of the results of [6], and made several conjectures about both problems in domains of $\mathbb{R}^n$. These conjectures were proved independently by Han [48] and Rey [79], who extended their results to general domains of $\mathbb{R}^n$. In [48], Han proved the existence of a pointwise estimate for the subcritical solutions which are also minimizers of the Sobolev quotient; in fact, after normalization, he showed that the solutions are bounded from above by a minimizer for the Sobolev quotient (3) in $\mathbb{R}^n$.

More precisely, using the fact that $\max_{\Omega} u_\varepsilon \to \infty$, let $x_\varepsilon$ be a point of maximum for $u_\varepsilon$ and define $\mu_\varepsilon = u_\varepsilon^{-(n-2)/2}(x_\varepsilon)$. Since $u_\varepsilon = 0$ on $\partial \Omega$, clearly $x_\varepsilon$ is an interior point of $\Omega$ and, moreover, it can be shown through the method of moving planes and an interior integral estimate that the $x_\varepsilon$ stay away from the boundary when $\varepsilon \to 0$. Set
\[
v_\varepsilon(x) = \mu_\varepsilon^{(n-2)/2}u_\varepsilon(\mu_\varepsilon^{1-(n-2)/4}x + x_\varepsilon)
\]
where $x \in \Omega_\varepsilon = \mu_\varepsilon^{(n-2)/4-1}(\Omega - x_\varepsilon)$. Then elliptic theory implies that $v_\varepsilon$ converges
uniformly in compact sets of $\mathbb{R}^n$ to a function $v$ which is the unique solution of
\begin{equation}
\begin{cases}
-\Delta v = n(n-2)v^{2^*-1} & \text{in } \mathbb{R}^n, \\
v(0) = 1 & \\
0 \leq v \leq 1 & \text{in } \mathbb{R}^n.
\end{cases}
\end{equation}

Therefore,
\[ v(x) = \frac{1}{\left(1 + |x|^2\right)^{\frac{n-2}{2}}} \cdot \]

After some careful estimates (see Lemma 3 in [48]), Han proved that there exists a constant $C > 0$ such that
\[ v_\varepsilon(x) \leq C v(x) \quad \text{for all } x \in \Omega_\varepsilon. \]

Analogously, when proving the validity of optimal Sobolev inequalities on Riemannian manifolds (in [55], [56], [65], [38] and [12]), pointwise estimates for blow-up solutions of an equation with critical exponent and a singular perturbation of a subcritical nonlinear term similar to (4) have been obtained.

In [56], Hebey and Vaugon considered the ball $B_2 \subset \mathbb{R}^n$ of radius 2 and centered at the origin, and for each $\alpha > 0$ a metric $g^\alpha$ on $B_2$ satisfying
\[ \frac{1}{4} \delta_{ij} \leq g^\alpha_{ij} \leq 4 \delta_{ij} \quad \text{in } B_2, \]
\[ |g^\alpha_{ij}|, |\partial_k g^\alpha_{ij}| \leq K \quad \text{in } \overline{B_2}. \]

For each $\alpha > 0$, they further considered a minimizer positive solution $u_\alpha$ (whose existence can be established through the Yamabe method) of the elliptic equation
\begin{equation}
-\Delta_{g^\alpha} u + \alpha u = n(n-2)u^{2^*-1}
\end{equation}

in certain starshaped sets $\omega_\alpha$ with the property that $\overline{\omega_\alpha} \subset B_2$. For $\mu_\alpha = \max_{\omega_\alpha} u_\alpha^{-(n-2)/2} \to 0$, they defined the normalized sequence
\[ v_\alpha(y) = \mu_\alpha^{(n-2)/2} u_\alpha(\mu_\alpha y) \]

for $y \in \Omega_\alpha = \mu_\alpha^{-1} \omega_\alpha$. As in [48], Hebey and Vaugon proved that the sequence $v_\alpha$ converges uniformly in compact sets of $\mathbb{R}^n$ to the function $v$ defined by (6), and then
showed that for all large $\alpha$ there exists a constant $C$ independent from $\alpha$ such that

$$v_\alpha(y) \leq Cv(y) \quad \text{for all } y \in \Omega_\alpha.$$  

On Riemannian manifolds with boundary, similar estimates were established by Li and Zhu [65], for a Laplacian-type equation with Neumann boundary condition, in order to prove the validity of a certain optimal Sobolev trace inequality (see Section 0.5 for more details on this inequality).

Aubin and Li obtained a pointwise estimate for blow-up solutions of a more general $p$-Laplacian type equation in [12]. Specifically, assume that there exists a Riemannian metric $g$ on $B_2$ such that $B_2$ is convex under this metric and such that the curvature and its first covariant derivative satisfy $|R_{ijkl}|, |\nabla R_{ijkl}| \leq \delta^*$, for some $\delta^* > 0$ that can be considered as small as needed. Now, if $(M, g)$ is any Riemannian manifold, with or without boundary, and $1 < p < n$, $1 \leq r < p^*$ and $0 < a \leq p$, let

$$I_\alpha(u) = \frac{||\nabla g u||_{L^r(M)}^a + \alpha \|u\|_{L^r(M)}^a}{\|u\|_{L^p(M)}^a}.$$  

If

$$\lambda_\alpha := \inf_{H^1_0(M)} I_\alpha < \frac{1}{K^a},$$  

it follows that for every large $\alpha$ there exists $u_\alpha \in H^1_0(M) \cap C(M)$ such that $\|u_\alpha\|_{L^p(M)} = 1$, $I_\alpha(u_\alpha) = \lambda_\alpha$ and, therefore, $u_\alpha$ satisfies the Euler-Lagrange equation

$$-\Delta_{p,g} u_\alpha + \alpha \|u_\alpha\|_{L^r(M)}^{p-r} \|\nabla g u_\alpha\|_{L^p(M)}^{p-a} u_\alpha^{r-1} = \lambda_\alpha \|\nabla g u_\alpha\|_{L^p(M)}^{p-a} u_\alpha^{r-1}, \quad (8)$$  

where $\Delta_{p,g} u = \text{div}_g \left( (|\nabla g u|^{p-2} \nabla g u \right)$ denotes the $p$-Laplacian with respect to the metric $g$.

It can be easily shown then that max $u_\alpha \to \infty$, which allows one to “blow-up” the sequence $(u_\alpha)$. We now return to considering $M = B_1$. Taking some maximum point $x_\alpha \in B_1$ of $u_\alpha$, it follows that $x_\alpha \notin \partial B_1$ (since $u_\alpha = 0$ on $\partial B_1$). We define the normalization

$$v_\alpha(y) = \mu_\alpha^{n/p} u_\alpha(\psi_\alpha(y)),$$

where $\mu_\alpha = u_\alpha(x_\alpha)^{-p/n} \to 0$, $\psi_\alpha(y) = \exp_{x_\alpha}(\mu_\alpha y)$ and $y \in \Omega_\alpha := \psi_\alpha^{-1}(B_1)$. Then, if

$$v(y) = \frac{c_1}{(c_2 + |y|^{p-1})^{n-p/p}}$$
be the unique minimizer of the Sobolev quotient (3) which satisfies $v(0) = 1$, $\nabla v(0) = 0$ and $\|v\|_{L^p_*(\mathbb{R}^n)} = 1$, it follows that $v_\alpha \to v$ in $C^1_{loc}(\mathbb{R}^n)$. Under these conditions, Aubin and Li proved ([12], Proposition 3.1) the following result:

**Theorem 1.1.** [12] Let $g$ be a Riemannian metric on $B_2$ such that $B_2$ is convex under this metric and such that $|R_{ijkl}|, |\nabla R_{ijkl}| \leq \delta^*$, for some $\delta^* > 0$. Let $1 < p < n$, $1 < s \leq p$, $x_\alpha \in B_1$, $\varepsilon_\alpha \geq 0$, $0 \leq \lambda_\alpha \leq \Lambda$, $\mu_\alpha \to 0^+$, $\psi_\alpha(y) = \exp_{x_\alpha}(\mu_\alpha y)$ for $y \in \Omega_\alpha := \psi_\alpha^{-1}(B_1)$, $g_\alpha = \mu_\alpha^2 \psi_\alpha^* g$, and $v_\alpha \in H^1_{p}(\Omega_\alpha)$ be a solution of

\[-\Delta_{g_\alpha} v_\alpha + \varepsilon_\alpha \|v_\alpha\|_{L^p(\Omega_\alpha, g_\alpha)}^{p-1} v_\alpha^{s-1} = \lambda_\alpha v_\alpha^{p-1} \text{ in } \Omega_\alpha,\]  

satisfying $v_\alpha(0) = 1$, $0 \leq v_\alpha \leq 1$ and

\[
\int_{\Omega_\alpha} |v_\alpha - v|^{p^*} \, dv_{g_\alpha} \to 0, \\
\int_{\Omega_\alpha} |\nabla g_\alpha v_\alpha - \nabla g_\alpha v|^{p} \, dv_{g_\alpha} \to 0.
\]

Assume further that

\[\frac{\text{dist}_g(x_\alpha, \partial B_1)}{\mu_\alpha} = \text{dist}_{g_\alpha}(0, \partial \Omega_\alpha) \to \infty.\]

Then, there exist positive constants $C = C(n, p, s, \Lambda, \delta^*)$ and $D = D(n, p, s, \Lambda)$ such that for all large $\alpha$ we have

\[v_\alpha(y) \leq Cv(y)^{1-D\delta^*} \text{ for all } y \in \Omega_\alpha.\]

All the estimates for $p = 2$ in [55], [56] and [65] were sharp ($D = 0$), but relied heavily on the conformal invariance of the Laplacian, which is not present for $p \neq 2$. Aubin and Li established their pointwise estimate by a different method, consisting in using the Moser iteration technique in order to get a precise bound for the $p$-Laplacian of the $v_\alpha$ and then applying the comparison principle for $p$-Laplacian type equations to a suitable test function (see Lema 1.1 of Chapter 1 for a sketch of their ideas).

In Chapter 1, we make Aubin and Li’s estimate sharp, proving the following result:

**Theorem 1.2.** Under the conditions of Theorem 1.1, if $\delta^*$ is sufficiently small, there exists a positive constant $C = C(n, p, s, \Lambda, \delta^*)$ such that for all large $\alpha$ we have

\[v_\alpha(y) \leq Cv(y) \text{ for all } y \in \Omega_\alpha.\]
The method of proof of Theorem 1.2 also relies on the comparison principle for \( p \)-Laplacian type equations, plus the choice of a suitable test function in geodesic polar coordinates. Our starting point is Aubin and Li’s result, which allows us to bound the \( v_\alpha \) and their \( p \)-Laplacians by known functions, and thus is of great help in finding the appropriate test function. We give separate proofs for the \( p = 2 \) and the Euclidean metric cases. In particular, Theorem 1.1 and Theorem 1.2 taken together furnish a new proof for the \( p = 2 \) case which does not rely on the conformal invariance of the Laplacian.

0.4 Chapter 2: Nonvalidity of the optimal inequalities

In Chapter 2, we investigate ranges where the optimal Sobolev inequalities \((I^p_r),(I^n_p)\) and \((I^n_p)\) are not valid, and compare them with the ranges obtained by Aubin and Li [12] where these optimal inequalities hold. In order to do this, we first enunciate their results more precisely.

Set, for \( n = 2, 3 \),

\[
r^*(n, p) = \begin{cases} \frac{np}{n - p + 2} & \text{if } p \in \left(1, \frac{n + 2}{3}\right) \cup (2, n), \\ \frac{n - p - 1}{n - p} & \text{if } p \in \left(\frac{n + 2}{3}, \sqrt{n}\right), \\ p & \text{if } p \in [\sqrt{n}, 2], \\ \end{cases}
\]

and, for \( n \geq 4 \) and \( 1 < p < n \),

\[
r^*(n, p) = \frac{np}{n - p + 2}.
\]

**Theorem.** [12] Let \((M, g)\) be a \( n \)-dimensional complete smooth Riemannian manifold with positive injectivity radius \( d > 0 \) and whose Riemann curvature satisfies

\[
|R_{ijkl}|, |
abla_m R_{ijkl}| \leq k.
\]

For \( n \geq 4 \) and \( 1 < p < n \), let \( r > r^*(n, p) \); for \( n = 2, 3 \) and \( p \in (1, \sqrt{n}) \cup (2, n) \), let \( r > r^*(n, p) \), or \( p \in [\sqrt{n}, 2] \), let \( r \geq r^*(n, p) \). Then there exists a constant \( A = A(n, p, r, d, k) \) such that inequality \((I^p_r)\) holds for all \( u \in H^{1,p}(M)\).
As a consequence from this theorem, it follows that for \( n \)-dimensional complete smooth Riemannian manifolds \((M, g)\) with positive injectivity radius and Riemann curvature bounded up to first order, inequality \((\text{I}_p^a)\) holds for all \( u \in H^{1,p}(M) \) if \( 1 < p \leq 2 \). We recall that this result was obtained independently by Druet [39] for compact manifolds.

**Theorem.** [12] Let \((M, g)\) be a \( n \)-dimensional complete smooth Riemannian manifold with positive injectivity radius \( d > 0 \) and whose Riemann curvature satisfies

\[
|R_{ijkl}|, |\nabla_m R_{ijkl}| \leq k.
\]

For \( n = 3, 4 \) and \( 2 < p < n \), let \( 0 < a < p \); for \( n > 4 \) and

(i) \( 2 < p \leq \sqrt{n} \), let \( 0 < a \leq 2 \);

(ii) \( \sqrt{n} < p \leq \frac{n + 2}{3} \), let \( 0 < a < \frac{2p}{-3p^2 + np + 2n} \);

(iii) \( \frac{n + 2}{3} \leq p < n \), let \( 0 < a < p \).

Then there exists a constant \( A = A(n, p, a, d, k) \) such that inequality \((\text{I}_p^a)\) holds for all \( u \in H^{1,p}(M) \).

In particular, it follows that for any \( 1 < p < n \) there exists \( a > 1 \) such that \((\text{I}_p^a)\) holds for all \( u \in H^{1,p}(M) \). Raising both sides to \( 1/a \), we conclude that the inequality \((\text{I}_p^a)\) holds for all \( u \in H^{1,p}(M) \) for any \( 1 < p < n \), if \((M, g)\) is a \( n \)-dimensional complete smooth Riemannian manifold with positive injectivity radius and Riemann curvature bounded up to first order.

We show that several of the results obtained by Aubin and Li are sharp and, moreover we extend Druet’s nonvalidity results [38] for \( n > 4 \) to \( \sqrt{n} < p < \frac{n + 2}{3} \) by using sharper estimates. Indeed, we prove the following:

**Theorem 2.1.** Let \((M^n, g)\) be a complete Riemannian manifold with positive scalar curvature somewhere. Then the optimal Sobolev inequality \((\text{I}_p^r)\) is not valid if

\[
1 < p < \frac{n + 2}{3} \quad \text{and} \quad r < \frac{np}{n - p + 2}, \quad \text{or if} \quad p = \frac{n + 2}{3} \quad \text{and} \quad r \leq \frac{np}{n - p + 2}.
\]

Theorem 2.1 shows that in the range \( 1 < p \leq \frac{n + 2}{3} \), the values of \( r \) that Aubin and Li obtained for which the optimal inequality is valid are sharp, except for the marginal
value \( r = \frac{np}{n - p + 2} \) in the range \( 1 < p < \frac{n + 2}{3} \), for which it is not known whether the optimal inequality is true or not. Moreover, it extends the range of nonvalidity of inequality (I_p^n) obtained by Druet [38] (for \( n > 4 \) and \( 2 < p < \sqrt{n} \), cited above in Section 0.2), by showing that for \( r = p \) this optimal inequality does not hold for \( \sqrt{n} \leq p < \frac{n + 2}{3} \) also (this same result can be alternatively obtained as a consequence of Theorem 2.2 next).

**Theorem 2.2.** Let \((M^n, g)\) be a complete Riemannian manifold with positive scalar curvature somewhere. Then the optimal Sobolev inequality (I_p^n) is not valid in any of the following situations:

- \( n = 3 \) and \( \begin{cases} 1 < p < \frac{5}{3} \text{ and } a > 2, \\ p = \frac{5}{3} \text{ and } a \geq 2, \end{cases} \)
- \( n = 4 \), \( 1 < p \leq 2 \) and \( a > 2; \)
- \( n > 4 \) and \( \begin{cases} 1 < p \leq \sqrt{n} \text{ and } a > 2, \\ \sqrt{n} < p < \frac{n + 2}{3} \text{ and } a > 2p \frac{p - 1}{n - p}, \\ p = \frac{n + 2}{3} \text{ and } a \geq p. \end{cases} \)

Theorem 2.2 shows that some results obtained by Aubin and Li are sharp, while others are not known if they can be improved or not. See Chapter 2 for a thorough discussion.

### 0.5 Chapter 3: Best constants in Sobolev trace inequalities

Let \((M, g)\) be a compact \( n \)-dimensional Riemannian manifold with boundary. If \( 1 \leq p < n \) and \( p^* = n \frac{n - 1}{n - p} \), the Sobolev trace embedding \( H^{1,p}(M) \hookrightarrow L^q(\partial M) \) is compact for \( 1 \leq q < p^* \), while \( H^{1,p}(M) \hookrightarrow L^{p^*}(\partial M) \) is only continuous. Therefore, there exist positive constants \( A_1, B_1 \), which may a priori depend on \( M \) and \( g \), such that the inequality

\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{1/p^*} \leq A_1 \left( \int_M |\nabla u|^p \, dv_g \right)^{1/p} + B_1 \left( \int_M |u|^p \, dv_g \right)^{1/p} \tag{12}
\]

holds for every \( u \in H^{1,p}(M) \). From this inequality, several other Sobolev-type trace inequalities follow: for instance (see Proposition 3.1), there exists a constant \( C > 0 \)
such that
\[
\int_M |u|^p \, dv_g \leq C \left( \int_M |\nabla u|^p \, dv_g + \int_{\partial M} |u|^p \, ds_g \right)^{1/p}
\] (13)
for all \( u \in H^{1,p}(M) \). From (12) and (13), there follows immediately that there exist positive constants \( A, B \), which may depend on \( M \) and \( g \), such that
\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{1/p^*} \leq A \left( \int_M |\nabla u|^p \, dv_g \right)^{1/p} + B \left( \int_{\partial M} |u|^p \, ds_g \right)^{1/p}
\] (1p,gen)
for every \( u \in H^{1,p}(M) \). This last inequality is our main focus of interest in this chapter.

Following the terminology and notation of Section 0.2, we call this inequality the \textit{generic Sobolev trace inequality of order} \( p \), and study the best constants associated to \((I_{1p,gen})\):
\[
\alpha_p(M, g) = \inf \left\{ A \in \mathbb{R} : \text{there exists } B \in \mathbb{R} \text{ such that } (I_{1p,gen}) \text{ is satisfied} \right\},
\]
\[
\beta_p(M, g) = \inf \left\{ B \in \mathbb{R} : \text{there exists } A \in \mathbb{R} \text{ such that } (I_{1p,gen}) \text{ is satisfied} \right\}.
\]
If the first best constant \( \alpha_p(M, g) \) is attained, that is, if \( \alpha_p(M) \in A_p(M) \), then there exists \( B > 0 \) such that
\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{1/p^*} \leq \alpha_p(M, g) \left( \int_M |\nabla u|^p \, dv_g \right)^{1/p} + B \left( \int_{\partial M} |u|^p \, ds_g \right)^{1/p}
\] (1p)
for all \( u \in H^{1,p}(M) \). This inequality will be called the \textit{first optimal Sobolev trace inequality of order} \( p \). Similarly, if the second best constant is attained, then there exists \( A > 0 \) such that
\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{1/p^*} \leq A \left( \int_M |\nabla u|^p \, dv_g \right)^{1/p} + \beta_p(M, g) \left( \int_{\partial M} |u|^p \, ds_g \right)^{1/p}
\] (Jp)
for all \( u \in H^{1,p}(M) \) and this inequality will be called the \textit{second optimal Sobolev trace inequality of order} \( p \). We also consider the scaled optimal inequalities
\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{p/p^*} \leq \alpha_p^p(M, g) \int_M |\nabla u|^p \, dv_g + B \int_{\partial M} |u|^p \, ds_g,
\] (Ip)
and
\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{p/p^*} \leq A \int_M |\nabla u|^p \, dv_g + \beta_p^p(M, g) \int_{\partial M} |u|^p \, ds_g.
\] (Jp)
\((Ip)\) and \((Jp)\) imply, respectively, \((I_1)\) and \((J_1)\). However, it is possible that either \((Ip)\) or \((Jp)\) is valid but, respectively, \((I_1)\) or \((J_1)\) is not. Indeed, as in case of optimal Sobolev
inequalities on Riemannian manifolds without boundary, there are ranges of $p$ for which $(\mathcal{J}_1^p)$ is valid, but $(\mathcal{J}_p)$ is not (Theorem 3.2).

The first result in these directions was obtained by P. L. Lions [70] in 1985, using his concentration-compactness principle. Lions proved that the best constant in the Sobolev trace embedding for the Euclidean half-space $\mathbb{R}^n_+ = \{(x', t) : t \geq 0\}$, i.e.,

$$
\frac{1}{K(n, p)} = \inf_{\nabla u \in L^p(\mathbb{R}^n_+)} \left( \frac{\int_{\mathbb{R}^n_+} |\nabla u|^p dx}{(\int_{\partial \mathbb{R}^n_+} |u|^p \, dx')^{1/p}} \right)^{1/p},
$$

is always attained. In 1988, using geometric methods, Escobar [44] and Beckner [17] found independently the minimizers for this Sobolev quotient in the case $p = 2$, and thereby were able to compute explicitly the value of $K(n, 2)$.

Using the explicit shape of the minimizers and via a blow-up method, Adimurthi and Yadava [3] proved the validity of the first optimal trace inequality of order 2 for Euclidean bounded domains in $\mathbb{R}^n$ with $n \geq 5$. Later, Y. Y. Li and M. Zhu [65] proved its validity for any compact Riemannian manifold with boundary of dimension $n \geq 3$.

They showed that $\bar{\sigma}_2(M, g) = K(n, 2)$. In particular, it follows that the first best constant does not depend on the metric $g$ when $p = 2$.

In this chapter, we show that the first best constant does not depend on either the metric $g$ or the manifold $M$ for any $1 < p < n$. Indeed, we prove the following result:

**Theorem 3.1.** [18] Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold with boundary and $1 < p < n$. Then

$$
\bar{\sigma}_p(M, g) = K(n, p).
$$

Different from the first best constant, as in the case of compact Riemannian manifolds without boundary, the second best constant depends very strongly on the metric $g$ of the manifold and the determination of the validity or not of $(\mathcal{J}_1^p)$ and $(\mathcal{J}_p)$ is simpler:

**Theorem 3.2.** [18] Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold with boundary and $1 < p < n$. Then $(\mathcal{J}_p^1)$ is valid with

$$
\bar{\beta}_p(M, g) = \text{vol}_p(\partial M) \cdot \frac{p-1}{p(n-1)},
$$
where $\bar{g} = g|_{\partial M}$. On the other hand, $(\mathcal{J}_p^g)$ is valid if and only if $n = 2$ and $1 < p < 2$, or if $n \geq 3$ and $1 < p \leq 2$.

Therefore, $(\mathcal{J}_p^g)$ is not valid if $2 < p < n$.

0.6 Chapter 4: Best constants in second order Sobolev inequalities and applications

Although some open problems still remain in the study of best constants and optimal Sobolev inequalities of first order on Riemannian manifolds, with or without boundary, the next step forward has already been taken and questions related to second order Sobolev inequalities have begun to be investigated very recently, particularly in connection with Paneitz-Branson type operators, which were introduced in [76] and [21]. We mention the works [4], [5], [35], [45], among others. As the earlier research was motivated by a second order invariant related to the Yamabe problem, so the later research has been motivated by the study of a fourth order invariant related to the prescribed scalar curvature problem.

Let $(M, g)$ be a smooth compact Riemannian manifold, with or without boundary, of dimension $n \geq 3$. For $1 < p < n/2$, we denote by $H_0^{1,p}(M)$ the standard first order Sobolev space defined as the completion of $C_0^\infty(M)$ with respect to the norm

$$
\|u\|_{H^1,p(M)} = \left( \int_M |\nabla_g u|^p \, dv_g + \int_M |u|^p \, dv_g \right)^{1/p},
$$

and by $H^{2,p}(M)$ and $H_0^{2,p}(M)$ the standard second order Sobolev spaces defined as the completion, respectively, of $C_0^\infty(M)$ and $C^\infty(M)$ with respect to the norm

$$
\|u\|_{H^2,p(M)} = \left( \int_M |\nabla^2_g u|^p \, dv_g + \int_M |\nabla_g u|^p \, dv_g + \int_M |u|^p \, dv_g \right)^{1/p}.
$$

In this work we consider the following Sobolev spaces:

$$
E_1 = H^{2,p}(M),
$$

if $M$ has no boundary, and

$$
E_2 = H_0^{2,p}(M),
$$
$E_3 = H^{2,p}(M) \cap H_0^{1,p}(M),$

if $M$ has boundary. Denoting by $\Delta_g u = \text{div}_g(\nabla u)$ the Laplacian with respect to the metric $g$, a norm on $E_i$ equivalent to $\|\cdot\|_{H^{2,p}(M)}$ is (see Appendix A to Chapter 4):

$$\|u\|_{E_i} = \left( \int_M |\Delta_g u|^p \, dv_g + \int_M |u|^p \, dv_g \right)^{1/p}.$$  

The Sobolev embedding theorem ensures that the inclusion $E_i \subset L^{p^*_2}(M)$ is continuous for $p^*_2 = \frac{np}{n - 2p}$. Thus, there exist constants $A, B \in \mathbb{R}$ such that

$$\|u\|_{L^{p^*_2}(M)}^p \leq A \|\Delta_g u\|_{L^p(M)}^p + B \|u\|_{L^p(M)}^p \quad (14)$$

for all $u \in E_i$. Consider, for each $i$, the first and second best constants associated to this inequality:

$$\alpha^i_p(M) = \inf \{A \in \mathbb{R} : \text{there exists } B \in \mathbb{R} \text{ such that inequality (14) holds}\}$$

and

$$\beta^i_p(M) = \inf \{B \in \mathbb{R} : \text{there exists } A \in \mathbb{R} \text{ such that inequality (14) holds}\},$$

respectively. Two natural questions in this context are the dependence or not of the best constants on the geometry of the manifold $M$, and the validity or not of the associated optimal inequalities:

$$\|u\|_{L^{p^*_2}(M)}^p \leq \alpha^i_p(M) \|\Delta_g u\|_{L^p(M)}^p + B \|u\|_{L^p(M)}^p \quad (15)$$

and

$$\|u\|_{L^{p^*_2}(M)}^p \leq A \|\Delta_g u\|_{L^p(M)}^p + \beta^i_p(M) \|u\|_{L^p(M)}^p \quad (16)$$

for all $u \in E_i$.

Concerning the second best constant and optimal inequality (16), work done by Bakry [15] and by Druet and Hebey (presented in [51]; see also Section 3.5 here) on first order Sobolev inequalities immediately generalizes to the second order case, and one finds that

$$\beta^i_p(M) = \text{vol}_g(M)^{-p/2n}$$
and that (16) is valid if and only if \( n = 3, 4 \) or if \( n \geq 5 \) and \( 1 < p \leq 2 \).

Similarly to what happens in the first order case, the study of the first best constant \( \alpha^i_{p}(M) \) and the optimal inequality (15) is more delicate. Recently, Djadli, Hebey and Ledoux [35] and, independently, Caraffa [30], established for \( p = 2 \) and \( M \) without boundary of dimension \( n \geq 5 \) the independence of the first best constant with respect to the geometry. We show that \( \alpha^i_{p}(M) \) is independent of the metric for \( 1 < p < \frac{n}{2} \) on any compact Riemannian manifold, with or without boundary, of dimension \( n \geq 3 \).

In order to state our results precisely, let us fix some notations. Let \( D^{2,p}(\mathbb{R}^n) \) be the completion of \( C^\infty_0(\mathbb{R}^n) \) under the norm

\[
\|u\|_{D^{2,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\Delta u|^p \, dx \right)^{1/p}.
\]

This space is characterized as the set of functions in \( L^{p^*}(\mathbb{R}^n) \) whose second order partial derivatives in the distributional sense are in \( L^p(\mathbb{R}^n) \). The inclusion \( D^{2,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n) \) is continuous by the Sobolev embedding theorem. Denote by \( K_2 = K_2(n, p) \) the best constant of this embedding, that is,

\[
\frac{1}{K_2(n, p)} = \inf_{u \in D^{2,p}(\mathbb{R}^n) \setminus \{0\}} \frac{\|\Delta u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}}.
\]  

(17)

Since Lions [69], it is known that the infimum is achieved and that minimizers are positive, radially symmetric decreasing functions, up to translation and multiplication by a nonzero constant. For \( p = 2 \), it was shown by Edmunds, Fortunato and Janelli [43] and Lieb [67] that

\[
K_2(n, 2) = \frac{16}{n(n - 4)(n^2 - 4)\omega_n^{4/n}}
\]

where \( \omega_n \) denotes the volume of the unit \( n \)-dimensional sphere \( S^n \subset \mathbb{R}^{n+1} \), and that the set of extremal functions is precisely

\[
z(x) = c_1 \left( \frac{1}{c_2 + |x - x_0|^2} \right)^{\frac{n-4}{2}}
\]  

(18)

where \( c_2 > 0, c_1 \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^n \). Although the explicit value of \( K_2(n, p) \) and the exact shape of minimizers are not known for \( p \neq 2 \), the asymptotic behaviors of the extremal functions and their Laplacians were determined by Hulshof and van der Vorst [57] for any \( 1 < p < \frac{n}{2} \) (see Appendix B).
The first result we prove is the following:

**Theorem 4.1.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold, with or without boundary, of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\). Then \(\alpha_p^1(M) = K_2^p\). In particular, given \(\varepsilon > 0\), there exists a real constant \(B = B(M, g, \varepsilon)\) such that

\[
\| u \|_{L^p(M)}^p \leq (K_2^p + \varepsilon) \| \Delta_g u \|_{L^p(M)}^p + B \| u \|_{L^p(M)}^p
\]

for all \(u \in E_i\).

Concerning the validity of the optimal inequality, contrary to what happens in the first order case, one cannot hope (15) to hold for \(p = 2\), as was shown in [35] for standard spheres of dimension \(n \geq 6\). We prove the nonvalidity of (15) for \(p = 2\) and compact Riemannian manifolds, with or without boundary, which have positive scalar curvature somewhere. More precisely, we have the following:

**Theorem 4.2.** [20] Let \((M, g)\) be a smooth compact \(n\)-dimensional Riemannian manifold, with or without boundary, with positive scalar curvature somewhere. Then, the optimal inequality (15) is not valid if \(n \geq 6\) and \(p = 2\).

The proof of this theorem, in the same spirit of Druet in the first order case [38], depends on knowing the explicit form of the extremal functions. We remark that the optimal second order Sobolev inequality which includes the first order term

\[
\| u \|_{L^{2n-4}(M)}^2 \leq K_2^2 \| \Delta_g u \|_{L^2(M)}^2 + A \| \nabla_g u \|_{L^2(M)}^2 + B \| u \|_{L^2(M)}^2
\]

was recently shown by Hebey [53] to be valid on compact Riemannian manifolds without boundary of dimension \(n \geq 5\).

As a subsequent step, we apply the asymptotically sharp inequality (19) to the study of fourth order partial differential equations with critical growth on compact Riemannian manifolds, with and without boundary. Specifically, given \(a, b, f \in C^0(M)\), if \(M\) has no boundary, we seek solutions to the equation

\[
\Delta_g \left( |\Delta_g u|^{p-2} \Delta_g u \right) - \text{div}_g \left( a(x) |\nabla_g u|^{p-2} \nabla_g u \right) + b(x) |u|^{p-2} u = f(x) |u|^{p^*_2} u \text{ in } M, \quad (P_1)
\]
and if $M$ has boundary, solutions to the Dirichlet problem
\[
\begin{cases}
\Delta_g \left(|\Delta_g u|^{p-2} \Delta_g u\right) - \text{div}_g \left(a(x)|\nabla_g u|^{p-2} \nabla_g u\right) + b(x)|u|^{p-2} u = f(x)|u|^{p-2} u & \text{in } M, \\
u = \nabla_g u = 0 & \text{on } \partial M,
\end{cases}
\tag{P_1}
\]
and to the Navier problem
\[
\begin{cases}
\Delta_g \left(|\Delta_g u|^{p-2} \Delta_g u\right) - \text{div}_g \left(a(x)|\nabla_g u|^{p-2} \nabla_g u\right) + b(x)|u|^{p-2} u = f(x)|u|^{p-2} u & \text{in } M, \\
u = \Delta_g u = 0 & \text{on } \partial M.
\end{cases}
\tag{P_2}
\]

For $p = 2$, equation (P_1) appears in conformal geometry. Indeed, given a Riemannian manifold $(M, g)$ of dimension $n \geq 5$ with scalar curvature $\text{Scal}_g$ and Ricci curvature $\text{Ric}_g$, the following so-called Paneitz-Branson operator is conformally invariant:
\[
P_g u = \Delta^2_g u - \text{div}_g \left(\frac{(n-2)^2 + 4}{2(n-1)(n-2)} \text{Scal}_g g - \frac{4}{n-2} \text{Ric}_g\right) du + \frac{n-4}{2} Q_g u,
\]
where
\[
Q_g = -\frac{1}{2(n-1)} \text{Scal}_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} \text{Scal}^2_g - \frac{2}{(n-2)^2} |\text{Ric}_g|^2.
\]
Existence of a conformal metric $\tilde{g} = u^{-4 \frac{1}{n-4}} g$ with scalar curvature $\text{Scal}_{\tilde{g}}$ and $\text{Ric}_{\tilde{g}}$ is equivalent to finding a positive solution for the fourth order equation
\[
P_{\tilde{g}} u = \frac{n-4}{2} Q_{\tilde{g}} u^{\frac{n+4}{n-4}} \quad \text{in } M.
\]
When $(M, g)$ is Einstein and $p = 2$, this last equation becomes (P_1). Our motivation for investigating (P_1), (P_2) and (P_3) arises from the desire of understanding the role of the geometry in these problems. Problem (P_1) for $p = 2$ was studied by Djadli, Hebey and Ledoux [35], with constant coefficients and special emphasis on spheres, and by Esposito and Robert [45], with subcritical perturbation and more general second order terms, on compact manifolds.

Nontrivial weak solutions of (P_1) correspond, modulo nonzero constant multiples, to critical points of the functional
\[
J(u) = \int_M |\Delta_g u|^p dv_g + \int_M a(x)|\nabla_g u|^p dv_g + \int_M b(x)|u|^p dv_g
\]
on the manifold
\[
V_i = \left\{u \in E_i : \int_M f(x)|u|^{p^*_2} dv_g = 1\right\}.
\]
The functional $J$ is said to be coercive on $E_i$ if there exists some positive constant $C$, dependent only on $a$ and $b$, such that
\[
J(u) \geq C \|u\|_{E_i}^p
\]
for all \( u \in E_i \). We say that \((H_i)\) holds if

\[
\inf_{V_i} J < \frac{1}{K^p_i \left( \max_M f \right)^{\frac{p}{p-2}}}.
\]

Under these conditions, we have the following results:

**Theorem 4.3A.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold without boundary of dimension \( n \geq 3 \) and \( 1 < p < \frac{n}{2} \). Assume that \( a, b, f \in C^0(M) \) are such that the functional \( J \) is coercive on \( E_1 \) and \((H_1)\) holds. Then \((P_1)\) possesses a nontrivial weak solution \( u \). Moreover, if \( p = 2 \) and \( a \in C^{1,\gamma}(M), b, f \in C^{\gamma}(M) \), then \( u \in C^{4,\gamma}(M) \); if, in addition, \( f \geq 0 \) and \( a > 0 \) is a constant such that \( b(x) \leq \frac{a^2}{4} \), then \((P_1)\) admits a positive solution.

**Theorem 4.3B.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold with boundary of dimension \( n \geq 3 \) and \( 1 < p < \frac{n}{2} \). Assume that \( a, b, f \in C^0(M) \) are such that the functional \( J \) is coercive on \( E_2 \) and \((H_2)\) holds. Then \((P_2)\) possesses a nontrivial weak solution \( u \). Moreover, if \( p = 2 \) and \( a \in C^{1,\gamma}(M), b, f \in C^{\gamma}(M) \), then \( u \in C^{4,\gamma}(M) \).

**Theorem 4.3C.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold with boundary of dimension \( n \geq 3 \) and \( 1 < p < \frac{n}{2} \). Assume that \( a, b, f \in C^0(M) \) are such that the functional \( J \) is coercive on \( E_3 \) and \((H_3)\) holds. Then \((P_3)\) possesses a nontrivial weak solution \( u \). Moreover, if \( b, f \in C^{\gamma}(M) \) and either \( a \equiv 0 \) or \( p = 2 \) and \( a \) is a nonnegative constant, then \( u \in C^{4,\gamma}(M) \); if, in addition, \( f \geq 0 \) and \( b(x) \leq \frac{a^2}{4} \), then \((P_3)\) admits a positive solution.

Since \( V_i \) is not weakly closed in the \( E_i \) topology, the direct variational method does not apply. One also encounters difficulties in establishing the regularity of weak solutions, since the Moser iterative scheme fails in our case. We remark that the case \( p = 2 \) and \( n \geq 7 \) of Theorem 4.3A was proved by Caraffa [30], using the Yamabe method.

An interesting application of Theorems 4.3A-C is the following result, which relates the geometry of the manifold at a point of maximum of \( f \) and the behavior of \( f \) up to the second order at this point. A version of this result was originally obtained by Druet [40] for the \( p \)-Laplacian. Let \( z \) be the minimizer of the Sobolev quotient defined in (18) with \( c_1 = c_2 = 1 \) and
denote

\begin{align*}
I_1 &= I_1(n, p) = \int_{\mathbb{R}^n} z^{p_2} \, dx, \\
I_2 &= I_2(n, p) = \int_{\mathbb{R}^n} z^{p_1} r^2 \, dx, \\
I_3 &= I_3(n, p) = \int_{\mathbb{R}^n} |\Delta z|^p \, dx, \\
I_4^1 &= I_4^1(n, p) = \int_{\mathbb{R}^n} |\Delta z|^{p_1} r^2 \, dx, \\
I_4^2 &= I_4^2(n, p) = \int_{\mathbb{R}^n} |\Delta z|^{p_2} |z'(r)| r \, dx
\end{align*}

whenever the right-hand side makes sense, and set \( I_4 = I_4^1 + 2p I_4^2 \). We have:

**Corollary 4.2.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold, with or without boundary, of dimension \( n \geq 5 \) and \( \frac{n + 2}{n} < p < \frac{n + 2}{4} \). Let \( a \equiv 0 \) and \( b \in C^0(M) \) be such that the functional \( J \) is coercive on \( E_i \). Furthermore, assume that \( f \in C^2(M) \), \( \max_M f > 0 \) and \( f \) has a point of maximum \( x_0 \) outside the boundary. If

\[
\frac{\Delta_g f(x_0)}{f(x_0)} > \frac{1}{3} \left( 1 - \frac{p_2}{p} \frac{I_4}{I_2 I_3} \right) \text{Scal}_g(x_0),
\]

then \((P_i)\) possesses a nontrivial weak solution.

The methods used above are then applied to the study of the fourth order Brezis-Nirenberg problem on compact Riemannian manifolds. Indeed, consider the following one-parameter problems:

\[
\Delta_g \left(|\Delta_g u|^{p-2} \Delta_g u\right) = |u|^{p_2-2} u + \lambda |u|^{p-2} u \quad \text{in } M, \quad (\text{BN}_{i1})
\]

if \( M \) has no boundary,

\[
\begin{align*}
\left\{ \begin{array}{ll}
\Delta_g \left(|\Delta_g u|^{p-2} \Delta_g u\right) &= |u|^{p_2-2} u + \lambda |u|^{p-2} u \quad \text{in } M, \\
u &= \nabla_g u = 0 \quad \text{on } \partial M,
\end{array} \right. \quad (\text{BN}_{i2})
\]

and

\[
\begin{align*}
\left\{ \begin{array}{ll}
\Delta_g \left(|\Delta_g u|^{p-2} \Delta_g u\right) &= |u|^{p_2-2} u + \lambda |u|^{p-2} u \quad \text{in } M, \\
u &= \Delta_g u = 0 \quad \text{on } \partial M,
\end{array} \right. \quad (\text{BN}_{i3})
\]

if \( M \) has boundary. Denote by \( \lambda_1 \) the first eigenvalue associated to the equation

\[
\Delta_g \left(|\Delta_g u|^{p-2} \Delta_g u\right) = \lambda |u|^{p-2} u \quad \text{in } M,
\]

on \( E_i \). The variational characterization of \( \lambda_1 \) is given by

\[
\lambda_1 = \inf_{u \in E_i \setminus \{0\}} \frac{\int_M |\Delta_g u|^p \, dv_g}{\int_M |u|^p \, dv_g}.
\]
Clearly, $\lambda_1 = 0$ on $E_1$ and $\lambda_1 > 0$ on $E_2$ and on $E_3$.

In the spirit of Brezis and Nirenberg [24], we are interested in determining the range of values of $\lambda$ for which $(BN_1)$, $(BN_2)$ and $(BN_3)$ admit nontrivial solutions. With the aid of an eigenfunction associated to $\lambda_1$, it is always possible to find nontrivial solutions for $\lambda$ close to $\lambda_1$. A more difficult task is to obtain solutions for $\lambda$ far from $\lambda_1$. For $p = 2$ and in Euclidean bounded domains of dimension $n \geq 8$, Edmunds, Fortunato and Janelli [43] and van der Vorst [87] established, respectively, the existence of nontrivial solutions of $(BN_2)$ and the existence of positive solutions of $(BN_3)$ for any $0 < \lambda < \lambda_1$. In addition, still in this context, it is known that $(BN_2)$ has no nontrivial solutions for $\lambda < 0$ and $(BN_3)$ has no positive solutions for $\lambda \leq 0$ in star-shaped domains, and that $(BN_3)$ has no positive solutions for $\lambda \geq \lambda_1$ (see [77] and [74]).

The situation changes drastically when we consider compact Riemannian manifolds with boundary which have positive scalar curvature somewhere (similar phenomena occur in the second order Brezis-Nirenberg problem; see [20] and the references therein). Indeed, in this case, for $n \geq 6$ we establish the existence of nontrivial solutions for $(BN_2)$ and $(BN_3)$ for any $\lambda < \lambda_1$ and of positive solutions for $(BN_3)$ for $0 \leq \lambda < \lambda_1$. In particular, the existence of nontrivial solutions to $(BN_2)$ for $\lambda < 0$ and of positive solutions to $(BN_3)$ for $\lambda = 0$ contrasts with the results mentioned above for star-shaped Euclidean domains. Our results seem to point to the existence of only one critical dimension $n = 5$ in the case of manifolds with positive scalar curvature somewhere, in comparison with the Euclidean case, where $n = 5, 6, 7$ are the critical dimensions (see [78]). An analogous version of these results is proved on compact Riemannian manifolds without boundary in the case $p = 2$. Moreover, we also discuss $(BN_2)$ and $(BN_3)$ for other values of $p$ on compact manifolds of dimension $n \geq 6$ which are flat on a neighborhood, which include bounded domains in $\mathbb{R}^n$. Nontrivial solutions are found for $\frac{n}{n-2} < p \leq \sqrt{\frac{n}{2}}$ and $0 < \lambda < \lambda_1$. This generalizes Theorem 1.1 of [43] and Theorem 3 of [87]. These results are resumed in the following theorems:

**Theorem 4.4A.** [20] Let $(M, g)$ be a smooth compact Riemannian manifold without boundary of dimension $n \geq 6$. If $p = 2$ and $M$ has positive scalar curvature somewhere, then $(BN_1)$ has a nontrivial solution in $C^{4,\gamma}(M)$ for any $\lambda < \lambda_1$. If $\lambda \geq \lambda_1$, then $(BN_1)$ has no positive solution.

**Theorem 4.4B.** [20] Let $(M, g)$ be a smooth compact Riemannian manifold with boundary of dimension $n \geq 6$. Then:

(i) If $p = 2$ and $M$ has positive scalar curvature somewhere, $(BN_2)$ has a nontrivial solution $C^{4,\gamma}(M)$ for any $\lambda < \lambda_1$. 
(ii) If \( \frac{n}{n-2} < p \leq \sqrt{\frac{n}{2}} \) and \( M \) is flat in a neighborhood, \((\text{BN}_2)\) has a nontrivial solution in \( C^{4,\gamma}(M) \) for any \( 0 < \lambda < \lambda_1 \).

**Theorem 4.4C.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold with boundary of dimension \( n \geq 6 \). Then:

(i) If \( p = 2 \) and \( M \) has positive scalar curvature somewhere, \((\text{BN}_3)\) has a positive solution for any \( 0 \leq \lambda < \lambda_1 \) and a nontrivial solution for any \( \lambda < 0 \) in \( C^{4,\gamma}(M) \). If \( \lambda \geq \lambda_1 \), then \((\text{BN}_3)\) has no positive solution.

(ii) If \( \frac{n}{n-2} < p \leq \sqrt{\frac{n}{2}} \) and \( M \) is flat in a neighborhood, \((\text{BN}_3)\) has a positive solution in \( C^{4,\gamma}(M) \) for any \( 0 < \lambda < \lambda_1 \). If \( \lambda \geq \lambda_1 \), then \((\text{BN}_3)\) has no positive solution.

Theorem 4.4A was proved in [30] for \( n > 6 \); in fact, Caraffa considered a more general equation than \((\text{BN}_1)\) and obtained a sharper result.
Chapter 1
A Sharp Pointwise Estimate

1.1 Introduction

Denote by $\Delta_{p,g}u = \text{div}_g \left( |\nabla_g u|^{p-2} \nabla_g u \right)$ the $p$-Laplacian with respect to the metric $g$ and let $v(y) = \frac{c_1}{\left( c_2 + |y|^{\frac{2}{p-1}} \right)^{\frac{1}{p-1}}}$ be the unique minimizer of the Sobolev quotient which satisfies $v(0) = 1$, $\nabla v(0) = 0$ and $\|v\|_{L^p(R^n)} = 1$.

**Theorem 1.1.** [12] Let $g$ be a Riemannian metric on $B_2$ such that $B_2$ is convex under this metric and such that $|R_{ijkl}|, |\nabla R_{ijkl}| \leq \delta^*$, for some $\delta^* > 0$. Let $1 < p < n$, $1 < s \leq p$, $x_\alpha \in B_1$, $\varepsilon_\alpha \geq 0$, $0 \leq \lambda_\alpha \leq \Lambda$, $\mu_\alpha \to 0^+$, $\psi_\alpha(y) = \exp_{x_\alpha}(\mu_\alpha y)$ for $y \in \Omega_\alpha := \psi_\alpha^{-1}(B_1)$, $g_\alpha = \mu_\alpha^{-2} \psi_\alpha^* g$, and $v_\alpha \in H^1_{0,p}(\Omega_\alpha)$ be a solution of

$$-\Delta_{p,g_\alpha} v_\alpha + \varepsilon_\alpha \|v_\alpha\|_{L^s(\Omega_\alpha, g_\alpha)}^{p-s} v_\alpha^{s-1} = \lambda_\alpha v_\alpha^{p-1} \quad \text{in} \ \Omega_\alpha. \quad (1.1)$$

satisfying $v_\alpha(0) = 1$, $0 \leq v_\alpha \leq 1$ and

$$\int_{\Omega_\alpha} |v_\alpha - v|^{p^*} \, dv_{g_\alpha} \to 0,$$

$$\int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha - \nabla v| \, dv_{g_\alpha} \to 0.$$

Assume further that

$$\frac{\text{dist}_g(x_\alpha, \partial B_1)}{\mu_\alpha} = \text{dist}_{g_\alpha}(0, \partial \Omega_\alpha) \to \infty.$$

Then, there exist positive constants $C = C(n, p, s, \Lambda, \delta^*)$ and $D = D(n, p, s, \Lambda)$ such that for all large $\alpha$ we have

$$v_\alpha(y) \leq C v(y)^{1-D\delta^*} \quad \text{for all} \ y \in \Omega_\alpha. \quad (1.2)$$

**Theorem 1.2.** Under the conditions of Theorem 1.1, if $\delta^*$ is sufficiently small, there exists a positive constant $C = C(n, p, s, \Lambda, \delta^*)$ such that for all large $\alpha$ we have

$$v_\alpha(y) \leq C v(y) \quad \text{for all} \ y \in \Omega_\alpha. \quad (1.3)$$
1.2 Proof of Theorem 1.2

The starting point of the proof of Theorem 1.2 is Theorem 1.1 of [12]. Therefore, we will present a sketch of its proof here. For the full details, see [12], pp. 362–367.

**Sketch of the proof of Theorem 1.1.** Multiplying (1.1) by \(v_\alpha\) and integrating by parts over \(\Omega_\alpha\) leads to \(\epsilon_\alpha \|v_\alpha\|_{L^p(\Omega_\alpha, g_\alpha)}^{p-s} \leq C\). Hence, \(-\Delta_{p,g_\alpha} v_\alpha \leq C\) and from interior regularity results for degenerate quasilinear elliptic equations [84] we may assume \(v_\alpha \to v\) in \(C^{1,\gamma}_\text{loc}(\mathbb{R}^n)\) for some \(0 < \gamma < 1\). It follows that we need to prove (1.2) only for large \(|y|\). The central idea of the proof is to use the comparison principle for the \(p\)-Laplacian, choosing a suitable test function. In order to do this, one needs first to obtain a somewhat more precise estimate for the \(p\)-Laplacian of the \(v_\alpha\), in order to compare it with the \(p\)-Laplacian of the test function. This is done by obtaining the following initial pointwise estimate

\[
v_\alpha(y) \leq C v(y)\frac{1}{\|v\|_{L^p(B_{1 R-\epsilon}(y))}} \quad \text{for all } y \in \Omega_\alpha \text{ such that } |y| \geq 1,
\]

for sufficiently large \(\alpha\), with \(C = C(n, p, s, \Lambda, \delta^*)\). Again, for the same reasons given above, one needs to show this preliminary estimate only for large \(|y|\). Therefore, we fix \(R > 1\) and prove (1.4) for \(|y| > 2R\). Denoting \(R = \frac{|y|}{2}\), we consider the rescaling

\[
\tilde{v}_\alpha(z) = R^{\frac{p-s}{p}} v_\alpha(Rz + y) \quad \text{for } z \in B_1,
\]

and define in \(B_1\) the metric \(\tilde{g}_\alpha(z) = (\tilde{g}_\alpha)_{ij}(z)dz_idz_j\) with \((\tilde{g}_\alpha)_{ij}(z) = (g_\alpha)_{ij}(Rz + y)\). Then \(\tilde{v}_\alpha\) satisfies the equation

\[-\Delta_{p,\tilde{g}_\alpha} \tilde{v}_\alpha + \alpha \mu_p R^p \tilde{v}_\alpha^{p-1} = \lambda \tilde{v}_\alpha^{p^* - 1},\]

or

\[-\Delta_{p,\tilde{g}_\alpha} \tilde{v}_\alpha \leq \lambda \tilde{v}_\alpha^{p^* - 1}.
\]

Since

\[\|\tilde{v}_\alpha\|_{L^{p^*}(B_1, \tilde{g}_\alpha)} = \|v_\alpha\|_{L^{p^*}(B_{R^{-\epsilon}}(y), g_\alpha)} \leq \|v_\alpha\|_{L^{p^*}(\Omega_\alpha, g_\alpha)} \to 0,
\]

as \(\alpha \to \infty\), taking large enough \(\alpha\) we can ensure that \(\|\tilde{v}_\alpha\|_{L^{p^*}(B_1, \tilde{g}_\alpha)}\) is small enough. It follows then by the Moser’s iteration scheme that

\[\|\tilde{v}_\alpha\|_{L^\infty(B_1)} \leq C \|\tilde{v}_\alpha\|_{L^1(B_1)},\]

where the constant \(C\) depends only on \(n, p\) and \(\delta^*\). Hence,

\[
\left(\frac{|y|}{2}\right)^{\frac{p-s}{p}} v_\alpha(y) = \tilde{v}_\alpha(0) \leq C \|\tilde{v}_\alpha\|_{L^1(B_1, \tilde{g}_\alpha)} \leq C \|\tilde{v}_\alpha\|_{L^{p^*}(B_1, \tilde{g}_\alpha)} = \|v_\alpha\|_{L^{p^*}(B_{1 R-\epsilon}(y))}.
\]

whence (1.4) follows.
Having the initial pointwise estimate (1.4), one obtains easily that
\[-\Delta_{p,g} v_\alpha = \frac{\varepsilon_1}{|y|^p} v_\alpha^{p-1} \leq 0 \quad \text{in } \Omega_\alpha - B_{\hat{R}}, \tag{1.5}\]
where \(\varepsilon_1 > 0\) is small enough, when \(\hat{R} > 0\) is taken large enough. We then choose
\[w(y) = \frac{1}{|y|^{\frac{n-p}{p+1}}} .\]
as a test function, with \(\delta < n - p\). By using the following estimate for the \(p\)-Laplacian (see also Lemma 1.2 below)
\[-\Delta_{p,g} w(y) = -\Delta_p w(y) + |w'(r)|^{p-2} w'(r) O(r),\]
where \(|O(r)| \leq D\delta^* r\), one computes
\[-\Delta_{p,g} w \geq \left( \delta \left( \frac{n-p-\delta}{p-1} \right)^{p-1} - D\delta^* \right) \frac{1}{|y|^{n-p}} w^{p-1} .\]
Thus, choosing \(\delta^*\) small enough so that \(\delta \left( \frac{n-p-\delta}{p-1} \right)^{p-1} - D\delta^* \geq \frac{\delta}{2} \left( \frac{n-p-\delta}{p-1} \right)^{p-1} =: \varepsilon_1\), we get
\[-\Delta_{p,g} w - \frac{\varepsilon_1}{|y|^p} w^{p-1} \geq 0 \quad \text{in } \Omega_\alpha - B_{\hat{R}}. \tag{1.6}\]
Since \(v_\alpha \leq 1\) everywhere and \(v_\alpha = 0\) on \(\partial \Omega_\alpha\), multiplying \(w\) by some large constant \(\hat{C} = \hat{C}(\hat{R})\), the comparison principle yields
\[v_\alpha \leq \hat{C} w \quad \text{in } \Omega_\alpha - B_{\hat{R}},\]
whence (1.2) follows. \(\square\)

**Proof of Theorem 1.2.** According to Theorem 1.1, there exist positive constants \(C = C(n, p, s, \Lambda, \delta^*)\) and \(D = D(n, p, s, \Lambda)\) such that
\[v_\alpha(x) \leq c_1 \left( \frac{1}{c_2 + |x|^{\frac{n-p}{p+1}}} \right)^{\frac{n-p}{p} - D\delta^*} \quad \text{for all } x \in \Omega_\alpha,\]
where \(c_2\) is a constant uniquely determined by \(v\). We can simplify the above and find a constant \(C = C(n, p, s, \Lambda, \delta^*)\) such that
\[v_\alpha(x) \leq C \left( \frac{1}{1 + |x|^{\frac{n-p}{p+1}}} \right)^{\frac{n-p}{p} - \delta} \quad \text{for all } x \in \Omega_\alpha.\]
where \(\delta = O(\delta^*).\) Therefore, in order to prove (1.3), it is enough to find a positive constant \(C = C(n, p, s, \Lambda, \delta^*)\) such that
\[ v_\alpha(x) \leq C \left( \frac{1}{1 + |x|} \right)^{\frac{n-p}{p-1}} \text{ for all } x \in \Omega_\alpha. \]  \hspace{1cm} (1.7)

From now on, \( C \) will denote possibly different constants which depend only on \( n, p, s, \Lambda \) and \( \delta^* \).

Clearly, since \( v_\alpha \leq 1 \), this inequality is true in \( B_1 \). Thus, it suffices to prove it in \( \Omega_\alpha - B_1 \).

Once again, we do this by applying the comparison principle for the \( p \)-Laplacian: we will find a function \( \varphi_\alpha \) defined in \( \Omega_\alpha - B_1 \) satisfying

\[
\begin{cases}
-\Delta_{p,g_\alpha} \varphi_\alpha \geq -\Delta_{p,g_\alpha} v_\alpha \quad \text{in } \Omega_\alpha - B_1, \\
\varphi_\alpha \geq v_\alpha \quad \text{in } \partial \Omega_\alpha \cup \partial B_1,
\end{cases}
\]

and such that

\[ \varphi_\alpha(x) \leq \frac{C}{|x|^\frac{p-n}{p-1}} \quad \text{in } \Omega_\alpha - B_1; \]

by the comparison principle, it follows that

\[ v_\alpha(x) \leq \varphi_\alpha(x) \leq \frac{C}{|x|^\frac{p-n}{p-1}} \leq C \left( \frac{1}{1 + |x|} \right)^{\frac{n-p}{p-1}} \leq Cv(x) \quad \text{in } \Omega_\alpha - B_1. \]

Now, in \( \Omega_\alpha - B_1 \) we have

\[ -\Delta_{p,g_\alpha} v_\alpha \leq \frac{1}{K_p} \varphi_\alpha^{\varphi_\alpha - 1} \leq C \left( \left( \frac{1}{1 + |x|} \right)^{\frac{n-p-\delta}{p-1}} \right)^{\frac{(p-1)\varphi_\alpha \delta}{n-p}} \leq \frac{C}{|x|^{\frac{n-p}{p-1} - \frac{(p-1)\varphi_\alpha \delta}{n-p}}}.
\]

Thus, in order for us to have \( -\Delta_{p,g_\alpha} \varphi_\alpha \geq -\Delta_{p,g_\alpha} v_\alpha \) in \( \Omega_\alpha - B_1 \), it is enough to show that \( \varphi_\alpha \) satisfies

\[ -\Delta_{p,g_\alpha} \varphi_\alpha \geq \frac{C}{|x|^{\frac{n}{n+a}}} \quad \text{in } \Omega_\alpha - B_1, \]

for some \( a > 0 \) small enough. On the other hand, since \( v_\alpha = 0 \) on \( \partial \Omega_\alpha \) and \( v_\alpha \leq 1 \), it is also enough that

\[ \varphi_\alpha \geq 0 \quad \text{on } \partial \Omega_\alpha \]

and

\[ \varphi_\alpha \geq 1 \quad \text{on } \partial B_1. \]

Summing up, in order to prove Theorem 1.2, it is enough to find a sequence of functions \( \varphi_\alpha \) satisfying for every sufficiently large \( \alpha \),

\[
\begin{cases}
-\Delta_{p,g_\alpha} \varphi_\alpha \geq \frac{1}{|x|^{\frac{n}{n+a}}} \quad \text{in } \Omega_\alpha - B_1, \\
\varphi_\alpha \geq 0 \quad \text{on } \partial \Omega_\alpha, \\
\varphi_\alpha \geq 1 \quad \text{on } \partial B_1, \\
\varphi_\alpha \leq \frac{C}{|x|^\frac{n-p}{p-1}} \quad \text{in } \Omega_\alpha - B_1.
\end{cases}
\]

(1.8)
Since we want to bound \( \varphi_\alpha \) and its \( p \)-Laplacian by radial functions, it will be convenient to do this in a radial domain, that is, in an annulus containing \( \Omega_\alpha - B_1 \). Since \( \Omega_\alpha \subset B_{\frac{\Lambda}{\mu_\alpha}} \), where \( \Lambda = \text{diam}_g B_1 \), and we may assume that the metric \( g_\alpha \) is defined in this ball (for this to occur, it is necessary that \( \text{dist}(x_\alpha, \partial B_2) > \text{diam}_g B \), and this can be assumed from the start, by choosing neighborhoods of points in the manifold conveniently when localizing) we will find \( \varphi_\alpha \) satisfying
\[
\begin{cases}
-\Delta_{p,g_\alpha} \varphi_\alpha \geq \frac{1}{|x|^{n+a}} & \text{in } B_{\frac{\Lambda}{\mu_\alpha}} - B_1, \\
\varphi_\alpha \geq 0 & \text{on } \partial B_{\frac{\Lambda}{\mu_\alpha}}, \\
\varphi_\alpha \geq 1 & \text{on } \partial B_1, \\
\varphi_\alpha \leq \frac{C}{|x|^{\frac{n+a}{p}}} & \text{in } B_{\frac{\Lambda}{\mu_\alpha}} - B_1.
\end{cases}
\] (1.9)

We do a final reduction of the problem to the annulus \( B_{\Lambda} - B_{\mu_\alpha} \). Take
\[
\xi_\alpha(x) = \frac{1}{\mu_\alpha} \varphi_\alpha \left( \frac{1}{\mu_\alpha} x \right),
\] (1.10)
and \( h_\alpha(x) = \mu_\alpha^2 A_{\alpha} g_\alpha \), where \( A_{\alpha}(x) = \frac{1}{\mu_\alpha} x \). Then, in \( B_{\Lambda} - B_{\mu_\alpha} \), we have \( \xi_\alpha \leq \frac{C}{|x|^{\frac{n+a}{p}}} \) and
\[
\begin{align*}
-\Delta_{p,h_\alpha} \xi_\alpha(x) &= - \left( \frac{1}{\mu_\alpha} \right)^{p-1} \left( \frac{1}{\mu_\alpha} \right)^p \Delta_{p,g_\alpha} \varphi_\alpha \left( \frac{1}{\mu_\alpha} x \right) \\
&= - \frac{1}{\mu_\alpha} \Delta_{p,g_\alpha} \varphi_\alpha \left( \frac{1}{\mu_\alpha} x \right) \\
&\geq \frac{1}{\mu_\alpha} \frac{\mu_\alpha^{n+a}}{|x|^{n+a}} \\
&= \frac{\mu_\alpha^a}{|x|^{n+a}},
\end{align*}
\]
while on \( \partial B_{\mu_\alpha} \), \( \xi_\alpha \geq \frac{1}{\mu_\alpha} \). Therefore, Theorem 1.2 will be proved if we can show the following result to be true:

**Lemma 1.1.** For small enough \( \delta^* \), there exists a positive constant \( C = C(n, p, s, \Lambda, \delta^*) \) such that for all sufficiently large \( \alpha \) we can find \( z_\alpha \in C^\infty(B_{\Lambda} - B_{\mu_\alpha}) \) such that
\[
\begin{cases}
-\Delta_{p,g_\alpha} z_\alpha \geq \frac{\mu_\alpha^a}{|x|^{n+a}} & \text{in } B_{\Lambda} - B_{\mu_\alpha}, \\
z_\alpha \geq 0 & \text{on } \partial B_{\Lambda}, \\
z_\alpha \geq \frac{1}{\mu_\alpha} & \text{on } \partial B_{\mu_\alpha}, \\
z_\alpha \leq \frac{C}{|x|^{\frac{n+a}{p}}} & \text{in } B_{\Lambda} - B_{\mu_\alpha}.
\end{cases}
\] (1.11)

**Proof.** The computations involved in the proof of Lemma 1.1 will become clear if we examine first the case \( p = 2 \), that is, the case of the Laplacian. We present a proof using Green’s function, with no reference to the conformal invariance of the Laplacian (which was used in [56]).
Proof of Lemma 1.1 for \( p = 2 \): Let \( G \) be the Green’s function for the Laplacian \(-\Delta_g\) in \((B_2, g)\) at 0, that is,
\[
\begin{cases}
-\Delta_g G = n(n-2)\omega_n \delta_0 & \text{in } B_\Lambda, \\
G = 0 & \text{on } \partial B_\Lambda.
\end{cases}
\]
Then (see [65], Appendix B) there exists a function \( E \) such that
\[
G(y) = \frac{1}{|y|^{n-2}} + E(y)
\]
and
\[
|E(y)| \leq \frac{C}{|y|^{n-4}}
\]
for some positive constant \( C \) depending only on \( n \) and \( g \). In particular, \( G \) satisfies
\[
\begin{cases}
-\Delta_g G = n(n-2)\omega_n \delta_0 & \text{in } B_\Lambda - B_{\mu_n}, \\
G > 0 & \text{on } B_\Lambda - B_{\mu_n}.
\end{cases}
\]
Set
\[
\tilde{z}_\alpha(y) = G(y) - \frac{\mu_n^\alpha}{2|y|^{n-2+a}}, \quad y \in B_\Lambda - B_{\mu_n}.
\]
Then
\[
-\Delta_g \tilde{z}_\alpha(y) = -\Delta_g G(y) - \Delta_g \left( \frac{\mu_n^\alpha}{2|y|^{n-2+a}} \right) = -\frac{\mu_n^\alpha}{2} \Delta_g \left( \frac{1}{|y|^{n-2+a}} \right).
\]
It is a well-known fact (see [62], Corollary X.3.6, or Lemma 1.2, below) that for a radial function \( u = u(r) \) we have
\[
|\Delta_g u(r) - \Delta u(r)| \leq C_n \delta^* ru'(r).
\]
Therefore, since for a radial function the Laplacian is given by
\[
\Delta u(r) = u''(r) + \frac{n-1}{r} u'(r),
\]
we have
\[
-\Delta_g \left( \frac{1}{|y|^{n-2+a}} \right) \geq \frac{(n-2+a)(a - C_n \delta^* r^2)}{r^{n+a}} \geq \frac{C}{r^{n+a}},
\]
if \( \delta^* \) is sufficiently small. Then it is easy to verify, using the above and (1.12), that for some large constant \( C = C(n, p) \), \( z_\alpha = C \tilde{z}_\alpha \) will satisfy the desired properties in (1.11) for large \( \alpha \).

In the general case, we first have to deal with the fact that the \( p \)-Laplacian is not linear. Again, for reasons of clarity, we first present a proof of Lemma 1.1 when \( g \) is flat. We use \(-\Delta_p\) to denote the flat \( p \)-Laplacian. Here, the fundamental solution \( \frac{1}{|y|^{n-2}} \) for the \( p \)-Laplacian in
$\mathbb{R}^n$ will play the role of the Green’s function in the case $p = 2$. If $z$ is a radially symmetric function, say

$$z(y) = f(|y|)$$

for a one variable function $f(r)$, then it is easy to verify that

$$-\Delta_p z(y) = -|f'(r)|^{p-2} Lf,$$

where

$$L f = (p - 1)f''(r) + \frac{n - 1}{r} f'(r)$$

is linear in $f$.

**Proof of Lemma 1.1 when $g$ is the flat metric:** Let

$$f_\alpha(r) = \frac{1}{r^{\frac{n-p}{p-1}}} - \frac{\mu_\alpha}{2r^{\frac{n-p}{p-1} + a}}$$

where $a$ is small enough so that $\frac{n - p}{p - 1} - a > 0$, and let $\tilde{\psi}_\alpha(y) = f_\alpha(|y|)$. We have for $\mu_\alpha < r < 1$

$$\frac{1}{2r^{\frac{n-p}{p-1}}} \leq f_\alpha(r) \leq \frac{1}{r^{\frac{n-p}{p-1}}}$$

and

$$\frac{1}{2} \left( \frac{n - p}{p - 1} - a \right) \frac{1}{r^{\frac{n-p}{p-1}}} \leq |f'(r)| \leq \frac{n - p}{p - 1} \frac{1}{r^{\frac{n-p}{p-1}}}.$$ 

Since $\frac{1}{r^{\frac{n-p}{p-1}}}$ is the fundamental solution for $\Delta_p$, we have

$$L \left( \frac{1}{r^{\frac{n-p}{p-1}}} \right) = 0.$$ 

Therefore,

$$-Lf_\alpha = \frac{\mu_\alpha}{2} L \left( \frac{1}{r^{\frac{n-p}{p-1} + a}} \right) = \frac{\alpha}{2} (p - 1) \left( \frac{n - p}{p - 1} + a \right) \frac{\mu_\alpha}{r^{\frac{n-p}{p-1} + a}}.$$

It follows that for some positive constant $C_{n,p}$

$$\frac{\mu_\alpha}{C_{n,p} |y|^{n+a}} \leq -\Delta_p \tilde{\psi}_\alpha(y) \leq C_{n,p} \frac{\mu_\alpha}{|y|^{n+a}}$$

for all $y \in B_A - B_{\mu_\alpha}$.

Again, for some large constant $C = C(n, p)$, $z_\alpha = C \tilde{z}_\alpha$ will clearly satisfy the desired properties in (1.11). This proves Lemma 1.1 in the flat case.

For the case of a general metric, we will need the following estimate for the $p$-Laplacian of a radial function in geodesic polar coordinates:
Lemma 1.2. Let $(r,\theta)$ be geodesic polar coordinates and let $u = u(r)$ be a radial function. Then,

$$-\Delta_{p,g} u = -\Delta_p u + O(r) |\partial_r u|^{p-2} \partial_r u.$$  

**Proof.** Since $\partial_\theta u \equiv 0$ and in a geodesic coordinate system $g^{rr} = 1$ and $g^{r\theta} = 0$, we have

$$\Delta_{p,g} u = \text{div}_g \left( |\nabla_g u|^{p-2} \nabla_g u \right)$$

$$= \sum_{i=1}^n \partial_{r_i} \left( |\nabla_g u|^{p-2} \sum_{j=1}^n g^{ij} \partial_j u \right) + \sum_{k=1}^n \left( |\nabla_g u|^{p-2} \sum_{j=1}^n g^{kj} \partial_j u \right) \Gamma^i_{ki}$$

$$= \partial_r \left( |\partial_r u|^{p-2} \partial_r u \right) + |\partial_r u|^{p-2} \partial_r u \sum_{i=1}^n \Gamma^i_{r1}$$

$$= (p-1) |\partial_r u|^{p-2} \partial_r^2 u + |\partial_r u|^{p-2} \partial_r u \partial_r (\log \sqrt{\det g})$$

As $\sqrt{\det g} = r^{n-1} J(r,\theta)$, where $J(r,\theta) = 1 + O(r^2)$, it follows that

$$\Delta_{p,g} u = (p-1) |\partial_r u|^{p-2} \partial_r^2 u + |\partial_r u|^{p-2} \partial_r u \left( \frac{n-1}{r} + \partial_r \log J(r,\theta) \right)$$

$$= \Delta_p u + |\partial_r u|^{p-2} \partial_r u \frac{\partial_r J(r,\theta)}{J(r,\theta)}$$

whence the result. □

**Proof of Lemma 1.1 in the general case:** We have

$$-\Delta_{p,g} z = - \left| \frac{\partial z}{\partial r} \right|^{p-2} L(z) + O(r) \left| \frac{\partial z}{\partial r} \right|^{p-2} \frac{\partial z}{\partial r},$$

where

$$L(z(r)) = (p-1) \frac{\partial^2 z}{\partial r^2} + \frac{n-1}{r} \frac{\partial z}{\partial r},$$

and $|O(r)| \leq C_n \delta^*$. 

Take $z_\alpha(y) = f_\alpha(\|y\|)$ with

$$f_\alpha(r) = \frac{1}{r^{\frac{n-p}{p-1}} + \frac{1}{r^{\frac{2}{p-1}}} - \frac{\mu_\alpha}{2r^{\frac{p}{p-1}+\alpha}},}$$

(1.17)

where $a = \frac{n-p}{4}$ and $0 < b < \min \left\{ \frac{n-p}{p+1}, 1 \right\}$ is fixed.

Computing directly, we obtain

$$f_\alpha'(r) = \frac{n-p}{p-1} \frac{1}{r^{\frac{n-p}{p-1}+1}} - \left( \frac{n-p}{p-1} - b \right) \frac{1}{r^{\frac{n-p}{p-1}+1}} + \left( \frac{n-p}{p-1} + a \right) \frac{\mu_\alpha}{2r^{\frac{p}{p-1}+\alpha+1}}$$

$$= \frac{1}{r^{\frac{n-p}{p-1}+1}} \left[ - \frac{n-p}{p-1} + \left( \frac{n-p}{p-1} - b \right) r^b + \frac{1}{2} \left( \frac{n-p}{p-1} + a \right) \frac{\mu_\alpha}{r^{\frac{p}{p-1}+\alpha}} \right]$$

$$\leq \frac{1}{r^{\frac{n-p}{p-1}+1}} \left[ - \frac{n-p}{p-1} + \frac{1}{2} \left( \frac{n-p}{p-1} + a \right) \right]$$

$$= \frac{1}{4} \frac{n-p}{p-1} \frac{1}{r^{\frac{n-p}{p-1}}}.$$
(in particular, \( f'_\alpha(r) < 0 \)) and

\[
f'_\alpha(r) = \frac{1}{r^{\frac{n-p}{p-1}+1}} \left( - \frac{n-p}{p-1} - \left( \frac{n-p}{p-1} - b \right) r^b + \frac{1}{2} \left( \frac{n-p}{p-1} + a \right) \left( \frac{\mu_a}{r} \right)^a \right) \geq -2 \frac{n-p}{p-1} \frac{1}{r^{\frac{n-p}{p-1}}},
\]

so that

\[
\frac{1}{4} \frac{n-p}{p-1} \frac{1}{r^{\frac{n-p}{p-1}}} \leq |f'_\alpha(r)| \leq 2 \frac{n-p}{p-1} \frac{1}{r^{\frac{n-p}{p-1}}}, \tag{1.18}
\]

whence

\[
|f'_\alpha(r)|^{p-2} \geq C_{n,p} \frac{1}{r^{\frac{n-p}{p-1}+p-2}}, \tag{1.19}
\]

with \( C_{n,p} = \left( \frac{2n-p}{p-1} \right)^{p-2} \) if \( 1 < p < 2 \) and \( C_{n,p} = \left( \frac{1}{4} \right)^{p-2} \) if \( p \geq 2 \).

Moreover,

\[
f''_\alpha(r) = \frac{n-p}{p-1} \left( \frac{n-p}{p-1} + 1 \right) \frac{1}{r^{\frac{n-p}{p-1}+2}} + \left( \frac{n-p}{p-1} - b \right) \left( \frac{n-p}{p-1} - b + 1 \right) \frac{1}{r^{\frac{n-p}{p-1}-b+1}} - \left( \frac{n-p}{p-1} + a \right) \left( \frac{n-p}{p-1} + a + 1 \right) \frac{\mu_a}{2r^{\frac{n-p}{p-1}+a+2}}
\]

so that, since \( L \left( \frac{1}{r^{\frac{n-p}{p-1}}} \right) = 0 \) and \( L \) is linear, we have

\[
L (f_\alpha) = (p-1) \left( \frac{n-p}{p-1} - b \right) \left( \frac{n-p}{p-1} - b + 1 \right) \frac{1}{r^{\frac{n-p}{p-1}-b+1}} - (n-1) \left( \frac{n-p}{p-1} + a \right) \left( \frac{n-p}{p-1} + a + 1 \right) \frac{\mu_a}{2r^{\frac{n-p}{p-1}+a+2}} - \left( n-1 \right) \left( \frac{n-p}{p-1} - b \right) - (n-1) \left( \frac{n-p}{p-1} + a \right) \frac{\mu_a}{2r^{\frac{n-p}{p-1}+a+2}}
\]

\[
= \left[ (p-1) \left( \frac{n-p}{p-1} - b + 1 \right) \left( \frac{n-p}{p-1} - b \right) - (n-1) \left( \frac{n-p}{p-1} - b \right) \right] \frac{1}{r^{\frac{n-p}{p-1}-b+1}} + \left[ (n-1) \left( \frac{n-p}{p-1} + a \right) - (p-1) \left( \frac{n-p}{p-1} + a \right) \left( \frac{n-p}{p-1} + a + 1 \right) \right] \frac{\mu_a}{2r^{\frac{n-p}{p-1}+a+2}}
\]

\[
= -b(p-1) \left( \frac{n-p}{p-1} - b \right) \frac{1}{r^{\frac{n-p}{p-1}-b+1}} - a(p-1) \left( \frac{n-p}{p-1} + a \right) \frac{\mu_a}{2r^{\frac{n-p}{p-1}+a+2}}.
\]

Therefore,

\[
-L (f_\alpha) = a(p-1) \left( \frac{n-p}{p-1} + a \right) \frac{\mu_a}{2r^{\frac{n-p}{p-1}+a+2}} + b(p-1) \left( \frac{n-p}{p-1} - b \right) \frac{1}{r^{\frac{n-p}{p-1}-b+1}} \tag{1.20}
\]

and

\[
-\Delta_p z_\alpha(y) = -|f'_\alpha(r)|^{p-2} L [f_\alpha(r)] \geq C_{n,p} \frac{1}{r^{\frac{n-p}{p-1}+p-2}} a(p-1) \left( \frac{n-p}{p-1} + a \right) \frac{\mu_a}{2r^{\frac{n-p}{p-1}+a+2}} + |f'_\alpha(r)|^{p-2} b(p-1) \left( \frac{n-p}{p-1} - b \right) \frac{1}{r^{\frac{n-p}{p-1}-b+1}}
\]

\[
= C_{n,p} \frac{\mu_a}{2r^{n+a}} + |f'_\alpha(r)|^{p-2} b(p-1) \left( \frac{n-p}{p-1} - b \right) \frac{1}{r^{\frac{n-p}{p-1}-b+1}}.
\]
Thus,
\[-\Delta_{p,g} z_\alpha(y) \geq C_{n,p} \frac{\mu_\alpha}{\gamma \alpha^a + a} + |f'_i(r)|^{p-2} \frac{1}{r^{\gamma \alpha^a + b}} \left[ b(p-1) \left( \frac{n-p}{p-1} - b \right) - C_{n,p} \delta^* r^{1-b} \right] \]
\[\geq C_{n,p} \frac{\mu_\alpha}{\gamma \alpha^a + a},\]
if \(\delta^*\) is small enough. To verify that \(z_\alpha\) satisfies the remaining conditions of (1.11) write
\[z_\alpha(y) = \frac{1}{|y|^{\frac{\gamma \alpha^a}{\gamma \alpha^a + b}}} + \frac{1}{|y|^{\frac{\gamma \alpha^a}{\gamma \alpha^a + b}}} - \frac{\mu_\alpha}{2 |y|^{\frac{\gamma \alpha^a}{\gamma \alpha^a + b}}}
= 1 + |y|^b - \frac{1}{2} \left( \frac{\mu_\alpha}{|y|} \right)^a
= \frac{1 + |y|^b - \frac{1}{2} \left( \frac{\mu_\alpha}{|y|} \right)^a}{|y|^{\frac{\gamma \alpha^a}{\gamma \alpha^a + b}}},\]
and notice that
\[1 + |y|^b - \frac{1}{2} \left( \frac{\mu_\alpha}{|y|} \right)^a \leq 2\]
for all \(|y| \leq 1\), so that
\[z_\alpha(y) \leq \frac{2}{|y|^{\frac{\gamma \alpha^a}{\gamma \alpha^a + b}}},\]
for all \(|y| \leq 1\); if \(|y| = \mu_\alpha\), then
\[z_\alpha(y) = 1 + \mu_\alpha^b - \frac{1}{2} \left( \frac{\mu_\alpha}{\mu_\alpha} \right)^a \geq 1 - \frac{1}{2 \mu_\alpha^{\frac{\gamma \alpha^a}{\gamma \alpha^a + b}}},\]
and finally, if \(|y| = \Lambda\), then
\[z_\alpha(y) = 1 + \Lambda - \frac{1}{2} \frac{\mu_\alpha}{\Lambda^{\frac{\gamma \alpha^a}{\gamma \alpha^a + b}}} \geq 0\]
since \(\mu_\alpha \to 0\). ■
Chapter 2

Nonvalidity of Optimal Inequalities

2.1 Introduction

In this chapter we will show ranges for which the optimal inequality is not valid on manifolds which have positive scalar curvature somewhere.

Theorem 2.1. Let \((M^n, g)\) be a complete Riemannian manifold with positive scalar curvature somewhere. Then the optimal Sobolev inequality

\[
\|u\|_{L^p(M)}^p \leq K(n, p) \|\nabla_g u\|_{L^p(M)}^p + A \|u\|_{L^p(M)}^p \quad \text{for all } u \in H^1(M)
\]

is not valid if \(1 < p < \frac{n+2}{3}\) and \(r < \frac{np}{n-p+2}\), or if \(p = \frac{n+2}{3}\) and \(r \leq \frac{np}{n-p+2}\).

In particular, Theorem 2.1 improves the result obtained by Druet for \(n > 4\), by showing that for \(r = p\) the optimal inequality is not valid for \(\sqrt{n} \leq p < \frac{n+2}{3}\) (this is a consequence also of Theorem 2.2 below). Theorem 2.1 also shows that in the range \(1 < p \leq \frac{n+2}{3}\) the values of \(r\) that Aubin and Li obtained for which the optimal inequality is valid are sharp, except for the marginal value \(r = \frac{np}{n-p+2}\) in the range \(1 < p < \frac{n+2}{3}\), for which it is not known if the optimal inequality is true or not.

Theorem 2.2. Let \((M^n, g)\) be a complete Riemannian manifold with positive scalar curvature somewhere. Then the optimal Sobolev inequality

\[
\|u\|_{L^p(M)}^p \leq K(n, p)^a \|\nabla_g u\|_{L^p(M)}^p + A \|u\|_{L^p(M)}^p \quad \text{for all } u \in H^1(M)
\]

is not valid in any of the following situations:

\[
\begin{aligned}
n = 3 & \quad \text{and } \begin{cases} 1 < p < \frac{5}{3} \text{ and } a > 2, \\ p = \frac{5}{3} \text{ and } a \geq 2, \end{cases} \\
n = 4 & \quad 1 < p \leq 2 \text{ and } a > 2;
\end{aligned}
\]
Theorem 2.2 shows that some results obtained by Aubin and Li are sharp, while others are not known if they can be improved or not. If $n = 3, 4$, Aubin and Li showed that the optimal inequality is valid for $1 < p \leq 2$ and $0 < a \leq p$ and for $2 < p < n$ and $0 < a < p$; thus Theorem 2.2 allows us to conclude only that the results they obtained are sharp for $n = 4$ and $p = 2$.

For $n > 4$, according to the results obtained by Aubin and Li, the optimal inequality is valid in the following cases:

(i) $1 < p < 2$ and $0 < a \leq p$;

(ii) $2 \leq p \leq \sqrt{n}$ and $0 < a \leq 2$;

(iii) $\sqrt{n} < p < \frac{n + 2}{3}$ and $0 < a < 2p - \frac{n - p}{3p^2 + np + 2n}$;

(iv) $\frac{n + 2}{3} \leq p < n$ and $0 < a < p$.

Therefore, the situation obtained combining their results with Theorem 2.2 can be analyzed using the figure in the next page.
Below the black curve, which represents the function
\[ a^*(n, p) = \begin{cases} 
  p & \text{if } 1 < p < 2, \\
  2 & \text{if } 2 \leq p \leq \sqrt{n}, \\
  2p - \frac{n - p}{3p^3 + np + 2n} & \text{if } \sqrt{n} < p < \frac{n + 2}{3}, \\
  p & \text{if } \frac{n + 2}{3} \leq p < n, 
\end{cases} \]
lies a region where the optimal inequality is known to be valid, according to the results of Aubin and Li. Directly above the gray curve, and directly above the portion of the black curve corresponding to the range \( 2 \leq p \leq \sqrt{n} \), there lies a region where the inequality is not valid, according to Theorem 2.2. Between the gray and black curves, as well as directly above the black curve corresponding to the range \( \frac{n + 2}{3} < p < n \), there lie regions where the validity or not of the optimal inequality is unknown. The regions where it is not known if the optimal inequality is valid or not are, therefore:

1. \(1 < p < 2\) and \(p < a < 2\);
2. \(\sqrt{n} < p < \frac{n + 2}{3}\) and \(2p - \frac{n - p}{3p^3 + np + 2n} \leq a \leq 2p - \frac{1}{n - p}\);
3. \(\frac{n + 2}{3} < p < n\) and \(a \geq p\).

Notice that these regions of uncertainty include the black curve itself in the interval \(\sqrt{n} < p < n\), except for the point \(p = \frac{n + 2}{3}\). Otherwise, Theorem 2.2 proves that the results obtained by Aubin and Li are sharp for the case \(n > 4, 2 < p < \sqrt{n}\) and \(p = \frac{n + 2}{3}\).

### 2.2 Proof of Theorem 2.1

In order to show that the Sobolev inequality
\[
\|u\|_{L^p(M)}^p \leq K(n, p)^p \|\nabla_g u\|_{L^p(M)}^p + A \|u\|_{L^r(M)}^p
\]
for all \(u \in H^{1,p}(M)\), is false for the given values of \(r\) stated in Theorem 2.1, it suffices to prove that, for at least one Riemannian manifold \((M, g)\), there exists a sequence of functions \(u_\varepsilon\) such that
\[
\frac{\|u_\varepsilon\|_{L^p(M)}^p - K(n, p)^p \|\nabla_g u_\varepsilon\|_{L^p(M)}^p}{\|u_\varepsilon\|_{L^r(M)}^p} \to \infty \quad (2.1)
\]
as \(\varepsilon \to 0\). We will produce such a sequence for Riemannian manifolds which have positive scalar curvature at some point.

Let
\[
v(x) = \left( \frac{1}{1 + |x|^{\frac{2}{p-1}}} \right)^{\frac{p-2}{p}} \quad (2.2)
\]
be a minimizer for the Sobolev critical inequality in the Euclidean space \( \mathbb{R}^n \), so that
\[
\|v\|_{L^p(R^n)}^p = K^p(n, p) \|\nabla v\|_{L^p(R^n)}^p.
\] (2.3)

We consider a geodesic ball \( B_\delta(x_0) \subset (M, g) \), where \( \delta > 0 \) will be chosen sufficiently small later, and any smooth radial cutoff function \( 0 \leq \eta \leq 1 \) satisfying
\[
\eta(x) = \begin{cases} 
1 & \text{if } x \in B_\frac{\delta}{2}(x_0), \\
0 & \text{if } x \in M \setminus B_\delta(x_0),
\end{cases}
\]
and define
\[
u_\varepsilon(x) = \eta(x)v_\varepsilon(x),
\] (2.4)
where
\[
v_\varepsilon(x) = \varepsilon^{-\frac{n-p}{p}} v(\varepsilon^{-1} x) = \varepsilon^{-\frac{n-p}{p}} \left( \frac{1}{1 + \varepsilon^{-\frac{n}{p-1}} |x|^{p-1}} \right)^{\frac{n-p}{p}}.
\] (2.5)

This is precisely the scaling which ensures that
\[
\|v_\varepsilon\|_{L^p(R^n)} = \|v\|_{L^p(R^n)} + c_1 \varepsilon^{p_1} + o(\varepsilon^{p_1}),
\]
\[
\|\nabla v_\varepsilon\|_{L^p(M)} = \|\nabla v\|_{L^p(R^n)} + c_2 \varepsilon^{p_2} + o(\varepsilon^{p_2})
\]
and
\[
\|u_\varepsilon\|_{L^p} = c_3 \varepsilon^{p_3} + o(\varepsilon^{p_3}),
\]
where \( p_1, p_2 \) and \( p_3 \) are all positive real numbers. It follows from (2.3) that
\[
\frac{\|u_\varepsilon\|_{L^p(M)}^p - K^p(n, p) \|\nabla v_\varepsilon\|_{L^p(M)}^p}{\|u_\varepsilon\|_{L^p(M)}^p} = C \varepsilon^{-p_4} + o(\varepsilon^{-p_4}),
\]
where \( p_4 = p_3 - \min(p_1, p_2) \). Thus we obtain (2.1) if \( C > 0 \) and \( p_4 > 0 \); if \( C > 0 \) and \( p_4 \leq 0 \), the quotient is bounded and this method does not allow us to conclude whether the optimal Sobolev inequality is false or not.

For the following estimates, we introduce the notation:
\[
I_1 = I_1(n, p) = \int_{\mathbb{R}^n} v^p \, dx,
\]
\[
I_2 = I_2(n, p) = \int_{\mathbb{R}^n} v^p |x|^2 \, dx,
\]
\[
I_3 = I_3(n, p) = \int_{\mathbb{R}^n} |\nabla v|^p \, dx,
\]
\[
I_4 = I_4(n, p) = \int_{\mathbb{R}^n} |\nabla v|^p |x|^2 \, dx.
\] (2.7)
Notice that $K^p(n, p) = I_1^{p/p^*} I_3^{-1}$.

We will see that in the case $p > \frac{n + 2}{3}$ the quotient remains bounded, suggesting that in this case the result can be improved for all $1 < r < p^*$. If this proves to be true, it will not contradict Druet’s result (see the Introduction), because for $n \geq 4$ we always have $\frac{n + 2}{3} \geq \sqrt{n}$ (equality happens only when $n = 4$). On the other hand, in the case $1 < p \leq \frac{n + 2}{3}$, situation (2.1) will occur.

2.2.1 Estimate of $\|u_\varepsilon\|_{L^{p^*}(M)}^p$

In normal geodesic coordinates, the expansion of the element of volume up to the third order is given by (see [50], p. 97)

$$dv_g = \left[1 - \frac{1}{6} \sum_{i,j=1}^n \text{Ric}_{ij}(x_0)x_i x_j + O(|x|^3)\right] dx.$$

We also have

$$\eta^{p^*}(x) = 1 + O(|x|^3).$$

We write

$$\int_M v_\varepsilon^{p^*} dv_g = \int_{B_\delta} v_\varepsilon^{p^*} dx - \frac{1}{6} \sum_{i,j=1}^n \text{Ric}_{ij}(x_0) \int_{B_\delta} v_\varepsilon^{p^*} x_i x_j dx + \int_{B_\delta} v_\varepsilon^{p^*} O(|x|^3) dx$$

$$= \int_{\mathbb{R}^n} v^{p^*} dx - \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}} v^{p^*} dx - \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 \int_{B_{\frac{\delta}{2}}} v^{p^*} |x|^2 dx$$

$$+ \varepsilon^3 \int_{B_{\frac{\delta}{2}}} v^{p^*} O(|x|^3) dx.$$

By straightforward computations, we have (in the following, all orders are taken as $\varepsilon \to 0$):

$$\int_{\mathbb{R}^n - B_{\frac{\delta}{2}}} v^{p^*} dx = O(\varepsilon^{\frac{n+2}{3}})$$

and

$$\int_{B_{\frac{\delta}{2}}} v^{p^*} O(|x|^3) dx = \begin{cases} O(1) & \text{if } 1 < p < \frac{n + 3}{3}, \\ O(|\ln \varepsilon|) & \text{if } p = \frac{n + 3}{3}, \\ O(\varepsilon^{\frac{n+2}{3}}) & \text{if } p > \frac{n + 3}{3}. \end{cases}$$

If $1 < p < \frac{n + 2}{2}$, $\int_{\mathbb{R}^n} v^{p^*} |x|^2 dx$ converges, and we write

$$\int_{B_{\frac{\delta}{2}}} v^{p^*} |x|^2 dx = \int_{\mathbb{R}^n} v^{p^*} |x|^2 dx - \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}} v^{p^*} |x|^2 dx$$

$$= I_2 + O(\varepsilon^{\frac{n+2}{3}}).$$
Thus, for $1 < p < \frac{n+2}{2}$, we obtain
\[
\int_M |v^{|p}| \, dv_g = I_1 \left[ 1 - \frac{\text{Scal}_g(x_0)}{6n} I_2 \right] + o(\varepsilon^2),
\]
whence
\[
\|u_\varepsilon\|_{L^p(M)} = \|v\|_{L^p(\mathbb{R}^n)} - \frac{p}{p^*} \frac{\text{Scal}_g(x_0)}{6n} I_2 \varepsilon^2 + o(\varepsilon^2). \tag{2.9}
\]

### 2.2.2 Estimate of $\|\nabla u_\varepsilon\|_{L^p(M)}$

We write
\[
\int_M |\nabla_g u_\varepsilon|^p \, dv_g = \int_{B_{\frac{\varepsilon}{2}}(x_0)} |\nabla_g u_\varepsilon|^p \, dv_g + \int_{B_{\frac{\varepsilon}{2}}(x_0) \setminus B_{\frac{\varepsilon}{2}}(x_0)} |\nabla_g (\eta v_\varepsilon)|^p \, dv_g
\]
\[
= \int_{B_{\frac{\varepsilon}{2}}(x_0)} |\nabla v_\varepsilon|^p \, dv_g + \int_{B_{\frac{\varepsilon}{2}}(x_0) \setminus B_{\frac{\varepsilon}{2}}(x_0)} |\eta \nabla v_\varepsilon + v_\varepsilon \nabla \eta|^p \, dv_g,
\]
since $v$ and $\eta$ are radial with respect to the geodesic coordinates. By direct computation, we obtain
\[
|\nabla v|^p = \left( \frac{n-p}{p-1} \right)^p \left( \frac{1}{1 + |x|^{\frac{p-1}{n}}} \right)^n |x|^\frac{p}{p-1}.
\tag{2.10}
\]

We estimate the first integral. Using (2.8), we write
\[
\int_{B_{\frac{\varepsilon}{2}}(x_0)} |\nabla v_\varepsilon|^p \, dv_g = \int_{B_{\frac{\varepsilon}{2}}(x_0)} |\nabla v_\varepsilon|^p \, dx - \frac{1}{6} \sum_{i,j=1}^n \text{Ric}_{ij}(x_0) \int_{B_{\frac{\varepsilon}{2}}} |\nabla v_\varepsilon|^p \, x_i x_j \, dx
\]
\[
+ \int_{B_{\frac{\varepsilon}{2}}} |\nabla v_\varepsilon|^p \, O(|x|^3) \, dx
\]
\[
= \int_{\mathbb{R^n}} |\nabla v|^p \, dx - \int_{\mathbb{R^n} \setminus B_{\frac{\varepsilon}{2}}} |\nabla v|^p \, dx - \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 \int_{B_{\frac{\varepsilon}{2}}} |\nabla v|^p \, |x|^2 \, dx
\]
\[
+ \varepsilon^3 \int_{B_{\frac{\varepsilon}{2}}} |\nabla v|^p \, O(|x|^3) \, dx.
\]

By straightforward computations, from (2.10) we get
\[
\int_{\mathbb{R^n} \setminus B_{\frac{\varepsilon}{2}}} |\nabla v|^p \, dx = O(\varepsilon^{\frac{2n}{p-1}}) \tag{2.11}
\]
and
\[
\int_{B_{\frac{\varepsilon}{2}}} |\nabla v|^p \, O(|x|^3) \, dx = \begin{cases} O(1) & \text{if } 1 < p < \frac{n+3}{4}, \\ O(|\ln \varepsilon|) & \text{if } p = \frac{n+3}{4}, \\ O(\varepsilon^{\frac{2n}{p-1}-3}) & \text{if } p > \frac{n+3}{4}. \tag{2.12} \end{cases}
\]

If $1 < p < \frac{n+2}{3}$, $\int_{\mathbb{R^n}} |\nabla v|^p \, |x|^2 \, dx$ converges, and we write
\[
\int_{B_{\frac{\varepsilon}{2}}} |\nabla v|^p \, |x|^2 \, dx = \int_{\mathbb{R^n}} |\nabla v|^p \, |x|^2 \, dx - \int_{\mathbb{R^n} \setminus B_{\frac{\varepsilon}{2}}} |\nabla v|^p \, |x|^2 \, dx
\]
\[
= I_4 + O(\varepsilon^{\frac{2n}{p-1}}).
We now estimate the second integral:

$$
\int_{B_{x_0}} |\eta \nabla v_{e} + v_{e} \nabla \eta|^p \, dv_\eta = O(1) \left[ \int_{B_{1/2} \setminus B_{1/4}} |\nabla v|^p \, dx + \varepsilon^p \int_{B_{1/2} \setminus B_{1/4}} v^p \, dx \right] \tag{2.13}
$$

$$
= O(\varepsilon^{\frac{n+2}{3}}).
$$

Therefore, for $1 < p < \frac{n+2}{3}$, we obtain

$$
\| \nabla g u_{e} \|^p_{L^p(M)} = \| \nabla v \|^p_{L^p(\mathbb{R}^n)} - \frac{\text{Scal}_g(x_0)}{6n} I_4 \varepsilon^2 + o(\varepsilon^2). \tag{2.14}
$$

On the other hand, if $p = \frac{n+2}{3}$, then we write

$$
\int_{B_{1/2}} |\nabla v|^p |x|^2 \, dx = \left( \frac{n-p}{p-1} \right)^p \int_{B_{1/2}} \frac{|x|^{2+p-1}}{(1+|x|^{p-1})^n} \, dx
$$

$$
= \omega_{n-1} \left( \frac{n-p}{p-1} \right)^p \int_1^{\frac{1}{2^n}} \frac{s^{2+p-1} + n-1}{s^{n-1}} \frac{1}{1+s^{p-1}} \, ds + O(1)
$$

$$
= \omega_{n-1} \left( \frac{n-p}{p-1} \right)^p \int_1^{\frac{1}{2^n}} s^{-1} \left[ 1-ns^{p-1} + O\left(s^{-\frac{2n}{p-1}}\right) \right] \, ds + O(1)
$$

$$
= \omega_{n-1} \left( \frac{n-p}{p-1} \right)^p |\ln \varepsilon| + o(|\ln \varepsilon|).
$$

Therefore, if $p = \frac{n+2}{3}$, it follows from this and (2.11), (2.12) and (2.13) that

$$
\| \nabla g u_{e} \|^p_{L^p(M)} = \int_{\mathbb{R}^n} |\nabla v|^p \, dx - \omega_{n-1} \left( \frac{n-p}{p-1} \right)^p \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2 |\ln \varepsilon|). \tag{2.15}
$$

### 2.2.3 Estimate of $\| u_{e} \|^p_{L^p(M)}$

First we write

$$
\eta^r(x) = 1 + O(|x|^2)
$$

and

$$
dv_\eta = \left[ 1 + O(|x|^2) \right] \, dx.
$$

So,

$$
\int_M u_{e}^r \, dv_\eta = \int_{B_{1}(x_0)} v_{e}^r \eta^r \, dv_\eta = \int_{B_{1}} v_{e}^r \, dx + \int_{B_{1/2}} v_{e}^r O(|x|^2) \, dx
$$

$$
= \varepsilon^{n-\frac{2(n+2)}{p-1}} \int_{B_{1/2}} v_{e}^r \, dx + \varepsilon^{n-\frac{2(n+2)}{p-1}+2} \int_{B_{1/2}} v_{e}^r O(|x|^2) \, dx.
$$
Since
\[
\int_{B_{\frac{1}{\sqrt{n-p}}}} v^r \, dx = \begin{cases} 
O(\varepsilon^{\frac{n-p}{p-n} - n}) & \text{if } 1 < r \leq \frac{n(p-1)}{n-p}, \\
O(\ln \varepsilon) & \text{if } r = \frac{n(p-1)}{n-p}, \\
O(1) & \text{if } r > \frac{n(p-1)}{n-p},
\end{cases}
\]
and
\[
\int_{B_{\frac{1}{\sqrt{n-p}}}} v^r O(|x|^2) \, dx = \begin{cases} 
O(\varepsilon^{\frac{n-p}{p-n} - n-2}) & \text{if } 1 < r < \frac{(n+2)(p-1) - n-p}{n-p}, \\
O(\ln \varepsilon) & \text{if } r = \frac{(n+2)(p-1) - n-p}{n-p}, \\
O(1) & \text{if } r > \frac{(n+2)(p-1) - n-p}{n-p}.
\end{cases}
\]
it follows that
\[
\int_M u_\varepsilon \, dv_g = \begin{cases} 
O(\varepsilon^{\frac{n-p}{p-n}}) & \text{if } 1 < r < \frac{n(p-1)}{n-p}, \\
O(\varepsilon^{\frac{n-p-n+p}{p}} |\ln \varepsilon|) & \text{if } r = \frac{n(p-1)}{n-p}, \\
O(\varepsilon^{\frac{n-p-n+p}{p}}) & \text{if } r > \frac{n(p-1)}{n-p}.
\end{cases}
\]
In particular, if \( 1 < p < \frac{n+2}{3} \) and \( r < \frac{np}{n-p+2} \), then
\[
\|u_\varepsilon\|_{L^p(M)} = \begin{cases} 
O(\varepsilon^{\frac{n-p}{p-n}}) & \text{if } 1 < r < \frac{n(p-1)}{n-p}, \\
O(\varepsilon^{\frac{n-p-n+p}{p}} |\ln \varepsilon|) & \text{if } r = \frac{n(p-1)}{n-p}, \quad = o(\varepsilon^2). \tag{2.16}
\end{cases}
\]

### 2.2.4 Conclusion

Putting together (2.9) and (2.14), we obtain
\[
\|u_\varepsilon\|_{L^p(M)}^p = K^p(n,p) \|\nabla g u_\varepsilon\|_{L^p(M)}^p = C(n,p)\varepsilon^2 + o(\varepsilon^2), \tag{2.17}
\]
where
\[
C(n,p) = \frac{\text{Scal}_g(x_0)}{6n} \frac{1}{p^{p^*}} \left( \frac{I_4}{I_3} - \frac{p}{p^*} \frac{I_2}{I_1} \right).
\]
We assert that
\[
\frac{I_4}{I_3} - \frac{p}{p^*} \frac{I_2}{I_1} > 0
\]
if $1 < p < \frac{n+2}{3}$. Indeed, direct computation shows that

\[ I_1 = \omega_{n-1} \int_0^\infty \frac{r^{n-1} \Gamma(p)}{(1 + r^p)^n} dr = \omega_{n-1} \frac{p-1}{p} \frac{\Gamma\left(\frac{n}{p}\right) \Gamma\left(\frac{n}{p} - 1\right)}{\Gamma(n)}, \]

\[ I_2 = \omega_{n-1} \int_0^\infty \frac{r^{n+1} \Gamma(p)}{(1 + r^p)^n} dr = \omega_{n-1} \frac{p-1}{p} \frac{\Gamma\left(\frac{n}{p} + 2\right) \Gamma\left(\frac{n}{p} - 2\right)}{\Gamma(n)}, \]

\[ I_3 = \omega_{n-1} \left(\frac{n}{p-1}\right)^{p-1} \int_0^\infty \frac{r^{\frac{n}{p}-1}}{(1 + r^p)^n} dr = \omega_{n-1} \frac{n-p}{p-1} \frac{\Gamma\left(\frac{n}{p} - 1\right) \Gamma\left(\frac{n+1}{p} - 1\right)}{\Gamma(n)}, \]

\[ I_4 = \omega_{n-1} \left(\frac{n}{p-1}\right)^{p-1} \int_0^\infty \frac{r^{\frac{n}{p}+1}}{(1 + r^p)^n} dr = \omega_{n-1} \frac{n-p}{p-1} \frac{\Gamma\left(\frac{n+2}{p} - 3\right) \Gamma\left(\frac{n+1}{p} - 1\right)}{\Gamma(n)}. \]

Thus, using the fact that $\Gamma(x+1) = x\Gamma(x)$, we have

\[ \frac{I_1 I_4}{I_2 I_3} = \frac{(n-p)(n+2)}{n(n+2-3p)}, \]

whence

\[ \frac{p^* I_1 I_4}{p I_2 I_3} = \frac{n+2}{n+2-3p} > 1 \]

if $1 < p < \frac{n+2}{3}$.

In particular, if $\text{Scal}_g(x_0) > 0$, the numerator in (2.1) is positive. Thus, (2.1) follows from (2.16) and (2.17), proving the desired result.

If $p = \frac{n+2}{3}$, then (2.9) still holds and so

\[ \|u_{\varepsilon}\|_{L^p(M)}^p = \|v\|_{L^p(M)}^p + O(\varepsilon). \]

Also, if $p = \frac{n+2}{3}$ and $r \leq \frac{np}{n+2-p}$, from (2.16) we have

\[ \|u_{\varepsilon}\|_{L^p(M)}^p = O(\varepsilon^2). \]

Thus, in this case (2.1) follows from (2.15).

### 2.3 Proof of Theorem 2.2

As in the proof of Theorem 2.1, in order to show that the Sobolev inequality

\[ \|u\|_{L^p(M)}^p \leq K(n,p)\|\nabla g\|_{L^p(M)}^p + A \|u\|_{L^p(M)}^p \quad \text{for all } u \in H^{1,p}(M), \]

if $1 < p < \frac{n+2}{3}$. Indeed, direct computation shows that
is false for the values of $a$ stated in Theorem 2.2, we prove that the sequence of functions $u_{\varepsilon}$ defined in (2.4) satisfy

$$
\frac{\|u_{\varepsilon}\|_{L^p(M)}^a - K(n, p)^a \|\nabla g u_{\varepsilon}\|_{L^p(M)}^a}{\|u_{\varepsilon}\|_{L^p(M)}^a} \to \infty
$$

(2.18)
as $\varepsilon \to 0$, if $(M, g)$ is a Riemannian manifold which has positive scalar curvature at some point.

### 2.3.1 Estimate of $\|u_{\varepsilon}\|_{L^p(M)}^a$

From the computations made in Subsection 2.1.1, it follows that, if $1 < p < \frac{n+2}{2}$, then

$$
\|u_{\varepsilon}\|_{L^p(M)}^a = \|v\|_{L^p(M)}^a - \frac{a \text{ Scal}_g(x_0)}{p^*} \frac{I_2}{I_1^{1-p^*}} \varepsilon^2 + o(\varepsilon^2).
$$

(2.19)

On the other hand, by direct computation, we have

$$
\|u_{\varepsilon}\|_{L^p(M)}^a = \|v\|_{L^p(M)}^a + \begin{cases} 
O(\varepsilon^2 \ln \varepsilon) & \text{if } p = \frac{n+2}{2}, \\
O(\varepsilon^{p-1}) & \text{if } p > \frac{n+2}{2}.
\end{cases}
$$

### 2.3.2 Estimate of $\|\nabla g u_{\varepsilon}\|_{L^p(M)}^a$

From (2.14), if $1 < p < \frac{n+2}{3}$, we have

$$
\|\nabla g u_{\varepsilon}\|_{L^p(M)}^a = \|\nabla v\|_{L^p(M)}^a - \frac{a \text{ Scal}_g(x_0)}{p} \frac{I_4}{I_3} \varepsilon^2 + o(\varepsilon^2).
$$

(2.20)

If $p = \frac{n+2}{3}$, from (2.15) we have

$$
\|\nabla u_{\varepsilon}\|_{L^p(M)}^a = \|\nabla v\|_{L^p(M)}^a - w_{n-1} \frac{a}{p} \left( \frac{n-p}{p-1} \right) \frac{\text{ Scal}_g(x_0)}{6n} \varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2 |\ln \varepsilon|).
$$

(2.21)

If $p > \frac{n+2}{3}$, the term (2.13) does not allow us to know if the numerator of the Sobolev quotient is positive or not, and therefore the estimate in this case is inconclusive.

### 2.3.3 Estimate of $\|u_{\varepsilon}\|_{L^p(M)}^a$

We write

$$
\int_M u_{\varepsilon}^p \, dv_g = \varepsilon^p \int_{B_{\frac{1}{\varepsilon}}} v^p \, dx + \varepsilon^{p+2} \int_{B_{\frac{1}{\varepsilon}}} v^p O(|x|^2) \, dx.
$$

Since

$$
\int_{B_{\frac{1}{\varepsilon}}} v^p \, dx = \begin{cases} 
O(1) & \text{if } 1 < p < \sqrt{n}, \\
O(\ln \varepsilon) & \text{if } p = \sqrt{n}, \\
O(\varepsilon^{\frac{n-p^2}{p}}) & \text{if } p > \sqrt{n},
\end{cases}
$$
and
\[
\int_{B_r} v^p O(|x|^2) \, dx = \begin{cases} 
O(1) & \text{if } 1 < p < \sqrt{n+3} - 1, \\
O(|\ln \varepsilon|) & \text{if } p = \sqrt{n+3} - 1, \\
O(\varepsilon^{\frac{a-2}{p}}) & \text{if } p > \sqrt{n+3} - 1,
\end{cases}
\]
it follows that
\[
\int_M u^p \varepsilon \psi \, dv = \begin{cases} 
O(\varepsilon^p) & \text{if } 1 < p < \sqrt{n}, \\
O(\varepsilon^p |\ln \varepsilon|^{a/p}) & \text{if } p = \sqrt{n}, \\
O(\varepsilon^{\frac{a-2}{p}}) & \text{if } p > \sqrt{n}.
\end{cases}
\]
Thus,
\[
\|u_{\varepsilon}\|_{L^p(M)}^a = \begin{cases} 
O(\varepsilon^a) & \text{if } 1 < p < \sqrt{n}, \\
O(\varepsilon^a |\ln \varepsilon|^{a/p}) & \text{if } p = \sqrt{n}, \\
O(\varepsilon^{\frac{a-2}{p}}) & \text{if } p > \sqrt{n}.
\end{cases}
\]

### 2.3.4 Conclusion

Putting together (2.19) and (2.20), for \(1 < p < \frac{n+2}{3}\) we obtain
\[
\|u_{\varepsilon}\|_{L^p(M)}^a - K^a(n, p)\|\nabla g u_{\varepsilon}\|_{L^p(M)}^2 = C(n, p, a)\varepsilon^2 + o(\varepsilon^2),
\]
where
\[
C(n, p, a) = \frac{a \text{Scal}_g(x_0)}{6n} \left( \frac{I_4}{I_3} - \frac{p}{p^*} \right) > 0.
\]
If \(\text{Scal}_g(x_0) > 0\), as we have already seen in Subsection 2.1.4. Thus, in this case, if \(1 < p \leq \sqrt{n}\) (which always happens if \(n = 3, 4\)) and \(a > 2\), or if \(\sqrt{n} < p < \frac{n+2}{3}\) and \(a > 2p^* - 1\), (2.18) follows from (2.23) and (2.22).

If \(p = \frac{n+2}{3}\), we obtain from (2.19) and (2.21)
\[
\|u_{\varepsilon}\|_{L^p(M)}^a - K^a(n, p)\|\nabla g u_{\varepsilon}\|_{L^p(M)}^2 = o_{n-1} K^a(n, p) \frac{a}{p} \left( \frac{n-p}{p-1} \right)^p \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2 |\ln \varepsilon|).
\]
(2.24)

Thus, if \(n = 3\) (which corresponds to \(p = \frac{5}{3}\)) and \(a \geq 2\), or if \(n = 4\) (which corresponds to \(p = 2\)) and \(a > 2\), or if \(n > 4\) and \(a \geq 2p^* - 1 = p\), (2.18) follows from (2.24) and (2.22).
Chapter 3
Best Constants in Sobolev Trace Inequalities

3.1 Introduction

Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary. If \(1 \leq p < n\) and \(p^* = \frac{n-1}{n-p}\), the Sobolev trace embedding \(H^{1,p}(M) \hookrightarrow L^q(\partial M)\) is compact for \(1 \leq q < p^*\), while \(H^{1,p}(M) \hookrightarrow L^{p^*}(\partial M)\) is only continuous. Therefore, there exist positive constants \(A_1, B_1\), which may a priori depend on \(M\) and \(g\), such that the inequality

\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{\frac{1}{p^*}} \leq A_1 \left( \int_M |\nabla u|^p \, dv_g \right)^{\frac{1}{p}} + B_1 \left( \int_M |u|^p \, dv_g \right)^{\frac{1}{p}} \tag{3.1}
\]

holds for every \(u \in H^{1,p}(M)\). From (3.1), several other Sobolev-type trace inequalities follow. Our first example is the following:

**Proposition 3.1.** Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary and \(1 < p < n\). Then, there exists a constant \(C > 0\) such that

\[
\int_M |u|^p \, dv_g \leq C \left( \int_M |\nabla u|^p \, dv_g + \int_{\partial M} |u|^p \, ds_g \right) \tag{3.2}
\]

for all \(u \in H^{1,p}(M)\).

**Proof.** Suppose by contradiction that for all \(m \in \mathbb{N}\) we can find \(u_m \in H^{1,p}(M)\) such that

\[
\int_M |u_m|^p > m \left( \int_M |\nabla u_m|^p + \int_{\partial M} |u_m|^p \right). 
\]

We may assume \(\|u_m\|_{L^p(M)} = 1\) for all \(m\) (substituting \(v_m = u_m/\|u_m\|_{L^p(M)}\) for \(u_m\), if necessary). This implies

\[
\int_M |\nabla u_m|^p + \int_{\partial M} |u_m|^p < \frac{1}{m},
\]

whence \(\int_M |\nabla u_m|^p \to 0\) and \(\int_{\partial M} |u_m|^p \to 0\). In particular, \((\nabla u_m)\) is also bounded in \(L^p(M)\), so \((u_m)\) is bounded in \(H^{1,p}(M)\) and we may assume, up to a subsequence, that \(u_m \rightharpoonup u\) in \(H^{1,p}(M)\). By the compactness of the Sobolev embedding \(H^{1,p}(M) \hookrightarrow L^p(M)\), it follows that \(u_m \to u\) in \(L^p(M)\), and hence \(\|u\|_{L^p(M)} = 1\); in particular, \(u \neq 0\).
Moreover, $u_m \rightharpoonup u$ in $H^{1,p}(M)$ also implies
\[
\int_M |\nabla u|^p \leq \lim \inf \int_M |\nabla u_m|^p = 0,
\]
i.e., $\nabla u = 0$, and therefore $u$ is constant a.e. in $M$.

But, the compactness of the Sobolev trace embedding $H^{1,p}(M) \hookrightarrow L^p(\partial M)$ implies $u_m \rightharpoonup u$ in $L^p(\partial M)$, whence we get $u = 0$ on $\partial M$. In particular, it follows that $u \in H_{0}^{1,p}(M)$ and since the norm in $H_{0}^{1,p}(M)$ is equivalent to the $L^p$-norm of the gradient, we conclude that $u = 0$ in $M$, a contradiction. ■

From (3.1) and (3.2), there follows immediately that there exist positive constants $A, B,$ which may depend on $M$ and $g$, such that
\[
\left( \int_{\partial M} |u|^p \ ds_g \right)^\frac{1}{p} \leq A \left( \int_M |\nabla u|^p \ dv_g \right)^\frac{1}{p} + B \left( \int_{\partial M} |u|^p \ ds_g \right)^\frac{1}{p}
\]
for every $u \in H^{1,p}(M)$. This last inequality is the main focus of interest in this chapter.

Following the terminology and notation introduced by Hebey in [51], we call this inequality the \textit{generic Sobolev trace inequality of order $p$}. Our aim is to study the best constants associated with this inequality, i.e.,
\[
\overline{\alpha}_p(M, g) = \inf \left\{ A \in \mathbb{R} : \text{there exists } B \in \mathbb{R} \text{ such that (I)} \right\}
\]
and
\[
\overline{\beta}_p(M, g) = \inf \left\{ B \in \mathbb{R} : \text{there exists } A \in \mathbb{R} \text{ such that (I)} \right\}.
\]
If the first best constant $\overline{\alpha}_p(M, g)$ is attained, then there exists $B > 0$ such that
\[
\left( \int_{\partial M} |u|^p \ ds_g \right)^\frac{1}{p} \leq \overline{\alpha}_p(M, g) \left( \int_M |\nabla u|^p \ dv_g \right)^\frac{1}{p} + B \left( \int_{\partial M} |u|^p \ ds_g \right)^\frac{1}{p} \quad (I_p)
\]
for all $u \in H^{1,p}(M)$. This inequality will be called the \textit{first optimal trace Sobolev inequality of order $p$}. Similarly, if the second best constant is attained, then there exists $A > 0$ such that
\[
\left( \int_{\partial M} |u|^p \ ds_g \right)^\frac{1}{p} \leq A \left( \int_M |\nabla u|^p \ dv_g \right)^\frac{1}{p} + \overline{\beta}_p(M, g) \left( \int_{\partial M} |u|^p \ ds_g \right)^\frac{1}{p} \quad (\overline{I}_p)
\]
for all $u \in H^{1,p}(M)$ and this inequality will be called the \textit{second optimal Sobolev trace inequality of order $p$}. In this framework, one investigates what are the best constants $\overline{\alpha}_p(M, g)$ and $\overline{\beta}_p(M, g)$ and whether the optimal Sobolev trace inequalities are valid or not.

It turns out, as expected from other Sobolev-type inequalities already studied, that the first best constant does not depend on the metric $g$ or the manifold $M$ for any $1 < p < n$. Indeed, let
\[
\frac{1}{K(n, p)^p} = \inf_{u \in L^p(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p \ dx}{\left( \int_{\partial \mathbb{R}^n} |u|^p \ ds \right)^\frac{1}{p}}.
\]
The main result we prove in this chapter is the following:
Theorem 3.1. [18] Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary and \(1 < p < n\). Then

\[
\overline{\pi}_p(M, g) = K(n, p).
\]

The proof of this result is by contradiction. A version of the concentration-compactness principle of Lions for manifolds with boundary (Theorem 3.3) will play a fundamental role in it. Together with Ekeland’s variational principle, it will permit us to obtain that, for any positive \(\alpha\), the infimum for the functional

\[
\int_M |\nabla u|^p + \alpha \int_{\partial M} |u|^p
\]

is always attained by some function \(u_\alpha \in H^{1,p}(M)\). Another fact which will be needed is the almost everywhere convergence of the gradients of solutions of a \(p\)-Laplacian type equation (Proposition 3.4) which is satisfied by the functions \(u_\alpha\). This will allow us to apply the Brezis-Lieb lemma and conclude the proof (we remark that in the \(p = 2\) case this last part is trivial, due to Hilbert space properties).

We will also consider the following scaled optimal inequalities

\[
\left( \int_{\partial M} |u|^p \, ds_g \right)^{\frac{1}{p}} \leq \overline{\pi}^p_p(M, g) \int_M |\nabla u|^p \, dv_g + B \int_{\partial M} |u|^p \, ds_g, \quad (\overline{I}_p^p)
\]

and

\[
\left( \int_{\partial M} |u|^p \, ds_g \right)^{\frac{1}{p}} \leq A \int_M |\nabla u|^p \, dv_g + \overline{\pi}^p_p(M, g) \int_{\partial M} |u|^p \, ds_g. \quad (\overline{J}_p^p)
\]

Clearly, since \((a + b)^{1/p} \leq a^{1/p} + b^{1/p}\) for \(p \geq 1\), \((\overline{I}_p^p)\) and \((\overline{J}_p^p)\) imply, respectively, \((\overline{I}_1^p)\) and \((\overline{J}_1^p)\). However, it is possible that either \((\overline{I}_1^p)\) or \((\overline{J}_1^p)\) is valid but, respectively, \((\overline{I}_p^p)\) or \((\overline{J}_p^p)\) is not. Indeed, there are ranges of \(p\) for which \((\overline{I}_1^p)\) is valid, but \((\overline{J}_p^p)\) is not:

Theorem 3.2. [18] Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary and \(1 < p < n\). Then \((\overline{I}_1^p)\) is valid with

\[
\overline{\beta}_p(M, g) = \text{vol}_{g}(\partial M)^{-\frac{p-1}{p(n-1)}},
\]

where \(g = g|_{\partial M}\). On the other hand, \((\overline{J}_p^p)\) is valid if and only if \(n = 2\) and \(1 < p < 2\), or if \(n \geq 3\) and \(1 < p \leq 2\).

In particular, this result shows that the second best constant, different from the first best constant, depends very strongly on the metric \(g\) of the manifold. The mathematics involved in the proof of this theorem is simple and part of it follows the arguments developed by Bakry [15], Druet and Hebey [51] for the second best constant and the second optimal Sobolev inequality for compact manifolds without boundary.
This chapter is organized as follows. In Section 2, we obtain the first best constant for a scaled version of the trace inequality (3.1). This inequality is needed in order to prove Theorem 3.1, as well as in order to prove the version of the concentration-compactness principle for manifolds with boundary cited above, in Section 3. Section 4 is the proof of Theorem 3.1 and Section 5 is the proof of Theorem 3.2.

3.2 A Preliminary Result

In order to show that $K_p(n, p)$ is the first best constant for $(I_p^\text{gen})$, the first step is to prove that $K_p(n, p)$ is also the first best constant for the corresponding scaled version of the trace inequality (3.1). This follows from a standard partition of unity argument, which we present here for the reader’s convenience.

**Proposition 3.2.** [18] Let $(M, g)$ be a compact Riemannian manifold with boundary and $1 < p < n$. For any $\epsilon > 0$, there exists $B_\varepsilon = B(M, g, \varepsilon)$ such that

$$\|u\|_{L_p^\ast(\partial M)} \leq (K_p(n, p) + \varepsilon) \|\nabla u\|_{L_p(M)} + B_\varepsilon \|u\|_{L_p(M)}$$

(3.3)

for all $u \in H^{1,p}(M)$.

**Proof.** Throughout the remainder of this chapter, we will denote $K(n, p) = K$.

Choose a finite covering of $M$ by geodesic balls $B_k = B_k(p_k)$ such that: if the center $p_k$ of the ball lies in the interior of the manifold, then the entire ball lies in its interior (that is, if $p_k \in M - \partial M$, then $B_k \subset M - \partial M$) and thus $B_k$ is a normal geodesic neighborhood with normal geodesic coordinates $x_1, \ldots, x_n$; if $p_k \in \partial M$, then $B_k$ is a Fermi neighborhood with Fermi coordinates $x_1, \ldots, x_{n-1}, t$. In these neighborhoods we have

$$(1 - \varepsilon')I \leq (g_{ij}) \leq (1 + \varepsilon')I$$

and

$$1 - \varepsilon' \leq \sqrt{\det(g_{ij})} \leq 1 + \varepsilon',$$

where $\varepsilon'$ can be taken small enough, depending on the choosing of the covering.

Let $\{\phi_k\}$ be a partition of unity associated to the covering $\{B_k\}$; for convenience we will write $\phi_k = \rho_k^p$. First notice that

$$\|u\|_{L_p^\ast(\partial M)} = \left(\int_{\partial M} \left(\sum \rho_k^p |u|^p\right)^{\frac{p}{p^\ast}}\right)^{\frac{p^\ast}{p}} \leq \sum \|(\rho_k |u|)^p\|_{L_p^\ast(p(\partial M))} = \sum \|\rho_k u\|_{L_p^\ast(\partial M)}^{\frac{p^\ast}{p}}$$
so that the problem is reduced, up to the weight $\rho_k$, to find an estimate for $\|u\|_{L^p(\partial M)}$ in each normal neighborhood.

Using, $p/p^* < 1$ and the Sobolev embedding in $\partial \mathbb{R}^n_+$, we obtain

$$\|\rho_k u\|_{L^p(\partial M)} \leq \left(1 + \varepsilon'\right)^{\frac{p}{p^*}} \int_{\partial \mathbb{R}^n_+} |\rho_k u|^{p^*} dx' dt$$

$$\leq \left(1 + \varepsilon'\right)^{\frac{p}{p^*}} \int_{\partial M} |\nabla g(\rho_k u)|^p dv_g$$

$$\leq (1 + \varepsilon_1)^{\frac{p}{p^*}} \int_{\partial M} |\nabla g(\rho_k u)|^p dv_g,$$

where $\varepsilon_1$ is such that $1 + \varepsilon_1 = 1 + \frac{\varepsilon}{\varepsilon'}$.

Let $|\nabla g \rho| \leq C$ for all $k$, where $C$ is a positive constant that depends only on the chosen finite covering of the compact manifold $M$. From now on, $C$ will denote several positive constants which depend only on $M, g, n, p$. Recall the elementary inequality

$$(1 + t)^p \leq 1 + C_p pt + C_p t^p, \quad \text{for all } t \geq 0,$$

where $C_p = \max(1, 2p^{-2})$. Using this inequality, plus Hölder’s and Young’s inequalities, we get

$$\int_{\partial M} |\nabla g(\rho_k u)|^p dv_g \leq \int_{B_k \cap M} \rho_k^p |\nabla g u|^p dv_g + C \int_{B_k \cap M} (\rho_k |\nabla g u|^{p-1} |u| |\nabla g \rho_k|$$

$$+ C \int_{B_k \cap M} |\nabla g \rho_k|^p |u|^p dv_g$$

$$\leq (1 + \varepsilon_2) \int_{B_k \cap M} \rho_k^p |\nabla g u|^p dv_g + C \int_{B_k \cap M} |u|^p dv_g,$$

for some $\varepsilon_2$ which can be taken small enough. Thus, if we choose $\varepsilon'$ and $\varepsilon_2$ sufficiently small, we obtain

$$\|\rho_k u\|_{L^p(\partial M)} \leq (K^p + \varepsilon) \int_{B_k \cap M} \rho_k^p |\nabla g u|^p dv_g + C \int_{B_k \cap M} |u|^p dv_g,$$

and so

$$\|u\|_{L^p(\partial M)} \leq (K^p + \varepsilon) \int_{M} |\nabla g u|^p dv_g + B \int_{M} |u|^p dv_g.$$

$\blacksquare$
3.3 A Concentration-Compactness Principle for Manifolds with Boundary

In the proof of the concentration-compactness theorem given below, the main point is obtaining the two reverse Hölder inequalities (3.4) and (3.5). The rest of the argument follows closely the one given by Lions in [69], but we present the complete proof for the reader’s convenience.

**Theorem 3.3.** [18] (Concentration-Compactness Principle for Manifolds with Boundary) Let \((M, g)\) be a compact Riemannian manifold with boundary, \(u_m \rightharpoonup u \) in \(H^{1,p}(M)\) and

\[
|\nabla u_m|^p \, dv_g \rightharpoonup \mu,
\]
\[
|u_m|^{p^*} \, dv_g \rightharpoonup \nu,
\]
\[
|u_m|^{p^*} \, ds_g \rightharpoonup \pi,
\]

where \(\mu, \nu, \pi\) are bounded non-negative measures. Then, there exist at most a countable set \(J\), \(\{x_j\}_{j \in J} \subset M\) and positive numbers \(\{\mu_j\}_{j \in J}, \{\nu_j\}_{j \in J}, \{\pi_j\}_{j \in J}\) such that

\[
\mu \geq |\nabla u|^p \, dv_g + \sum_{j \in J} \mu_j \delta_{x_j},
\]
\[
\nu = |u|^{p^*} \, dv_g + \sum_{j \in J} \nu_j \delta_{x_j},
\]
\[
\pi = |u|^{p^*} \, ds_g + \sum_{j \in J} \pi_j \delta_{x_j},
\]

with

\[
\nu_j^{1/p^*} \leq 2^{1/n} K \mu_j^{1/p},
\]
\[
\pi_j^{1/p^*} \leq K \mu_j^{1/p}. \]

**Proof.** Set \(v_m = u_m - u\), so that \(v_m \rightharpoonup 0\) in \(H^{1,p}(M)\), and

\[
\omega_m := |u_m|^{p^*} \, dv_g - |u|^{p^*} \, dv_g,
\]
\[
\theta_m := |u_m|^{p^*} \, ds_g - |u|^{p^*} \, ds_g.
\]

By assumption,

\[
\omega_m \rightharpoonup \nu - |u|^{p^*} \, dv_g =: \omega,
\]
\[
\theta_m \rightharpoonup \pi - |u|^{p^*} \, ds_g =: \theta,
\]

and by the Brézis-Lieb lemma we have as well

\[
|v_m|^{p^*} \, dv_g \rightharpoonup \omega,
\]
\[|u_m|^p d\theta \to \theta.\]

Set also
\[\lambda_m := |\nabla v_m|^p d\nu_g \to \lambda,\]
i.e., this bounded sequence of measures must have a weak limit, up to a subsequence, which we call \(\lambda\).

Cherrier [33] showed (see also [51] and the comments there) that given \(\varepsilon > 0\), there exists
\[C_\varepsilon > 0\]
such that
\[\|u\|_{L^p(M)} \leq (2^{\frac{1}{p}} K + \varepsilon) \|\nabla u\|_{L^p(M)} + C_\varepsilon \|u\|_{L^p(M)},\]
for all \(u \in H^{1,p}(M)\), where \(K\) is the best constant in the Sobolev embedding in \(\mathbb{R}^n\), i.e.,
\[\frac{1}{K^p} = \inf_{\begin{subarray}{c} \nabla u \in L^p(\mathbb{R}^n) \end{subarray}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p}{\left(\int_{\mathbb{R}^n} |u|^p\right)^{\frac{p}{p^*}}}.\]

Let \(\xi \in C^\infty(M)\); then \(\xi v_m \in H^{1,p}(M)\) and so
\[
\left(\int_{\partial M} |\xi|^p |v_m|^p d\nu_g\right)^{1/p^*} \\
\leq (2^{\frac{1}{p}} K + \varepsilon) \|\nabla (\xi v_m)\|_{L^p(M)} + C_\varepsilon \|\xi v_m\|_{L^p(M)} \\
\leq (2^{\frac{1}{p}} K + \varepsilon) \left[\left(\int_M |\xi|^p |\nabla v_m|^p d\nu_g\right)^{1/p} + \left(\int_M |\nabla \xi|^p |v_m|^p d\nu_g\right)^{1/p}\right] \\
+ C_\varepsilon \left(\int_M |\xi|^p |v_m|^p d\nu_g\right)^{1/p},
\]
Taking the limit when \(m \to \infty\), since \(v_m \to 0\) in \(L^p(M)\), we obtain
\[
\left(\int_M |\xi|^p d\omega\right)^{1/p^*} \leq (2^{\frac{1}{p}} K + \varepsilon) \left(\int_M |\xi|^p d\lambda\right)^{1/p}
\]
for all \(\varepsilon > 0\) and for all \(\xi \in C^\infty(M)\). Making \(\varepsilon \to 0\), we conclude that
\[
\left(\int_M |\xi|^p d\omega\right)^{1/p^*} \leq 2^{\frac{1}{p}} K \left(\int_M |\xi|^p d\lambda\right)^{1/p} \quad \text{for all} \ \xi \in C^\infty(M),
\]
which is a type of reverse Hölder inequality for two different measures.

Given \(\varepsilon > 0\), by Proposition 3.2 there exists \(C_\varepsilon > 0\) such that
\[\|u\|_{L^p(M)} \leq (\overline{K} + \varepsilon) \|\nabla u\|_{L^p(M)} + C_\varepsilon \|u\|_{L^p(M)},\]
for all \( u \in H^{1,p}(M) \). Thus, if \( \xi \in C^\infty(M) \), then \( \xi v_m \in H^{1,p}(M) \) and so
\[
\left( \int_{\partial M} |\xi|^p |v_m|^p d\gamma \right)^{1/p^*} \\
\leq (K + \varepsilon) \|\nabla (\xi v_m)\|_{L^p(M)} + C \|\xi v_m\|_{L^p(M)} \\
\leq (K + \varepsilon) \left[ \left( \int_M |\xi|^p |\nabla v_m|^p d\gamma \right)^{1/p} + \left( \int_M |\nabla \xi|^p |v_m|^p d\gamma \right)^{1/p} \right] \\
+ C \varepsilon \left( \int_M |\xi|^p |v_m|^p d\gamma \right)^{1/p}.
\]
Taking the limit when \( m \to \infty \), since \( v_m \to 0 \) in \( L^p(M) \), we obtain
\[
\left( \int_{\partial M} |\xi|^p d\theta \right)^{1/p^*} \leq (K^p + \varepsilon) \left( \int_M |\xi|^p d\lambda \right)^{1/p}
\]
for all \( \varepsilon > 0 \) and for all \( \xi \in C^\infty(M) \). Making \( \varepsilon \to 0 \), we conclude that
\[
\left( \int_{\partial M} |\xi|^p d\theta \right)^{1/p^*} \leq K \left( \int_M |\xi|^p d\lambda \right)^{1/p}
\]
for all \( \xi \in C^\infty(M) \). (3.5)

Let \( \{x_j\}_{j \in J} \) be the atoms of the measure \( \omega \) (i.e., the points \( x_j \in M \) such that \( \omega(\{x_j\}) > 0 \)) and let \( \{y_i\}_{i \in I} \) be the atoms of the measure \( \theta \). Since \( \omega \) and \( \theta \) are finite measures, there are at most countably many such points. Set
\[
\nu_j = \omega(\{x_j\})
\]
and
\[
\pi_i = \theta(\{y_i\}).
\]
We decompose
\[
\omega = \omega_0 + \sum_{j \in J} \nu_j \delta_{x_j}
\]
and
\[
\theta = \theta_0 + \sum_{i \in I} \pi_i \delta_{y_i}
\]
where \( \omega_0 \) and \( \theta_0 \) are positive finite measures with no atoms. We assert that
\[
(\nu_j)^{1/p^*} \leq 2^{1/n} K [\lambda(\{x_j\})]^{1/p}
\]
and
\[
(\pi_i)^{1/p^*} \leq K [\lambda(\{y_i\})]^{1/p}
\]
for all \( j, i \) (in particular, the singularities of \( \omega \) and \( \theta \) are singularities of \( \lambda \)).

In fact, take a sequence \( \{\xi_k\} \subset C^\infty(M) \) such that \( \xi_k \searrow \chi_{\{x_j\}} \) pointwise. In particular, this sequence is dominated by \( \xi_1 \) which is obviously integrable with respect to both measures \( \omega \) and
λ, so we can apply Lebesgue’s Dominated Convergence Theorem to the sequence of inequalities (3.4) with \( \xi_k \) in place of \( \xi \) to obtain
\[
\left( \int_M \chi_{\{x_j\}} \, d\omega \right)^{1/p^*} \leq 2^{1/n} K \left( \int_M \chi_{\{x_j\}} \, d\lambda \right)^{1/p},
\]
and hence the desired result. Analogously, we take \( \left( \xi_k \right) \subset C^\infty(M) \), such that \( \xi_k \searrow \chi_{\{y_j\}} \) pointwise. The same type of argument now applied to (3.5) leads us to conclude that
\[
\left( \int_{\partial M} \chi_{\{y_j\}} \, d\theta \right)^{1/p^*} \leq K \left( \int_M \chi_{\{y_j\}} \, d\lambda \right)^{1/p}.
\]
Now, we show that if \( \mu_j = \lambda(\{x_j\}) \), then
\[
\mu \geq |\nabla u|^p \, dv_g + \sum_{j \in J} \mu_j \delta_{x_j}.
\]
In order to prove this, it is sufficient to show that
\[
\mu \geq |\nabla u|^p \, dv_g \text{ and } \mu_j = \mu(\{x_j\}).
\]
because the measures \( |\nabla u|^p \, dv_g \) and \( \sum_{j \in J} \mu_j \delta_{x_j} \) are mutually singular. In order to show the first inequality, it is enough to prove that
\[
\int_M \xi \, d\mu \geq \int_M |\nabla u|^p \, dv_g
\]
for all \( \xi \in C^\infty(M), \xi > 0 \). This can be done by noticing that the functional defined in \( H^{1,p}(M) \) by \( \nu \mapsto \left( \int_M \xi |\nabla \nu|^p \, dv_g \right)^{1/p} \) is weakly lower semicontinuous (since it is a seminorm), therefore
\[
\int_M \xi |\nabla \nu|^p \, dv_g \leq \liminf_{k \to \infty} \int_M \xi |\nabla u_k|^p \, dv_g = \int_M \xi \, d\mu.
\]
To prove that
\[
\mu(\{x_j\}) = \lambda(\{x_j\}),
\]
we just need to show that if \( \left( \xi_k \right) \subset C^\infty(M) \) is a sequence such that \( 0 \leq \xi_k \leq 1, \xi_k(x_j) = 1 \) and \( \xi_k \searrow \chi_{\{x_j\}} \) pointwise, then
\[
\left| \int_M \xi_k \, d\mu - \int_M \xi_k \, d\lambda \right| \to 0
\]
as \( k \to \infty \), since the first integral converges to \( \mu(\{x_j\}) \) and the second converges to \( \lambda(\{x_j\}) \). We write
\[
\left| \int_M \xi_k \, d\mu - \int_M \xi_k \, d\lambda \right| \leq \left| \int_M \xi_k \, d\mu - \int_M \xi_k |\nabla u_m|^p \, dv_g \right| + \left| \int_M \xi_k |\nabla u_m|^p \, dv_g - \int_M \xi_k |\nabla (u_m - u)|^p \, dv_g \right| + \left| \int_M \xi_k |\nabla v_m|^p \, dv_g - \int_M \xi_k \, d\lambda \right|,
\]
and choose \( m \) large enough to make the first and third integrals small enough. Then, the second term of the right-hand side of this inequality can be also be made arbitrarily small since (using a triple Hölder’s inequality and the fact that \((u_m)\) is a bounded sequence in \(H^{1,p}(M)\))

\[
\left| \int_M \xi_k \left( |\nabla u_m|^p - |\nabla (u_m - u)|^p \right) d\nu_g \right|
= \left| \int_M \xi_k \int_0^1 \frac{d}{dt} |\nabla (u_m - tu)|^p dt \; d\nu_g \right|
= p \left| \int_M \xi_k \int_0^1 |\nabla (u_m - tu)|^{p-2} \nabla u \; dt \; d\nu_g \right|
\leq p \int_0^1 \int_{\text{supp} \xi_k} |\nabla (u_m - tu)|^{p-2} \nabla u \; dv_g \; dt
\leq p \mu_g(M)^{1/p} \|\nabla (u_m - tu)\|_{L^p(\text{supp} \xi_k)}^{p-2} \|\nabla u\|_{L^p(\text{supp} \xi_k)}
\leq C \|\nabla u\|_{L^p(\text{supp} \xi_k)} \to 0
\]
as \( k \to \infty \). [This argument shows in effect that the singular parts of \( \lambda \) and \( \mu \) are the same.]

Next we show that \( \omega_0 = \theta_0 = 0 \), thereby finishing the proof of the theorem. Inequality (3.4) implies that \( \omega \ll \lambda \), i.e., \( \omega \) is absolutely continuous with respect to \( \lambda \). Thus, by the Radon-Nikodym theorem, there exists \( f \in L^1(M, \lambda) \) such that \( \omega = f \lambda \). Moreover, \( f \) has the property that, for \( \lambda \)-almost every \( x \in M \), if \( \lambda(\{x\}) = 0 \), then \( f(x) = 0 \). This follows from the theorem on differentiation of measures:

\[
f(x) = \lim_{r \to 0^+} \frac{\int_{B_r(x)} d\omega}{\int_{B_r(x)} d\lambda} \leq \liminf_{r \to 0^+} \frac{\left( \int_{B_r(x)} d\lambda \right)^{p'/p}}{\int_{B_r(x)} d\lambda} = 2^{p'/n} K^{p'} \liminf_{r \to 0^+} \left( \int_{B_r(x)} d\lambda \right)^{\frac{n}{p'}} \to 0.
\]

In particular, if \( \{z_j\}_{j \in J} \) are the atoms of the measure \( \lambda \), it follows that \( f = 0 \) \( \lambda \)-a.e. in \( M \setminus \{z_j\}_{j \in J} \). Therefore, if \( A \) is any Borel set in \( M \), we have

\[
\omega(A) = \int_A f \; d\lambda = \sum_{j \in J} f(z_j) \lambda(\{z_j\}) = \sum_{j \in J} f(z_j) \lambda(\{z_j\}) \delta_{z_j}(A)
\]

which implies that \( \omega_0 = 0 \).

For \( \theta \), the result is immediate. Let \( \{x_j\}_{j \in J_0} \) be the set of singularities on the boundary. Take a sequence \( \xi_k \subset C^\infty(M) \) such that \( \xi_k \searrow \chi_A \), where \( A = \partial M \setminus \{x_j\}_{j \in J_0} \); by (3.5) we have

\[
\left( \int_{\partial M} \chi_A d\theta \right)^{1/p} \leq K \left( \int_M \chi_A d\lambda \right)^{1/p} = 0,
\]

but if \( \theta_0 \neq 0 \), then \( \int_{\partial M} \chi_A d\theta = \theta_0(A) \neq 0 \), a contradiction. \( \blacksquare \)

### 3.4 Proof of Theorem 3.1

Theorem 3.1 will follow immediately from Proposition 3.3 and Theorem 3.4 below.
Proposition 3.3. [18] Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold with boundary and $1 < p < n$. Suppose that there exist real numbers $A, B$ such that for all $u \in H^{1,p}(M)$ we have

$$\|u\|_{L^p(\partial M)}^p \leq A \|\nabla u\|_{L^p(M)}^p + B \|u\|_{L^p(\partial M)}^p.$$ 

Then $A \geq \overline{K}(n, p)^p$.

**Proof.** Suppose by contradiction that there exist $A < \overline{K}^p, B \in \mathbb{R}$ such that the inequality is true for all $u \in H^{1,p}(M)$. Fix $x_0 \in \partial M$ and, given $\varepsilon > 0$, let $B_\delta(0) \subset \mathbb{R}^n_+$ be the image through a chart of $M$ of a convex neighborhood centered in $x_0$ such that in $B_\delta(0)$ we have

$$(1 - \varepsilon)I \leq (g_{ij}) \leq (1 + \varepsilon)I$$

and

$$1 - \varepsilon \leq \sqrt{\det(g_{ij})} \leq 1 + \varepsilon.$$ 

It follows that, if we choose $\varepsilon$ small enough, we have, for any $u \in C^\infty_0(\mathbb{R}^n_+)$,

$$\left(\int_{\partial B_\delta(0)} |u|^p \, dx'\right)^{1/p} \leq A' \left(\int_{\partial B_\delta(0)} |\nabla u|^p \, dx + B' \int_{\partial B_\delta(0)} |u|^p \, dx', \right)$$

for some real numbers $A' < \overline{K}^p, B'$.

On the other hand, by Hölder’s inequality,

$$\left(\int_{\partial B_\delta(0)} |u|^p \, dx'\right)^{1/p} \leq |\partial B_\delta(0)|^{1/n} \left(\int_{\partial B_\delta(0)} |u|^p \, dx'\right)^{1/p'}.$$ 

Hence, choosing $\delta$ small enough, we get that there exists $A'' < \overline{K}^p$ such that for any $u \in C^\infty_0(\mathbb{B}_\delta(0))$

$$\left(\int_{\partial B_\delta(0)} |u|^p \, dx'\right)^{1/p} \leq A'' \int_{\mathbb{R}^n_+} |\nabla u|^p \, dx.$$ 

Let $u \in C^\infty_0(\mathbb{R}^n_+)$ and set $u_\lambda(x) = \lambda^{n-1} u(\lambda x)$. If $\lambda > 0$ is sufficiently large, then $u_\lambda \in C^\infty_0(\mathbb{B}_\delta(0))$, and so

$$\left(\int_{\partial B_\delta(0)} |u_\lambda|^p \, dx'\right)^{1/p} \leq A'' \int_{\mathbb{R}^n_+} |\nabla u_\lambda|^p \, dx.$$ 

Since this is the rescaling that ensures $\|u_\lambda\|_{L^p(\partial B_\delta(0))} = \|u\|_{L^p(\partial B_{\lambda\delta}(0))}$ and $\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n_+)} = \|\nabla u\|_{L^p(\mathbb{R}^n_+)}$, it follows that

$$\left(\int_{\partial B_\delta(0)} |u|^p \, dx'\right)^{1/p} \leq A'' \int_{\mathbb{R}^n_+} |\nabla u|^p \, dx$$

for all $u \in C^\infty_0(\mathbb{R}^n_+)$, contradicting the fact that $\overline{K}$ is the best constant for the Sobolev inequality in $\mathbb{R}^n_+$. ■
Theorem 3.4. [18] Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary and \(1 < p < n\). Then, for any \(\varepsilon > 0\), there exists \(A_\varepsilon = A(M, g, \varepsilon)\) such that

\[
\|u\|_{L^p(\partial M)}^p \leq (\overline{K}(n, p) + \varepsilon) \|\nabla u\|_{L^p(M)}^p + A_\varepsilon \|u\|_{L^p(\partial M)}^p
\]

for all \(u \in H^{1,p}(M)\).

Proof. Assume by contradiction that there exists \(\varepsilon_0 > 0\) such that for all \(\alpha > 0\) we can find \(u \in H^{1,p}(M)\) with

\[
\|u\|_{L^p(\partial M)}^p > (\overline{K}^p + \varepsilon_0) \int_M |\nabla g u|^p dv_g + \alpha \int_{\partial M} |u|^p dv_g,
\]

or, equivalently,

\[
\frac{\int_M |\nabla u|^p + \alpha \int_{\partial M} |u|^p}{\|u\|_{L^p(\partial M)}^p} < \frac{1}{\overline{K}^p + \varepsilon_0}.
\]

Of course, this inequality will be true for any \(0 < \varepsilon \leq \varepsilon_0\).

It follows that, for all \(\alpha > 1\), there exists \(\delta > 0\) independent of \(\alpha\) such that

\[
\xi_\alpha := \inf_{u \in H^{1,p}(M), u \neq 0} \frac{\int_M |\nabla u|^p + \alpha \int_{\partial M} |u|^p}{\|u\|_{L^p(\partial M)}^p} < \frac{1}{\overline{K}^p + \varepsilon_0} = \frac{1}{\overline{K}^p} - \delta. \tag{3.7}
\]

Claim 1. The infimum is reached by some function \(u_\alpha \in H^{1,p}(M)\) for each \(\alpha\).

As the quotient in (3.7) is homogeneous, for any fixed \(\alpha\) we can take a minimizing sequence \((u_m) \subset H^{1,p}(M)\) for it satisfying \(\|u_m\|_{L^p(\partial M)}^p = 1\). As

\[
\int_M |\nabla u_m|^p + \alpha \int_{\partial M} |u_m|^p \to \xi_\alpha, \tag{3.8}
\]

we conclude that \((u_m)\) is bounded in \(L^p(\partial M)\) and \((\nabla u_m)\) is bounded in \(L^p(M)\). By inequality (3.2), these two facts together imply that \((u_m)\) is bounded in \(H^{1,p}(M)\), and we may assume \(u_m \to u\) in \(H^{1,p}(M)\), \(u_m \to u\) in \(L^p(M)\) and \(u_m \to u\) in \(L^p(\partial M)\). In particular, we have

\[
|\nabla u_m|^p \to \mu, \\
|u_m|^p \to \nu, \\
|u_m|^p \to \pi,
\]

for some bounded, non-negative measures \(\mu, \nu, \pi\). By the concentration-compactness principle (Theorem 3.3), there exist at most a countable set \(J, \{x_j\}_{j \in J} \subset M\) and positive numbers
\{\mu_j\}_{j \in J}, \{\nu_j\}_{j \in J}, \{\pi_j\}_{j \in J} \text{ such that }

\mu \geq |\nabla u|^p \, dv + \sum_{j \in J} \mu_j \delta_{x_j},

\nu = |u|^{p^*} \, dv + \sum_{j \in J} \nu_j \delta_{x_j}, \quad (3.9)

\pi = |u|^{p^*} \, ds + \sum_{j \in J} \pi_j \delta_{x_j},

with

\nu_j^{1/p^*} \leq 2^{1/n} K\mu_j^{1/p}, \quad (3.10)

\pi_j^{1/p^*} \leq K\mu_j^{1/p}.

Now we recall a version of Ekeland’s Variational Principle (see, for instance, [72]):

**Lemma 3.1.** If $E$ is a Banach space, $M \subset E$ a closed differentiable manifold and $G : M \to \mathbb{R}$ is a continuously differentiable functional bounded from below, there exists a minimizing sequence $(v_m)$ for $G$ in $M$ such that

$$G(v_m) \to \inf_M G,$$

$$\|G'(v_m)\|_{(T_{v_m}M)^*} \to 0.$$

We apply Lemma 3.1 to $M = g^{-1}(1)$, where $g : H^{1,p}(M) \to \mathbb{R}$ is defined by $g(u) = \|u\|_L^{p^*}(\partial M)$ (because of the continuity of the Sobolev embedding $H^{1,p}(M) \hookrightarrow L^{p^*}(\partial M)$, $M$ is closed in $H^{1,p}(M)$) and $G : M \to \mathbb{R}$ defined by

$$G(u) = \frac{1}{p} \int_M |\nabla u|^p + \frac{\alpha}{p} \int_{\partial M} |u|^p,$$

so that we may assume that $\|G'(u_m)\|_{(T_{u_m}M)^*} \to 0$. The tangent space $T_{u_m}M$ consists exactly of the functions $\varphi$ such that $Dg(u_m)\varphi = 0$, that is,

$$\int_{\partial M} |u_m|^{p^* - 2} u_m \varphi = 0. \quad (3.11)$$

We can thus decompose $H^{1,p}(M) = T_{u_m}M \oplus \langle u_m \rangle$, by writing any $\psi \in H^{1,p}(M)$ in the form $\psi = (\psi - \lambda u_m) + \lambda u_m$ with $\lambda = \int_{\partial M} |u_m|^{p^* - 2} u_m \, \psi$. Now, if $\varphi \in T_{u_m}M$, then by Lemma 3.1

$$\int_M |\nabla u_m|^{p^* - 2} \nabla u_m \nabla \varphi + \alpha \int_{\partial M} |u_m|^{p^* - 2} u_m \varphi \to 0. \quad (3.12)$$

Take $x_k \in M$ in the support of the singular parts of $\mu, \nu$ and $\pi$. Let $\varphi \in C_0^\infty(M)$ satisfy $\text{supp} \varphi \subset B_{2\varepsilon}(x_k), \varphi \equiv 1$ on $B_{\varepsilon}(x_k)$ and $|\nabla \varphi| < \frac{2}{\varepsilon}$. Clearly the sequence $(\varphi u_m)$ is bounded in $H^{1,p}(M)$. Write

$$\varphi u_m = \left[ \varphi - \left( \int_{\partial M} |u_m|^{p^*} \right) \right] u_m + \left( \int_{\partial M} |u_m|^{p^*} \varphi \, dx \right) u_m,
so that
\[ \zeta_m := \left[ \varphi - \left( \int_{\partial M} |u_m|^p \varphi \right) \right] u_m \in T_{u_m, M} \]

is a bounded sequence in \( H^{1,p}(M) \). It follows from (3.12), (3.8) and (3.9) that
\[
\lim_{m \to \infty} \left[ \int_M |\nabla u_m|^{p-2} \nabla u_m \nabla (\varphi u_m) + \alpha \int_{\partial M} |u_m|^{p-2} u_m \varphi u_m \right] \\
= \lim_{m \to \infty} \left[ \int_M |\nabla u_m|^{p-2} \nabla u_m \nabla \zeta_m + \alpha \int_{\partial M} |u_m|^{p-2} u_m \zeta_m \right] \\
+ \lim_{m \to \infty} \left( \int_{\partial M} |u_m|^p \varphi \right) \left[ \int_M |\nabla u_m|^p + \alpha \int_{\partial M} |u_m|^p \right] \\
= \xi_\alpha \int_{\partial M} \varphi \, d\pi.
\]

On the other hand, we also have
\[
\lim_{m \to \infty} \left[ \int_M |\nabla u_m|^{p-2} u_m \nabla \varphi \right] \\
= \lim_{m \to \infty} \left[ \int_M |\nabla u_m|^{p-2} u_m \nabla \varphi + \int_{\partial M} \nabla u_m \nabla \varphi + \alpha \int_{\partial M} |u_m|^p \varphi \right] \\
= \lim_{m \to \infty} \int_M |\nabla u_m|^{p-2} u_m \nabla \varphi + \int_{\partial M} \varphi \, d\mu + \alpha \int_{\partial M} |u|^p \varphi.
\]

Therefore, we conclude that
\[
\lim_{m \to \infty} \int_M |\nabla u_m|^{p-2} u_m \nabla \varphi + \int_{\partial M} \varphi \, d\mu + \alpha \int_{\partial M} |u|^p \varphi = \xi_\alpha \int_{\partial M} \varphi \, d\pi.
\]
We assert that
\[
\lim_{m \to \infty} \left| \int_M |\nabla u_m|^{p-2} u_m \nabla \varphi \right| \to 0
\]
as \( \varepsilon \to 0 \). Indeed, by H"older’s Inequality,
\[
\lim_{m \to \infty} \left| \int_M |\nabla u_m|^{p-2} u_m \nabla \varphi \right| \\
\leq \lim_{m \to \infty} \int_{B_{2\varepsilon}(x_k)} |\nabla u_m|^{p-1} |u_m \nabla \varphi| \\
\leq \lim_{m \to \infty} \left( \|
abla u_m\|_{L^p(M)}^{p-1} \int_{B_{2\varepsilon}(x_k) \setminus B_{\varepsilon}(x_k)} |u_m|^p |\nabla \varphi|^p \right) \\
\leq C \lim_{m \to \infty} \left( \|
abla \varphi\|_{L^{p/p^*}(B_{2\varepsilon}(x_k) \setminus B_{\varepsilon}(x_k))} \right)^\frac{p}{p^*} \left( \int_{B_{2\varepsilon}(x_k) \setminus B_{\varepsilon}(x_k)} |u_m|^p \right)^{\frac{p}{p^*}} \\
= C \left( \int_{B_{2\varepsilon}(x_k) \setminus B_{\varepsilon}(x_k)} |\nabla \varphi| \right)^{\frac{p}{p^*}} \lim_{m \to \infty} \left( \int_{B_{2\varepsilon}(x_k) \setminus B_{\varepsilon}(x_k)} |u_m|^p \right)^{\frac{p}{p^*}} \\
\leq C \left\{ \frac{1}{\varepsilon^m} \text{vol}_g B_{2\varepsilon}(x_k) \setminus B_{\varepsilon}(x_k) \right\}^{\frac{p}{p^*}} \lim_{m \to \infty} \left( \int_{B_{2\varepsilon}(x_k) \setminus B_{\varepsilon}(x_k)} |u_m|^p \right)^{\frac{p}{p^*}} \\
\leq C \left( \int_{B_{2\varepsilon}(x_k) \setminus B_{\varepsilon}(x_k)} d\nu \right)^{\frac{p}{p^*}} \\
\to 0
\]
as $\varepsilon \to 0$.

Therefore, taking the limit as $\varepsilon \to 0$, we conclude that
\[ \mu_k = \xi_k \pi_k. \]

It follows from (3.10) and (3.7) that
\[ \mu_k > \frac{1}{K^p}. \]

In particular, since $\mu$ is a bounded measure, only a finite number of the $\mu_j$ are nonzero. We assert that $\mu_j = 0$ for all $j$. Indeed, if $\mu_k \neq 0$ for at least one indice $k$, then
\[
\frac{1}{K^p} > \xi_k = \lim_{m \to \infty} \left( \int_M |\nabla u_m|^p + \alpha \int_{\partial M} |u_m|^p \right) \\
\geq \int_M |\nabla u|^p \, dx + \sum_j \mu_j + \alpha \int_{\partial M} |u|^p \\
\geq \mu_k > \frac{1}{K^p},
\]
a contradiction. Therefore, we conclude that $\mu_j = 0$ for all $j$, whence, from (3.10),
\[ \pi_j = 0 \quad \text{for all } j, \]
whence, from (3.9),
\[ \|u_m\|_{L^p(\partial M)} \to \|u\|_{L^p(\partial M)}. \]

In particular, $u_m \to u$ in $L^p(\partial M)$ and $\|u\|_{L^p(\partial M)} = 1$. Thus, since
\[
\int_M |\nabla u|^p + \alpha \int_{\partial M} |u|^p \leq \liminf_{m \to \infty} \int_M |\nabla u_m|^p + \alpha \int_{\partial M} |u|^p \\
\leq \liminf_{m \to \infty} \left( \int_M |\nabla u_m|^p + \alpha \int_{\partial M} |u_m|^p \right) \\
= \xi_\alpha,
\]
we finally conclude that $u$ is a minimum for $\xi_\alpha$, thereby proving the claim.

Now, for each $\alpha$, let $u_\alpha \in H^{1,p}(M)$ satisfy $\|u_\alpha\|_{L^p(\partial M)} = 1$ and
\[
\int_M |\nabla u_\alpha|^p + \alpha \int_{\partial M} |u_\alpha|^p = \xi_\alpha \leq \frac{1}{K} - \delta.
\]
Using the same arguments we used above to prove Claim 1, we conclude that $(u_\alpha)$ is bounded in $H^{1,p}(M)$, and therefore we may assume, up to a subsequence, that $u_\alpha \to u$ in $H^{1,p}(M)$, $u_\alpha \to u$ in $L^p(M)$ and $u_\alpha \to u$ in $L^p(\partial M)$. But, at the same time,
\[
\int_{\partial M} |u_\alpha|^p \leq \frac{1}{\alpha} \left( \frac{1}{K} - \delta \right) \to 0,
\]
as \( \alpha \to \infty \), so we conclude that

\[
u = 0 \quad \text{on } \partial M.
\]

Claim 2. \( \nabla u_\alpha \to \nabla u \) a. e.

Claim 2 will follow from the fact that \( u_\alpha \) satisfies the Neumann boundary value problem

\[
\begin{cases}
-\Delta_p u_\alpha = 0 & \text{in } M, \\
|\nabla u_\alpha|^{p-2} \frac{\partial u_\alpha}{\partial \nu} + \alpha |u_\alpha|^{p-2} u_\alpha = \xi_\alpha |u_\alpha|^{p-2} u_\alpha & \text{on } \partial M.
\end{cases}
\]  

(3.13)

and Proposition 3.4 below, which is a generalization of Theorem 3, p. 40, in [46] for the case \( p = 2 \). In order to prove Proposition 3.4, two elementary lemmas are needed. The first of them is well known (see [75],[82]).

Lemma 3.2. Let \( a, b \in \mathbb{R}^n \) and let \( \langle \cdot , \cdot \rangle \) denote the standard scalar product. Then

\[
\langle |a|^{p-2} a - |b|^{p-2} b, a - b \rangle \geq \begin{cases}
c_p |a - b|^p & \text{if } p \geq 2, \\
c_p \frac{|a - b|^2}{|a| + |b|^{2-p}} & \text{if } 1 < p < 2.
\end{cases}
\]

Lemma 3.3. [18] If \( f, g, h \in H^{1,p}(M) \), then

\[
\int_M (-\Delta_p g) h dv_g = \int_M h \langle |\nabla f|^{p-2} \nabla f, \nabla g \rangle_g dv_g + \int_M g \langle |\nabla f|^{p-2} \nabla f, \nabla h \rangle_g dv_g - \int_{\partial M} gh \langle |\nabla f|^{p-2} \nabla f, \nu \rangle ds_g.
\]

Proof. The proof is a straightforward application of the divergence theorem.

\[
\begin{align*}
\int_M (-\Delta_p g) h dv_g &= -\int_M \text{div}_g(|\nabla f|^{p-2} \nabla f) gh dv_g \\
&= \int_M \langle |\nabla f|^{p-2} \nabla f, \nabla (gh) \rangle_g dv_g - \int_M \text{div}_g(gh|\nabla f|^{p-2} \nabla f) dv_g \\
&= \int_M \langle |\nabla f|^{p-2} \nabla f, h \nabla g + g \nabla h \rangle_g dv_g - \int_{\partial M} \langle gh|\nabla f|^{p-2} \nabla f, \nu \rangle ds_g.
\end{align*}
\]

\[
\blacksquare
\]

Proposition 3.4. [18] Let \( p > 1 \) and \( (u_m) \subset H^{1,p}(M) \) be a bounded sequence of weak solutions of

\[
-\Delta_p u_m = f_m \quad \text{in } M,
\]

with \( (f_m) \) bounded in \( L^1(M) \). Then \( (u_m) \) has a convergent subsequence in \( H^{1,q}(M) \) for each \( 1 \leq q < p \).
\textbf{Proof.} We may assume, up to a subsequence, that \( u_m \to u \) in \( H^{1,p}(M) \) and \( u_m \to u \) in \( L^p(M) \).

Fix \( \sigma, j > 0 \) and define

\[
\beta_\sigma(t) = \begin{cases} 
-\sigma & \text{if } t \leq -\sigma, \\
t & \text{if } -\sigma \leq t \leq \sigma, \\
\sigma & \text{if } t \geq \sigma.
\end{cases}
\]

According to Egoroff’s Theorem, there exists a measurable set \( E_j \subset M \setminus \partial M \) such that \( |M \setminus E_j| < \frac{1}{j} \) and \( u_m \to u \) uniformly on \( E_j \). Let \( \varphi \in C_0^\infty(M \setminus \partial M) \) be a cutoff function satisfying \( \varphi \equiv 1 \) on \( E_j \). For all \( m \) sufficiently large we have \( |u_m - u| \leq \frac{\sigma}{2} \), so that for all such \( m \) we can use Lemma 3.2 to write, if \( p \geq 2 \)

\[
\int_{E_j} |\nabla u_m - \nabla u|^p \, dv_g \leq C_p \int_{E_j} \left\langle |\nabla u_m|^{p-2} \nabla u_m - |\nabla u|^{p-2} \nabla u, \nabla u_m - \nabla u \right\rangle \, dv_g
\]

\[= C_p \int_M \varphi \left\langle |\nabla u_m|^{p-2} \nabla u_m, \nabla \beta_\sigma(u_m - u) \right\rangle \, dv_g - C_p \int_M \varphi \left\langle |\nabla u|^{p-2} \nabla u, \nabla \beta_\sigma(u_m - u) \right\rangle \, dv_g.
\]

Now, since \( \beta_\sigma(u_m - u) \to 0 \) in \( H^{1,p}(M) \), we have

\[
\int_M \varphi \left\langle |\nabla u|^{p-2} \nabla u, \nabla \beta_\sigma(u_m - u) \right\rangle \, dv_g \to 0.
\]

On the other hand, by Lemma 3.3, we can write

\[
\int_M \varphi \left\langle |\nabla u_m|^{p-2} \nabla u_m, \nabla \beta_\sigma(u_m - u) \right\rangle \, dv_g = \int_M \varphi f_m \beta_\sigma(u_m - u) \, dv_g
\]

\[\quad - \int_M \left\langle |\nabla u_m|^{p-2} \nabla u_m, \nabla \varphi \right\rangle \beta_\sigma(u_m - u) \, dv_g.
\]

Hölder’s inequality and the fact that \( \beta_\sigma(u_m - u) \to 0 \) in \( L^p(M) \) (for \( \beta_\sigma \) is Lipschitz) imply that

\[
\int_M \left\langle |\nabla u_m|^{p-2} \nabla u_m, \nabla \varphi \right\rangle \beta_\sigma(u_m - u) \, dv_g \leq C \|\nabla u_m\|_{L^p(M)}^{p-1} \|\beta_\sigma(u_m - u)\|_{L^p(M)}
\]

\[\quad \leq C \|\beta_\sigma(u_m - u)\|_{L^p(M)} \to 0.
\]

Therefore,

\[
\limsup_{m \to \infty} \int_{E_j} |\nabla u_m - \nabla u|^p \, dv_g \leq \sigma \sup \|f_m\|_{L^1(M)} = O(\sigma).
\]

Hence, passing to a further subsequence if necessary, we deduce that \( \nabla u_m \to \nabla u \) a. e. on \( E_j \).

This is true for each \( j \), so we conclude that

\[\nabla u_m \to \nabla u \quad \text{a. e. in } M.\]

The sequence \( (\nabla u_m) \) is uniformly integrable for any \( 1 \leq q < p \), as by Hölder’s inequality we have for any \( E \subset M \)

\[
\int_E |\nabla u_m|^q \leq \left( \int_M |\nabla u_m|^p \right)^{q/p} |E|^{1-q/p} \leq C |E|^{1-q/p}.
\]
From this and the fact that almost everywhere convergence implies convergence in measure in finite measure spaces, it follows from Vitali’s Convergence Theorem that $\nabla u_m \to \nabla u$ in $L^q(M)$ for each $1 \leq q < p$.

If $1 < p < 2$, we write, using Lemma 3.2,

$$\int_{E_j} \frac{|\nabla u_m - \nabla u|^2}{(|\nabla u_m| + |\nabla u|)^{2-p}} dv_g \leq C_p \int_{E_j} \left( |\nabla u_m|^{p-2} \nabla u_m - |\nabla u|^{p-2} \nabla u, \nabla u_m - \nabla u \right) dv_g$$

and the same argument proves that

$$\limsup_{m \to \infty} \int_{E_j} \frac{|\nabla u_m - \nabla u|^2}{(|\nabla u_m| + |\nabla u|)^{2-p}} dv_g = O(\sigma).$$

which allows us to conclude again that $\nabla u_m \to \nabla u$ a.e. on $E_j$. The rest of the argument is the same. \[\Box\]

Now, since $\nabla u_\alpha \to \nabla u$ a.e. and $(\nabla u_\alpha)$ is bounded in $L^p(M)$, Brézis-Lieb lemma ([22]) implies that

$$\int_M |\nabla u_\alpha|^p = \int_M |\nabla(u_\alpha - u)|^p + \int_M |\nabla u|^p + o(1). \quad (3.14)$$

Thus, for any $\varepsilon \leq \varepsilon_0$ we have, using Proposition 3.3,

$$\xi_\alpha \geq \int_M |\nabla u_\alpha|^p$$

$$\geq \int_M |\nabla(u_\alpha - u)|^p + o(1)$$

$$\geq \frac{1}{K^p - \varepsilon} \|u_\alpha - u\|_{L^p(\partial M)} + o(1)$$

$$= \frac{1}{K^p + \varepsilon} \|u_\alpha\|_{L^p(\partial M)} + o(1)$$

$$= \frac{1}{K^p + \varepsilon} + o(1).$$

Hence, sending $\alpha \to \infty$, we obtain

$$\frac{1}{K^p - \delta} \geq \frac{1}{K^p + \varepsilon},$$

which is a contradiction for all $\varepsilon$ sufficiently small. \[\Box\]

### 3.5 Proof of Theorem 3.2

In order to study the second best constant and its associated optimal inequalities, first we derive the following Poincaré-type inequality for compact manifolds with boundary. We set

$$\bar{u} = \frac{1}{\text{vol}_g(\partial M)} \int_{\partial M} u \, ds_g.$$
Lemma 3.4. (Poincaré’s Inequality) Let \((M,g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary. If \(p > 1\), then there exists a constant \(C > 0\) such that
\[
\|u - \overline{u}\|_{L^p(\partial M)} \leq C \|\nabla u\|_{L^p(M)}
\]
for all \(u \in H^{1,p}(M)\).

Proof. It suffices to prove the result when \(M\) is connected. Suppose by contradiction that there exists a sequence \((u_m) \subset H^{1,p}(M)\) such that
\[
\left( \int_{\partial M} |u_m - \overline{u}_m|^p \, ds_g \right)^{\frac{1}{p}} > m \left( \int_M |\nabla u_m|^p \, dv_g \right)^{\frac{1}{p}}.
\]
Define
\[
v_m = \frac{u_m - \overline{u}_m}{\|u_m - \overline{u}_m\|_{L^p(\partial M)}}.
\]
Then \((v_m)\) satisfies
\[
\overline{v}_m = 0, \quad \|v_m\|_{L^p(\partial M)} = 1 \quad \text{and} \quad \left( \int_M |\nabla v_m|^p \, dv_g \right)^{\frac{1}{p}} < \frac{1}{m}.
\]
It follows from Proposition 3.1 that \((v_m)\) is bounded in \(H^{1,p}(M)\), thus we may assume that \(v_m \rightharpoonup v\) in \(H^{1,p}(M)\) and \(v_m \rightarrow v\) in \(L^p(\partial M)\). In particular, \(\overline{v} = 0\) and \(\|v\|_{L^p(\partial M)} = 1\).

Moreover, since
\[
\int_M |\nabla v|^p \leq \liminf \int_M |\nabla v_m|^p = 0,
\]
it follows that \(\nabla v = 0\), hence \(v\) is constant. This, together with the fact that \(\overline{v} = 0\), implies that \(v = 0\) on \(\partial M\), contradicting \(\|v\|_{L^p(\partial M)} = 1\). \(\blacksquare\)

Using Lemma 3.4, we obtain a Sobolev-Poincaré-type inequality for compact manifolds with boundary:

Lemma 3.5. (Sobolev-Poincaré’s Inequality) Let \((M,g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary. If \(1 < p < n\), then there exists a constant \(C > 0\) such that
\[
\|u - \overline{u}\|_{L^p(\partial M)} \leq C \|\nabla u\|_{L^p(M)}
\]
for all \(u \in H^{1,p}(M)\).

Proof. By the Sobolev trace inequality \((T_{p,\mathrm{gen}}^1)\), there exists a constant \(C_1 > 0\) such that
\[
\|u - \overline{u}\|_{L^p(\partial M)} \leq C_1 \left( \|\nabla u\|_{L^p(M)} + \|u - \overline{u}\|_{L^p(\partial M)} \right)
\]
for all \(u \in H^{1,p}(M)\). On the other hand, by the preceding lemma, there exists a constant \(C_2 > 0\) such that
\[
\|u - \overline{u}\|_{L^p(\partial M)} \leq C_2 \|\nabla u\|_{L^p(M)}
\]
for all $u \in H^{1,p}(M)$. Putting these two together gives the desired inequality. ■

The first part of Theorem 3.2 follows from the following result:

**Proposition 3.5.** Let $(M,g)$ be a compact $n$-dimensional Riemannian manifold with boundary. If $1 < p < n$, then

$$
\overline{\beta}_p(M,g) = \text{vol}_g(\partial M)^{-\frac{p-1}{n-1}}
$$

and there exists a constant $A > 0$ such that

$$
\|u\|_{L^p(\partial M)} \leq A \|\nabla u\|_{L^p(M)} + \text{vol}_g(\partial M)^{-\frac{p-1}{n-1}} \|u\|_{L^p(M)}
$$

for all $u \in H^{1,p}(M)$.

**Proof.** Taking $u = 1$ in $(I^p_{p,\text{gen}})$, we obtain $\text{vol}_g(\partial M)^{\frac{1}{p}} \leq B \text{vol}_g(\partial M)^{\frac{1}{p}}$ so that $B \geq \text{vol}_g(\partial M)^{\frac{1}{p}} - \frac{1}{p}$. Thus, $\overline{\beta}_p(M) \geq \text{vol}_g(\partial M)^{-\frac{p-1}{n-1}}$. On the other hand, by the Sobolev-Poincaré inequality, there exists a constant $A > 0$ such that

$$
\|u\|_{L^p(\partial M)} \leq \|u - \pi\|_{L^p(\partial M)} + \|\pi\|_{L^p(\partial M)} \leq A \|\nabla u\|_{L^p(M)} + \text{vol}_g(\partial M)^{\frac{1}{p}} - 1 \int_{\partial M} |u| \ ds_g.
$$

for all $u \in H^{1,p}(M)$. Applying Hölder’s inequality to the last term of the above inequality, we get

$$
\|u\|_{L^p(\partial M)} \leq A \|\nabla u\|_{L^p(M)} + \text{vol}_g(\partial M)^{\frac{1}{p} - 1 + \frac{q-1}{q}} \|u\|_{L^p(M)}
$$

$$
= A \|\nabla u\|_{L^p(M)} + \text{vol}_g(\partial M)^{-\frac{p-1}{n-1}} \|u\|_{L^p(M)}.
$$

Following the arguments of Bakry [15], Druet and Hebey [51] for the second optimal Sobolev inequality for compact manifolds without boundary, we will now show that $(J^p_p)$ is valid if and only if $n = 2$ and $1 < p < 2$, or if $n \geq 3$ and $1 < p \leq 2$. Therefore, $(J^p_p)$ is not valid if $p > 2$.

**Lemma 3.6.** Let $(M,g)$ be a compact $n$-dimensional Riemannian manifold with boundary. For all $u \in L^q(\partial M)$ there holds

$$
\left( \int_{\partial M} |u|^q \ ds_g \right)^{\frac{2}{q}} \leq \text{vol}_g(\partial M)^{-\frac{2q-1}{q}} \left( \int_{\partial M} u \ ds_g \right)^2
$$

$$
+ (q-1) \left( \int_{\partial M} |u - \pi|^q \ ds_g \right)^{\frac{2}{q}}
$$

if $q \geq 2$, and

$$
\left( \int_{\partial M} |u|^q \ ds_g \right)^{\frac{2}{q}} \leq \text{vol}_g(\partial M)^{-\frac{2q-1}{q}} \left| \int_{\partial M} u \ ds_g \right|^p
$$

$$
+ (1 + q(q-1)^{q-1})^{\frac{2}{q}} \left( \int_{\partial M} |u - \pi|^q \ ds_g \right)^{\frac{2}{q}}
$$

if $p \leq q \leq 2$. 
Theorem 3.5. Let \((M,g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary.

If \(n = 2\) and \(1 < p < 2\), or if \(n \geq 3\) and \(1 < p \leq 2\) then \((\mathcal{T}_p)\) is valid.

Proof. The proof is divided in two cases: \(p \geq \frac{2n}{n+1}\) and \(1 < p < \frac{2n}{n+1}\). The case \(p \geq \frac{2n}{n+1}\) will follow from the first inequality of the previous lemma with \(q = \frac{p}{n}\), since in this case we have \(p^* \geq 2\). Since \(p \leq 2\), we obtain

\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{\frac{p}{p^*}} \leq \left( \frac{\text{vol}_g(\partial M)}{\text{vol}_g(M)} \right)^{2 - \frac{n}{p-1}} \left( \int_{\partial M} |u|^p \, ds_g \right)^2 + \left( \int_{\partial M} |u - \varpi|^{p^*} \, ds_g \right)^{\frac{p}{p^*}} \]

\[
\leq \text{vol}_g(\partial M)^{-p} \left( \int_{\partial M} u \, ds_g \right)^p + (p^* - 1)^{\frac{p}{2}} \left( \int_{\partial M} |u - \varpi|^{p^*} \, ds_g \right)^{\frac{p}{p^*}}
\]

for all \(u \in H^{1,p}(M)\). Then, by the inequalities of Hölder and Sobolev-Poincaré, there exists a constant \(A > 0\) independent of \(u\) such that

\[
\left( \int_{\partial M} |u|^{p^*} \, ds_g \right)^{\frac{p}{p^*}} \leq \text{vol}_g(\partial M)^{-p} \left( \int_{\partial M} |u|^p \, ds_g \right)^{\frac{p}{p^*}} + (p^* - 1)^{\frac{p}{2}} A \int_M |\nabla u|^p \, dv_g
\]

If \(1 < p < \frac{2n}{n+2}\), we use the second inequality with \(q = \frac{p}{n}\) and again the Hölder’s and Sobolev-Poincaré’s inequalities in order to obtain

\[
\left( \int_{\partial M} |u|^{p^*} \, dv_g \right)^{\frac{p}{p^*}} \leq \text{vol}_g(\partial M)^{-p} \left( \int_{\partial M} u \, ds_g \right)^p
\]

\[
+ \left( 1 + \frac{p^*}{p} (p^* - 1)^{p^* - 1} \right) \left( \int_{\partial M} |u - \varpi|^{p^*} \, ds_g \right)^{\frac{p}{p^*}} \leq \text{vol}_g(\partial M)^{-p} \int_{\partial M} |u|^p \, ds_g
\]

\[
+ \left( 1 + \frac{p^*}{p} (p^* - 1)^{p^* - 1} \right)^{\frac{p}{p^*}} \int_M |\nabla u|^p \, dv_g
\]

for all \(u \in H^{1,p}(M)\). ■

Theorem 3.6. Let \((M,g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary.

If \(n \geq 3\) and \(2 < p < n\), then \((\mathcal{T}_p)\) is not valid.

Proof. Fix a function \(v \in C^\infty(M)\) nonconstant in \(\partial M\). Define \(\varphi, \psi : [0, \infty) \to [0, \infty)\) as

\[
\varphi(t) = \int_{\partial M} |1 + tv|^p \, ds_g
\]

and

\[
\psi(t) = \left( \int_{\partial M} |1 + tv|^{p^*} \, ds_g \right)^{\frac{p}{p^*}}.
\]
Expanding these functions until second order terms, we find

$$\int_{\partial M} |1 + tv|^p \, ds_g = \text{vol}_{\gamma}(\partial M) + p \left( \int_{\partial M} v \, ds_g \right) t + \frac{p(p-1)}{2} \left( \int_{\partial M} v^2 \, ds_g \right) t^2 + o(t^2)$$

and

$$\left( \int_{\partial M} |1 + tv|^{p^*} \, ds_g \right)^{\frac{p^*}{p}} = \text{vol}_{\gamma}(\partial M)^{\frac{p^*}{p^*}} \left( \int_{\partial M} v \, ds_g \right) t + \frac{p(p - p^*)}{2} \text{vol}_{\gamma}(\partial M)^{\frac{p^*}{p^*} - 2} \left( \int_{\partial M} v \, ds_g \right)^2 t^2 + \frac{p(p^* - 1)}{2} \text{vol}_{\gamma}(\partial M)^{\frac{p^*}{p^*} - 1} \left( \int_{\partial M} v^2 \, ds_g \right)^2 t^2 + o(t^2).$$

Moreover,

$$\int_M |\nabla (1 + tv)|^p \, dv_g = t^p \int_M |\nabla v|^p \, dv_g = o(t^2)$$

since $p > 2$. Assume by contradiction that $(\mathcal{P}_p)$ is valid. Then it follows that

$$\text{vol}_{\gamma}(\partial M)^{\frac{p^*}{p^*}} + p \text{vol}_{\gamma}(\partial M)^{\frac{p^*}{p^*} - 1} \left( \int_{\partial M} v \, ds_g \right) t + \frac{p(p - p^*)}{2} \text{vol}_{\gamma}(\partial M)^{\frac{p^*}{p^*} - 2} \left( \int_{\partial M} v \, ds_g \right)^2 t^2 + \frac{p(p^* - 1)}{2} \text{vol}_{\gamma}(\partial M)^{\frac{p^*}{p^*} - 1} \left( \int_{\partial M} v^2 \, ds_g \right)^2 t^2$$

$$\leq \text{vol}_{\gamma}(\partial M)^{1 - \frac{p^*}{n+1}} + p \text{vol}_{\gamma}(\partial M)^{- \frac{n-1}{n+1}} \left( \int_{\partial M} v \, ds_g \right) t + \frac{p(p - 1)}{2} \text{vol}_{\gamma}(\partial M)^{- \frac{n-1}{n+1}} \left( \int_{\partial M} v^2 \, ds_g \right)^2 t^2 + o(t^2).$$

Using $\frac{p^*}{p} = 1 - \frac{p-1}{n-1}$, this inequality simplifies to

$$(p^* - 1) \left( \int_{\partial M} v^2 \, ds_g \right) \leq (p^* - p) \frac{1}{\text{vol}_{\gamma}(\partial M)} \left( \int_{\partial M} v \, ds_g \right)^2 + (p - 1) \left( \int_{\partial M} v^2 \, ds_g \right),$$

whence

$$\text{vol}_{\gamma}(\partial M) \left( \int_{\partial M} v^2 \, dv_g \right) \leq \left( \int_{\partial M} v \, dv_g \right)^2.$$}

This inequality implies that the Cauchy-Schwarz inequality

$$\int_{\partial M} v \, dv_g \leq \|1\|_{L^2(\partial M)} \|v\|_{L^2(\partial M)} = \text{vol}_{\gamma}(\partial M)^{1/2} \left( \int_{\partial M} v^2 \, dv_g \right)^{1/2},$$

is an equality. Therefore, $v$ must be constant on $\partial M$, contradicting the choice of $v$. ■
Chapter 4

Best Constants in Second Order Sobolev Inequalities on Riemannian Manifolds and Applications

4.1 Introduction

Let \((M, g)\) be a smooth compact Riemannian manifold, with or without boundary, of dimension \(n \geq 3\). For \(1 < p < n/2\), we denote by \(H^1_p(M)\) the standard first order Sobolev space defined as the completion of \(C^\infty_0(M)\) with respect to the norm
\[
\|u\|_{H^1_p(M)} = \left( \int_M |\nabla_g u|^p \, dv_g + \int_M |u|^p \, dv_g \right)^{1/p},
\]
and by \(H^2_p(M)\) and \(H^{2,p}(M)\) the standard second order Sobolev spaces defined as the completion, respectively, of \(C^\infty_0(M)\) and \(C^\infty(M)\) with respect to the norm
\[
\|u\|_{H^2_p(M)} = \left( \int_M |\nabla_g u|^p \, dv_g + \int_M |\nabla_g^2 u|^p \, dv_g + \int_M |u|^p \, dv_g \right)^{1/p}.
\]

In this chapter we consider the following Sobolev spaces:

\(E_1 = H^2_p(M)\),

if \(M\) has no boundary, and

\(E_2 = H^2_p(M)\),

\(E_3 = H^{2,p}(M) \cap H^1_p(M)\),

if \(M\) has boundary. Denoting by \(\Delta_g u = \text{div}_g(\nabla u)\) the Laplacian with respect to the metric \(g\), a norm on \(E_i\) equivalent to \(\|\cdot\|_{H^2_p(M)}\) is:
\[
\|u\|_{E_i} = \left( \int_M |\Delta_g u|^p \, dv_g + \int_M |u|^p \, dv_g \right)^{1/p},
\]
as is shown in the first appendix to this chapter (Propositions A.1 and A.2). If \(p_2^* = np/(n-2p)\), the Sobolev embedding theorem ensures that the inclusion \(E_i \subset L^{q}(M)\) is compact for \(1 < q < p_2^*\) and continuous for \(q = p_2^*\). Thus, there exist constants \(A, B \in \mathbb{R}\) such that
\[
\|u\|_{L^{p_2^*(M)}}^p \leq A \|\Delta_g u\|_{L^p(M)}^p + B \|u\|_{L^p(M)}^p \tag{4.1}
\]
for all \( u \in E_i \). Consider, for each \( i \), the best constant associated to this inequality:

\[
\alpha^i_p(M) = \inf \{ A \in \mathbb{R} : \text{there exists } B \in \mathbb{R} \text{ such that inequality (4.1) holds} \}
\]

As in the first order case, two natural questions are the dependence or not of the best constant on the geometry of the manifold \( M \), and the validity or not of the associated optimal inequality:

\[
\|u\|_{L^p(M)}^p \leq \alpha^i_p(M) \|\Delta_g u\|_{L^p(M)}^p + B \|u\|_{L^p(M)}^p
\]  

for all \( u \in E_i \).

The first result of this chapter (Theorem 4.1, below) shows that \( \alpha^i_p(M) \) is independent of the metric for \( 1 < p < n/2 \) on any compact Riemannian manifold, with or without boundary, of dimension \( n \geq 3 \). In order to state the results precisely, let us fix some notations. Let \( D^{2,p}(\mathbb{R}^n) \) be the completion of \( C_0^\infty(\mathbb{R}^n) \) under the norm

\[
\|u\|_{D^{2,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\Delta u|^p \, dx \right)^{1/p}.
\]

This space is characterized as the set of functions in \( L^{p^*}(\mathbb{R}^n) \) whose second order partial derivatives in the distributional sense are in \( L^p(\mathbb{R}^n) \). The inclusion \( D^{2,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n) \) is continuous by the Sobolev embedding theorem. Denote by \( K_2 = K_2(n, p) \) the best constant of this embedding, that is,

\[
\frac{1}{K_2(n, p)} = \inf_{u \in D^{2,p}(\mathbb{R}^n) \setminus \{0\}} \frac{\|\Delta u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}}.
\]

Since Lions [69], it is known that the infimum is achieved and that minimizers are positive, radially symmetric decreasing functions, up to translation and multiplication by a nonzero constant. For \( p = 2 \), it was shown by Edmunds, Fortunato and Janelli [43] and Lieb [67] that

\[
K_2(n, 2) = \frac{16}{n(n - 4)(n^2 - 4)\omega_n^{4/n}}
\]

where \( \omega_n \) denotes the volume of the unit \( n \)-dimensional sphere \( S^n \subset \mathbb{R}^{n+1} \), and that the set of extremal functions is precisely

\[
z(x) = c_1 \left( \frac{1}{c_2 + |x - x_0|^2} \right)^{n-4}
\]

where \( c_1 > 0, c_2 \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^n \). Although the explicit value of \( K_2(n, p) \) and the exact shape of minimizers are not known for \( p \neq 2 \), the asymptotic behaviors of the extremal functions and their Laplacians were determined by Hulshof and van der Vorst [57] for any \( 1 < p < n/2 \) (see Appendix B of this chapter).

The first result of this chapter is the following:
Theorem 4.1. [20] Let \((M, g)\) be a smooth compact Riemannian manifold, with or without boundary, of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\). Then \(\alpha_p^p(M) = K_p^p\). In particular, given \(\varepsilon > 0\), there exists a real constant \(B = B(M, g, \varepsilon)\) such that

\[
\|u\|_{L^p(M)}^p \leq (K_p^p + \varepsilon) \|
\Delta_g u\|_{L^p(M)}^p + B \|u\|_{L^p(M)}^p
\]  

(4.6)

for all \(u \in E_i\).

The proof of this theorem in the case \(p = 2\) by Djadli, Hebey and Ledoux, was based on a partition of unity argument involving harmonic charts and on the Bochner-Lichnerowicz-Weitzenböck integral formula (see [35]). This integral identity is no longer available in the case \(p \neq 2\). In its place, we use Calderon-Zygmund inequalities from the theory of singular integrals and \(L^p\) theory of elliptic operators, which demand only standard charts. The case \(E_3 = H^{2,p}(M) \cap H^{1,p}_0(M)\) requires an additional result about the sharp Sobolev inequality on bounded Euclidean domains (see Lemma 4.1).

Concerning the validity of the optimal inequality, contrary to what happens in the first order case, one cannot hope (4.2) to hold for \(p = 2\), as was shown in [35] for standard spheres of dimension \(n \geq 6\). We prove the nonvalidity of (4.2) for \(p = 2\) and compact Riemannian manifolds, with or without boundary, which have positive scalar curvature somewhere. More precisely, we have the following:

Theorem 4.2. [20] Let \((M, g)\) be a smooth compact \(n\)-dimensional Riemannian manifold, with or without boundary, with positive scalar curvature somewhere. Then, the optimal inequality (4.2) is not valid if \(n \geq 6\) and \(p = 2\).

The proof of this theorem, in the same spirit of Druet in the first order case [38], is done by constructing a family of local test functions for which the appropriate Sobolev quotient is unbounded. Such test functions have compact support centered on a point of positive scalar curvature, and locally are scalings of a minimizer for (4.3); the expansion of the element of volume up to second order is used.

In the remainder of this chapter, we apply the asymptotically sharp inequality (4.6) to the study of fourth order elliptic problems with critical growth on compact Riemannian manifolds, with and without boundary. Specifically, given \(a, b, f \in C^0(M)\), if \(M\) has no boundary, we seek solutions to the equation

\[
\Delta_g \left( |\Delta_g u|^{p-2} \Delta_g u \right) - \text{div}_g \left( a(x) |\nabla_g u|^{p-2} \nabla_g u \right) + b(x) |u|^{p-2} u = f(x) |u|^{p^*_2 - 2} u \text{ in } M, \quad (P_1)
\]
and if \( M \) has boundary, solutions to the Dirichlet problem
\[
\begin{cases}
\Delta_g \left( |\Delta_g u|^{p-2} \Delta_g u \right) - \text{div}_g \left( a(x) |\nabla_g u|^{p-2} \nabla_g u \right) + b(x) |u|^{p-2} u = f(x) |u|^{p^*_2 - 2} u & \text{in } M, \\
u = \nabla_g u = 0 & \text{on } \partial M,
\end{cases}
\]
and to the Navier problem
\[
\begin{cases}
\Delta_g \left( |\Delta_g u|^{p-2} \Delta_g u \right) - \text{div}_g \left( a(x) |\nabla_g u|^{p-2} \nabla_g u \right) + b(x) |u|^{p-2} u = f(x) |u|^{p^*_2 - 2} u & \text{in } M, \\
u = \Delta_g u = 0 & \text{on } \partial M.
\end{cases}
\]

Nontrivial weak solutions of \((P_i)\) correspond, modulo nonzero constant multiples, to critical points of the functional
\[
J(u) = \int_M |\Delta_g u|^p \, dv_g + \int_M a(x) |\nabla_g u|^p \, dv_g + \int_M b(x) |u|^p \, dv_g
\]
on the manifold
\[
V_i = \left\{ u \in E_i : \int_M f(x) |u|^{p^*_2} \, dv_g = 1 \right\}.
\]
The functional \( J \) is said to be coercive on \( E_i \) if there exists some positive constant \( C \), dependent only on \( a \) and \( b \), such that
\[
J(u) \geq C \|u\|_{E_i}^p
\]
for all \( u \in E_i \). This happens, for instance, if \( a \geq 0 \), \( b > 0 \) and \( M \) has no boundary, or if \( a \geq 0 \), \( b \geq 0 \) and \( M \) has boundary. We say that \((H_i)\) holds if
\[
\inf_{V_i} \frac{1}{K_n^2 \left( \max_M f \right)^{p^*_2}} > 0,
\]
\( (H_i) \)

Under these conditions, we have the following results:

**Theorem 4.3A.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold without boundary of dimension \( n \geq 3 \) and \( 1 < p < \frac{n}{2} \). Assume that \( a, b, f \in C^0(M) \) are such that the functional \( J \) is coercive on \( E_1 \) and \((H_i)\) holds. Then \((P_1)\) possesses a nontrivial weak solution \( u \). Moreover, if \( p = 2 \) and \( a \in C^{1,\gamma}(M), b, f \in C^{\gamma}(M) \), then \( u \in C^{4,\gamma}(M) \); if, in addition, \( f \geq 0 \) and \( a > 0 \) is a constant such that \( b(x) \leq \frac{a^2}{4} \), then \((P_1)\) admits a positive solution.
**Theorem 4.3B.** [20] Let \((M,g)\) be a smooth compact Riemannian manifold with boundary of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\). Assume that \(a, b, f \in C^0(M)\) are such that the functional \(J\) is coercive on \(E_2\) and \((H_2)\) holds. Then \((P_2)\) possesses a nontrivial weak solution \(u\). Moreover, if \(p = 2\) and \(a \in C^{1,\gamma}(M), b, f \in C^{\gamma}(M),\) then \(u \in C^{4,\gamma}(M)\).

**Theorem 4.3C.** [20] Let \((M,g)\) be a smooth compact Riemannian manifold with boundary of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\). Assume that \(a, b, f \in C^0(M)\) are such that the functional \(J\) is coercive on \(E_3\) and \((H_3)\) holds. Then \((P_3)\) possesses a nontrivial weak solution \(u\). Moreover, if \(b, f \in C^{\gamma}(M)\) and either \(a \equiv 0\) or \(p = 2\) and \(a\) is a nonnegative constant, then \(u \in C^{4,\gamma}(M)\); if, in addition, \(f \geq 0\) and \(b(x) \leq \frac{a^2}{4}\), then \((P_3)\) admits a positive solution.

Since \(V_i\) is not weakly closed in the \(E_i\) topology, the direct variational method does not apply. One also encounters difficulties in establishing the regularity of weak solutions, since the Moser iterative scheme fails. The existence part of these theorems is proved through a minimization argument involving Ekeland’s variational principle together with a version of the concentration-compactness principle which is a consequence of (4.6), an idea derived from the similar argument utilized in Chapter 3. The regularity part is obtained through a construction inspired on the work done by van der Vorst [86] in connection with the biharmonic operator.

An immediate application of the preceding theorems, noticing that \(u \equiv (\int_M f \, dv_g)^{-\frac{p}{p-2}} \in V_i\), is the following corollary:

**Corollary 4.1.** [20] Let \((M,g)\) be a smooth compact Riemannian manifold, with or without boundary, of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\). Assume that \(a, b \in C^0(M)\) are such that the functional \(J\) is coercive on \(E_i\) and \(f \in C^0(M)\) is such that \(\int_M f \, dv_g > 0\). If

\[
\left( \frac{\max_M f}{\int_M f \, dv_g} \right)^{\frac{p}{p-2}} \int_M b \, dv_g < \frac{1}{K_2^p},
\]

then \((P_i)\) possesses a nontrivial weak solution.

As another application of Theorems 4.3A-C, we obtain a result which relates the geometry of the manifold at a point of maximum of \(f\) and the behavior of \(f\) up to the second order at this point. The proof involves estimates on the growth of the standard bubbles localized at a maximum point of \(f\), which are obtained from the asymptotic behavior of the minimizers of
(4.3) Fix a positive radially symmetric minimizer \( z = z(r) \) for (4.3). Denote
\[
I_1 = I_1(n, p) = \int_{\mathbb{R}^n} z^{p^*_2} \, dx,
I_2 = I_2(n, p) = \int_{\mathbb{R}^n} z^{p^*_2} r^2 \, dx,
I_3 = I_3(n, p) = \int_{\mathbb{R}^n} |\Delta z|^p \, dx,
I_4^1 = I_4^1(n, p) = \int_{\mathbb{R}^n} |\Delta z|^p r^2 \, dx,
I_4^2 = I_4^2(n, p) = \int_{\mathbb{R}^n} |\Delta z|^{p-1} |z'(r)| r \, dx
\]
whenever the right-hand side makes sense, and set \( I_4 = I_4^1 + 2pI_4^2 \). We have:

**Corollary 4.2.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold, with or without boundary, of dimension \( n \geq 5 \) and \( \frac{n + 2}{n} < p < \frac{n + 2}{4} \). Let \( a \equiv 0 \) and \( b \in C^0(M) \) be such that the functional \( J \) is coercive on \( E_i \). Furthermore, assume that \( f \in C^2(M) \), \( \max_M f > 0 \) and \( f \) has a point of maximum \( x_0 \) outside the boundary. If
\[
\frac{\Delta_g f(x_0)}{f(x_0)} > \frac{1}{3} \left( 1 - \frac{p^*_2 I_1 I_4}{p I_2 I_3} \right) \text{Scal}_g(x_0),
\]
then \((P_i)\) possesses a nontrivial weak solution.

We remark that the quotient \( \frac{I_1 I_4}{I_2 I_3} \) in (4.8) does not depend on the choice of \( z \).

The methods used above are then applied to the study of the fourth order Brezis-Nirenberg problem on compact Riemannian manifolds. Indeed, consider the following one-parameter problems:
\[
\Delta_g \left( |\Delta_g u|^{p-2} \Delta_g u \right) = |u|^{p^*_2-2} u + \lambda |u|^{p-2} u \text{ in } M,
\]
if \( M \) has no boundary,
\[
\begin{cases}
\Delta_g \left( |\Delta_g u|^{p-2} \Delta_g u \right) = |u|^{p^*_2-2} u + \lambda |u|^{p-2} u & \text{in } M, \\
u = \nabla_g u = 0 & \text{on } \partial M,
\end{cases}
\]
and
\[
\begin{cases}
\Delta_g \left( |\Delta_g u|^{p-2} \Delta_g u \right) = |u|^{p^*_2-2} u + \lambda |u|^{p-2} u & \text{in } M, \\
u = \Delta_g u = 0 & \text{on } \partial M,
\end{cases}
\]
if \( M \) has boundary. Denote by \( \lambda_1 \) the first eigenvalue associated to the equation
\[
\Delta_g \left( |\Delta_g u|^{p-2} \Delta_g u \right) = \lambda |u|^{p-2} u \text{ in } M,
\]
on \( E_i \). The variational characterization of \( \lambda_1 \) is given by
\[
\lambda_1 = \inf_{u \in E_i \setminus \{0\}} \frac{\int_M |\Delta_g u|^p \, dv_g}{\int_M |u|^p \, dv_g}.
\]
Clearly, $\lambda_1 = 0$ on $E_1$ and $\lambda_1 > 0$ on $E_2$ and on $E_3$.

As in [24], we investigate the range of values of $\lambda$ for which (BN$_1$), (BN$_2$) and (BN$_3$) admit nontrivial solutions and obtain the following results:

**Theorem 4.4A.** [20] Let $(M, g)$ be a smooth compact Riemannian manifold without boundary of dimension $n \geq 6$. If $p = 2$ and $M$ has positive scalar curvature somewhere, then (BN$_1$) has a nontrivial solution in $C^{4,\gamma}(M)$ for any $\lambda < \lambda_1$. If $\lambda \geq \lambda_1$, then (BN$_1$) has no positive solution.

**Theorem 4.4B.** [20] Let $(M, g)$ be a smooth compact Riemannian manifold with boundary of dimension $n \geq 6$. Then:

(i) If $p = 2$ and $M$ has positive scalar curvature somewhere, (BN$_2$) has a nontrivial solution $C^{4,\gamma}(M)$ for any $\lambda < \lambda_1$.

(ii) If $\frac{n}{n-2} < p \leq \sqrt{\frac{n}{2}}$ and $M$ is flat in a neighborhood, (BN$_2$) has a nontrivial solution in $C^{4,\gamma}(M)$ for any $0 < \lambda < \lambda_1$.

**Theorem 4.4C.** [20] Let $(M, g)$ be a smooth compact Riemannian manifold with boundary of dimension $n \geq 6$. Then:

(i) If $p = 2$ and $M$ has positive scalar curvature somewhere, (BN$_3$) has a positive solution for any $0 \leq \lambda < \lambda_1$ and a nontrivial solution for any $\lambda < 0$ in $C^{4,\gamma}(M)$. If $\lambda \geq \lambda_1$, then (BN$_3$) has no positive solution.

(ii) If $\frac{n}{n-2} < p \leq \sqrt{\frac{n}{2}}$ and $M$ is flat in a neighborhood, (BN$_3$) has a positive solution in $C^{4,\gamma}(M)$ for any $0 < \lambda < \lambda_1$. If $\lambda \geq \lambda_1$, then (BN$_3$) has no positive solution.

The arguments utilized in the proof of these results are again based on the minimization technique and estimates of the growth of standard bubbles. In the case $p = 2$, the more precise estimates are used.

The structure of this chapter is as follows. In Section 4.2 we prove the asymptotically sharp Sobolev inequality. In Section 4.3, we prove that this is the best we can have for $p = 2$ for manifolds with positive scalar curvature somewhere. In Section 4.4 we prove Theorems 4.3A-C and Corollary 4.2, and in Section 4.5 we consider the fourth-order Brezis-Nirenberg problem, proving Theorems 4.4A-C.

### 4.2 The asymptotically sharp Sobolev inequality

The proof of Theorem 4.1 will follow from Propositions 4.1 and 4.4 below.
Proposition 4.1. [20] Let \((M, g)\) be a compact Riemannian manifold, with or without boundary, of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\). Let \(A, B \in \mathbb{R}\) be such that
\[
\|u\|_{L^p(M)}^p \leq A \|\Delta_g u\|_{L^p(M)}^p + B \|u\|_{L^p(M)}^p
\]
for all \(u \in E_i\). Then \(A \geq K^p_2\).

**Proof.** We proceed by contradiction. Assume that there exist \(A < K^p_2\) and \(B \in \mathbb{R}\) such that the above inequality is true for all \(u \in E_i\). Fix \(x_0 \in M \setminus \partial M\) and a geodesic ball \(B_\delta(x_0)\), where \(\delta > 0\) will be chosen later. Given \(\varepsilon_0 > 0\), there exists \(\delta_0 > 0\) such that for all \(0 < \delta < \delta_0\) there exists a normal coordinates system defined on \(B_\delta(x_0)\), satisfying
\[
\left|g^{ij} - \delta_{ij}\right| \leq \varepsilon_0,
\]
\[
\left|\Gamma^k_{ij}\right| \leq \varepsilon_0
\]
and
\[
(1 - \varepsilon_0)dx \leq dv_g \leq (1 + \varepsilon_0)dx.
\]
In the sequel, we will denote by \(\varepsilon_j\) several possibly different positive constants, independent of \(\delta\).

Denoting by \(B_\delta\) the Euclidean ball of center 0 and radius \(\delta\), it follows that for any \(u \in C_0^\infty(B_\delta)\) we have
\[
\left(\int_{B_\delta} |u|^{p_2} dx\right)^{p/p_2} \leq \left(\frac{1}{1 - \varepsilon_0} \int_M |u|^{p_2} dv_g\right)^{p/p_2} \leq (1 + \varepsilon_1)A \int_M |\Delta_g u|^p dv_g + (1 + \varepsilon_1)B \int_M |u|^p dv_g \quad \text{(4.9)}
\]
\[
\leq (1 + \varepsilon_2)A \int_{B_\delta} |\Delta_g u|^p dx + (1 + \varepsilon_2)B \int_{B_\delta} |u|^p dx,
\]
for some positive numbers \(\varepsilon_1, \varepsilon_2 = O(\varepsilon_0)\). Writing
\[
\Delta_g u = \Delta u + \sum_{i,j=1}^n (g^{ij} - \delta_{ij})\partial_{ij} u + \sum_{i,j,k=1}^n g^{ij} \Gamma^k_{ij} \partial_k u, \quad \text{(4.10)}
\]
and using the elementary inequality \((a + b)^p \leq (1 + \varepsilon_3)a^p + C_{\varepsilon_3}b^p\), where \(\varepsilon_3\) will be chosen later, we find
\[
\int_{B_\delta} |\Delta_g u|^p dx \leq (1 + \varepsilon_3)\int_{B_\delta} |\Delta u|^p dx + \varepsilon_3^p C_{\varepsilon_3} \int_{B_\delta} |\partial^2 u|^p dx + \varepsilon_3^p C_{\varepsilon_3} \int_{B_\delta} |\partial u|^p dx. \quad \text{(4.11)}
\]
By the Calderon-Zygmund inequality (see [47]), there exists a positive constant \(C_{n,p}\), dependent only on \(n\) and \(p\), such that
\[
\int_{B_\delta} |\partial^2 u|^p dx \leq C_{n,p} \int_{B_\delta} |\Delta u|^p dx, \quad \text{(4.12)}
\]
while interpolation of lower-order derivatives yields

\[
\int_{B_{\delta}} |\partial u|^p \, dx \leq \varepsilon_4 \int_{B_{\delta}} |\partial^2 u|^p \, dx + C_{\varepsilon_4, \delta} \int_{B_{\delta}} |u|^p \, dx
\]

(4.13)

for any \( \varepsilon_4 > 0 \), with \( C_{\varepsilon_4, \delta} = O(\delta^{-p}) \). Therefore, putting together (4.9), (4.11), (4.12) and (4.13), and choosing \( \varepsilon_0, \varepsilon_3 \) and \( \varepsilon_4 \) sufficiently small, we find \( \delta > 0 \) such that for all \( u \in C_0^\infty(B_{\delta}) \) there holds

\[
\left( \int_{B_{\delta}} |u|^p \, dx \right)^{p/p_2} \leq A_1 \int_{B_{\delta}} |\Delta u|^p \, dx + B_1 \int_{B_{\delta}} |u|^p \, dx
\]

(4.14)

for some real numbers \( A_1 < K_2^p \) and \( B_1 \in \mathbb{R} \). On the other hand, by Hölder’s inequality,

\[
\int_{B_{\delta}} |u|^p \, dx \leq |B_{\delta}|^{2p/n} \left( \int_{B_{\delta}} |u|^{p_2} \, dx \right)^{p/p_2},
\]

where \( |B_{\delta}| \) stands for the Euclidean volume of \( B_{\delta} \). Thus, choosing \( \delta \) small enough so that

\[
|B_{\delta}|^{2p/n} B_1 = O(\delta^{p}) < 1
\]

and

\[
\frac{A_1}{1 - |B_{\delta}|^{2p/n} B_1} < K_2^p,
\]

it follows that there exists \( A_2 < K_2^p \) such that for all \( u \in C_0^\infty(B_{\delta}) \) there holds

\[
\left( \int_{B_{\delta}} |u|^{p_2} \, dx \right)^{p/p_2} \leq A_2 \int_{B_{\delta}} |\Delta u|^p \, dx.
\]

(4.15)

Now, given \( u \in C_0^\infty(\mathbb{R}^n) \) and \( \varepsilon > 0 \), define \( u_\varepsilon(x) = \varepsilon^{-n/p_2} u(\frac{x}{\varepsilon}) \). For \( \varepsilon \) small enough, we have \( u_\varepsilon \in C_0^\infty(B_{\delta}) \), and so

\[
\left( \int_{\mathbb{R}^n} |u_\varepsilon|^{p_2} \, dx \right)^{p/p_2} \leq A_2 \int_{\mathbb{R}^n} |\Delta u_\varepsilon|^p \, dx.
\]

Since this is precisely the rescaling such that

\[
\|u_\varepsilon\|_{L^{p_2}(\mathbb{R}^n)} = \|u\|_{L^{p_2}(\mathbb{R}^n)}
\]

and

\[
\|\Delta u_\varepsilon\|_{L^p(\mathbb{R}^n)} = \|\Delta u\|_{L^p(\mathbb{R}^n)},
\]

we conclude that

\[
\left( \int_{\mathbb{R}^n} |u|^{p_2} \, dx \right)^{p/p_2} \leq A_2 \int_{\mathbb{R}^n} |\Delta u|^p \, dx
\]

for all \( u \in C_0^\infty(\mathbb{R}^n) \), contradicting the definition of \( K_2 \). ■

The proof of Proposition 4.2 in the case \( E_3 = H^{2,p}(M) \cap H_0^{1,p}(M) \) requires the following lemma on the Euclidean sharp second order Sobolev inequality. For \( p = 2 \), this result was obtained by van der Vorst [86] using the concentration-compactness principle, Talenti’s comparison principle and a Pohozaev type identity. Our proof simplifies his argument for any \( 1 < p < \frac{n}{2} \).
Lemma 4.1. [20] Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with smooth boundary, $n \geq 3$ and $1 < p < \frac{n}{2}$. Then

$$||u||_{L^{p^*}(\Omega)} \leq K_2 ||\Delta u||_{L^p(\Omega)}$$  \hspace{1cm} (4.16)

for every $u \in H^{2,p}(\Omega) \cap H^{1,p}_0(\Omega)$. Moreover, $K_2$ is the best constant in this inequality.

Proof. Denote by $K_2(\Omega)$ the best constant in the embedding of $H^{2,p}(\Omega) \cap H^{1,p}_0(\Omega)$ into $L^{p^*}(\Omega)$, i.e.,

$$\frac{1}{K_2(\Omega)} = \inf_{u \in H^{2,p}(\Omega) \cap H^{1,p}_0(\Omega)} \frac{||\Delta u||_{L^p(\Omega)}}{||u||_{L^{p^*}(\Omega)}}.$$  

Proposition 4.2 implies that $K_2(\Omega) \geq K_2$. Assume by contradiction that $K_2(\Omega) > K_2$. Since the set $\{u \in C^2(\overline{\Omega}) : u = 0$ on $\partial\Omega\}$ is dense in $H^{2,p}(\Omega) \cap H^{1,p}_0(\Omega)$, it follows that there exists $u \in C^2(\overline{\Omega})$ in this set such that

$$\frac{||\Delta u||_{L^p(\Omega)}}{||u||_{L^{p^*}(\Omega)}} < \frac{1}{K_2}.$$

Set

$$f(x) = \begin{cases} -||\Delta u|| & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

and define

$$w = G * f,$$

where $*$ denotes the convolution operator and $G(x) = -\frac{1}{|x|^{n-2}}$ is the Green function of the Laplacian operator in $\mathbb{R}^n$. By the Hardy-Littlewood-Sobolev inequality (see [67] or [68]) there exists a constant $C(n,p)$ such that

$$\left| \int_{\mathbb{R}^n} w(y) h(y) \, dy \right| \leq C(n,p) ||f||_{L^p(\mathbb{R}^n)} ||h||_{L^r(\mathbb{R}^n)}$$

for every $h \in L^r(\mathbb{R}^n)$, where $r$ is defined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{p^*} = 1$. Hence, by Riesz representation theorem it follows that $w \in L^{p^*}(\mathbb{R}^n)$. Since $f \in L^\infty(\mathbb{R}^n)$ and has compact support, we conclude from Calderon-Zygmund estimates for singular integrals (see [47]), that $w \in D^{2,p}(\mathbb{R}^n) \cap C^{1,\gamma}(\mathbb{R}^n)$ and verifies

$$\Delta w = f \text{ in } \mathbb{R}^n.$$  

Moreover, since $G$ is a strictly negative function, we have $w > 0$ in $\mathbb{R}^n$. As

$$\Delta(w \pm u) \leq 0 \text{ in } \Omega,$$

$$w \pm u > 0 \text{ on } \partial\Omega,$$

the maximum principle provides us $w > ||u||$ in $\Omega$. Therefore,

$$||\Delta w||_{L^p(\mathbb{R}^n)} = ||\Delta u||_{L^p(\Omega)}.$$
and
\[ \|w\|_{L^p_2(\mathbb{R}^n)} > \|w\|_{L^{p^*}_2(\Omega)} \]
whence
\[ \frac{\|\Delta w\|_{L^p(\mathbb{R}^n)}}{\|w\|_{L^p_2(\mathbb{R}^n)}} < \frac{1}{K_2}, \]
a contradiction. ■

**Remarks.**

1. Since \( C_0^\infty(\Omega) \) is dense in \( H_0^{2,p}(\Omega) \) and zero extensions of functions in \( H_0^{2,p}(\Omega) \) belong to \( D^{2,p}(\mathbb{R}^n) \), one concludes directly from a scaling argument that Lemma 4.1 also holds for \( H_0^{2,p}(\Omega) \) in place of \( H_0^{2,p}(\Omega) \cap H_0^{1,p}(\Omega) \).

2. Using the Talenti comparison principle [83], a Pohozaev type identity for elliptic systems [74] and the regularity results of subsection 4.5.3, in the same spirit of [86] one proves that the best constant \( K_2 \) is never attained in \( H_0^{2,p}(\Omega) \cap H_0^{1,p}(\Omega) \).

**Proposition 4.2.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold, with or without boundary, of dimension \( n \geq 3 \) and \( 1 < p < \frac{n}{2} \). Then, given \( \varepsilon > 0 \), there exists a real constant \( B = B(M, g, \varepsilon) \) such that
\[ \|u\|_{L^{p^*}_2(M)}^p \leq (K_2^p + \varepsilon) \|\Delta g^i u\|_{L^p(M)}^p + B \|u\|_{L^p(M)}^p \]
for all \( u \in E_i \).

**Proof.** Let \( \varepsilon > 0 \) be given. Given \( \varepsilon_0 > 0 \), there exists \( \delta_0 > 0 \) such that for all \( 0 < \delta < \delta_0 \) there exists a finite covering \( \{B_k\}_{k=1,\ldots,N_\delta} \) of \( M \) by geodesic balls of radius \( \delta \) such that, in normal geodesic coordinates in each of these balls, we have
\[ |g^{ij} - \delta_{ij}| \leq \varepsilon_0, \]
\[ |\Gamma^k_{ij}| \leq \varepsilon_0 \]
and
\[ (1 - \varepsilon_0)dx \leq dv_g \leq (1 + \varepsilon_0)dx. \]
As in Proposition 4.3, we will denote by \( \varepsilon_j \) several possibly different positive constants independent of \( \delta \). Let \( \{\phi_k\}_{k=1,\ldots,N_\delta} \) be a partition of unity subordinated to the covering \( \{B_k\} \) such that \( \phi_k^{1/p} \in C_0^2(B_k) \) for each \( k \). First, we write
\[ \|u\|_{L^{p^*}_2(M)}^p = \|\sum_k \phi_k |u|^p\|_{L^{p^*}/p(M)} \leq \sum_k \|\phi_k |u|^p\|_{L^{p^*}/p(M)} \]
\[ \leq (1 + \varepsilon_0)^{p/p^*} \sum_k \left( \int_{B_k} \phi_k^{p^*/p} |u|^{p^*} \, dx \right)^{p/p^*}. \]
Then, decomposing $\Delta_{\phi}^{1/p}u$ as in (4.10), using the elementary inequality $(1 - \varepsilon_1)|a|^p \leq |a + b|^p + C_{\varepsilon_1}|b|^p$, where we choose $\varepsilon_1 = O(\varepsilon_0)$ small, the Calderon-Zygmund and the interpolation inequalities (4.12) and (4.13), with $\phi_k^{1/p}u$ in place of $u$, we find

$$\int_M |\Delta_{\phi}^{1/p}u|^p \, dv_g \geq (1 - \varepsilon_0) \int_{B_k} |\Delta_{\phi}^{1/p}u|^p \, dx$$

$$\geq (1 - \varepsilon_0)(1 - \varepsilon_1) \int_{B_k} |\Delta(\phi_k^{1/p}u)|^p \, dx - \varepsilon_0^p C_{\varepsilon_1} \int_{B_k} |\partial^2(\phi_k^{1/p}u)|^p \, dx$$

$$\geq (1 - \varepsilon_2) \int_{B_k} |\Delta(\phi_k^{1/p}u)|^p \, dx - C_{\varepsilon_0, \delta} \int_{B_k} \phi_k |u|^p \, dx,$$

for some positive number $\varepsilon_2 = O(\varepsilon_0)$. Noticing that $\phi_k^{1/p}u \in H^{2,p}(B_k)$, if $u \in E_1$ or $u \in E_2$, and $u \in H^{2,p}(B_k) \cap H^{1,p}(B_k)$ if $u \in E_3$, Lemma 4.1 implies

$$\int_M |\Delta_{\phi}^{1/p}u|^p \, dv_g \geq \frac{1 - \varepsilon_2}{K_2^p} \left( \int_{B_k} \phi_k^{1/p} |u|^{p_2} \, dx \right)^{p/p_2} - C_{\varepsilon_0, \delta} \int_{B_k} \phi_k |u|^p \, dx. \quad (4.18)$$

Putting together (4.17) and (4.18), and applying again the elementary inequality $(a + b)^p \leq (1 + \varepsilon_3)a^p + C_{\varepsilon_2}b^p$, choosing $\varepsilon_3 = O(\varepsilon_0)$ small, we obtain

$$\|u\|_{L^{p_2}(M)}^p \leq (1 + \varepsilon_4)K_2^p \sum_k \int_M |\Delta_{\phi}^{1/p}u|^p \, dv_g + C_{\varepsilon_0, \delta} \int_M |u|^p \, dv_g$$

$$\leq (1 + \varepsilon_4)K_2^p \sum_k \int_M |\Delta u|^p \, dv_g + C_{\varepsilon_0, \delta} \int_M |\nabla g(\phi_k^{1/p})|^p |\nabla g u|^p \, dv_g$$

$$+ C_{\varepsilon_0} \sum_k \int_M |\Delta_{\phi}^{1/p}u|^p |u|^p \, dv_g + C_{\varepsilon_0, \delta} \int_M |u|^p \, dv_g$$

$$\leq (1 + \varepsilon_5)K_2^p \int_M |\Delta u|^p \, dv_g + C_{\varepsilon_0, \delta} \int_M |\nabla g u|^p \, dv_g + C_{\varepsilon_0, \delta} \int_M |u|^p \, dv_g$$

for some positive numbers $\varepsilon_4, \varepsilon_5 = O(\varepsilon_0)$, since $|\nabla g(\phi_k^{1/p})|$ and $|\Delta_{\phi}^{1/p}u|$ are both bounded by a constant $C$ depending on $\delta$. Choosing $\varepsilon_0$ sufficiently small and correspondingly fixing $\delta > 0$, we get

$$\|u\|_{L^{p_2}(M)}^p \leq \left( K_2^p + \frac{\varepsilon_2}{2} \right) \|\Delta u\|_{L^p(M)}^p + \tilde{C}_{\varepsilon_0, \delta} \|\nabla g u\|_{L^p(M)}^p + \tilde{C}_{\varepsilon_0, \delta} \|u\|_{L^p(M)}^p. \quad (4.19)$$

On the other hand, by the $L^p$-theory of linear elliptic operators, there exists a positive constant $C_1(\delta)$ such that

$$\int_{B_k} |\partial^2(\phi_k^{1/p}u)|^p \, dx \leq C_1(\delta) \int_{B_k} |\Delta_{\phi}^{1/p}u|^p \, dx$$

$$\leq C_2(\delta) \left( \int_{B_k} |\Delta u|^p \, dx + \int_{B_k} |\nabla g u|^p \, dx + \int_{B_k} |u|^p \, dx \right) \quad (4.20)$$

$$\leq \frac{C_2(\delta)}{1 - \varepsilon_0} \left( \|\Delta u\|_{L^p(M)}^p + \|\nabla g u\|_{L^p(M)}^p + \|u\|_{L^p(M)}^p \right).$$
Using again the interpolation inequality of lower-order derivatives
\[
\int_{B_k} \left| \partial (\phi_k^{1/p} u) \right|^p \, dx \leq \theta \int_{B_k} \left| \partial^2 (\phi_k^{1/p} u) \right|^p \, dx + C_{\theta, \delta} \int_{B_k} \phi_k |u|^p \, dx,
\]

since \(|\nabla g u| \leq (1 + \varepsilon_0) |\partial u|\), it follows that
\[
\|\nabla g u\|_{L^p(M)}^p = \sum_k \left\| \phi_k^{1/p} \nabla g u \right\|_{L^p(M)}^p \leq (1 + \varepsilon_0)^{p+1} \sum_k \left\| \phi_k^{1/p} \partial u \right\|_{L^p(M)}^p \leq (1 + \varepsilon_0) \sum_k \left\| \partial (\phi_k^{1/p} u) \right\|_{L^p(M)}^p + C_{\epsilon_0, \delta, \theta} \|u\|_{L^p(M)}^p
\]

where \(\varepsilon_0 = O(\varepsilon_0)\). Thus, choosing \(\theta\) small enough, we obtain from (4.20) and (4.21)
\[
\frac{1}{2} \|\nabla g u\|_{L^p(M)}^p \leq \frac{\varepsilon}{4C_{\epsilon_0, \delta}} \|\Delta g u\|_{L^p(M)}^p + C_{\epsilon_0, \delta, \theta} \|u\|_{L^p(M)}^p.
\]

Finally, coupling (4.19) with (4.22), we find \(B > 0\) depending only on \(M, g\) and \(\varepsilon\) such that
\[
\|u\|_{L^p_z(M)}^p \leq (K_2^p + \varepsilon) \|\Delta g u\|_{L^p(M)}^p + B \|u\|_{L^p(M)}^p
\]
for all \(u \in E_\eta\).

4.3 The nonvalidity of the optimal inequality

Proof of Theorem 4.2. In order to prove this theorem, we construct a family of functions \((u_\varepsilon) \subset C^\infty_0(M)\) such that
\[
\frac{\|u_\varepsilon\|_{L^\infty_z(M)}^2 - K_2^2 \|\Delta g u_\varepsilon\|_{L^2(M)}^2}{\|u_\varepsilon\|_{L^2_z(M)}^2} \to +\infty
\]
as \(\varepsilon\) approaches zero. Fix \(x_0 \in M \setminus \partial M\) such that \(\text{Scal}_g(x_0) > 0\) and a geodesic ball \(B_\delta(x_0) \subset M \setminus \partial M\). Consider a radial cutoff function \(\eta \in C^\infty_0(B_\delta)\) satisfying \(\eta = 1\) in \(B_\frac{1}{2}\), \(\eta = 0\) outside \(B_\delta\) and \(0 \leq \eta \leq 1\) in \(B_\delta\). Define, up to the exponential chart \(\exp_{x_0}\),
\[
u_\varepsilon(x) = \eta(x) z_\varepsilon(x),
\]
where
\[
z_\varepsilon(x) = \varepsilon^{-\frac{n+1}{2}} z\left(\frac{x}{\varepsilon}\right)
\]
with
\[
z(x) = \frac{1}{(1 + |x|^2)^{\frac{n+1}{2}}}
\]
being an extremal function for the Sobolev quotient (4.3) in \(D^{2,2}(\mathbb{R}^n)\). In particular,
\[
\|z\|_{L^2_z(\mathbb{R}^n)}^2 = K_2^2 \|\Delta z\|_{L^2_z(\mathbb{R}^n)}^2.
\]
We will estimate the asymptotic behavior of $\|u_\varepsilon\|_{L^{2n}(M)}^2$, $\|u_\varepsilon\|_{L^2(M)}^2$ and $\|\Delta u_\varepsilon\|_{L^2(M)}^2$ with respect to $\varepsilon$ near the origin. The result of these computations will involve the scalar curvature $\text{Scal}_g(x_0)$ and the constants $I_1, I_2, I_3$ and $I_4$ introduced in (4.7).

1. Estimate of $\|u_\varepsilon\|_{L^{2n}(M)}^2$.

Write $\eta^{2n}(x) = 1 + O(r^3)$ and use the expansion of the metric in normal geodesic coordinates up to the third order in order to obtain (see [50])

$$\sqrt{\det g} = 1 - \frac{1}{6} \sum_{i,j=1}^n \text{Ric}_{ij}(x_0)x_i x_j + O(r^3), \tag{4.24}$$

where $\text{Ric}_{ij}$ denotes the components of the Ricci tensor in these coordinates. Then,

$$\int_M u_\varepsilon^{2n} dv_g = \int_{B_\delta} z_\varepsilon^{2n} dx - \frac{1}{6} \sum_{i,j=1}^n \text{Ric}_{ij}(x_0) \int_{B_\delta} z_\varepsilon^{2n} x_i x_j dx + \int_{B_\delta} z_\varepsilon^{2n} O(r^3) dx
= \int_{B_\delta} z_\varepsilon^{2n} dx - \frac{\text{Scal}_g(x_0)}{6n} \int_{B_\delta} z_\varepsilon^{2n} r^2 dx + \int_{B_\delta} z_\varepsilon^{2n} O(r^3) dx
= \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}} z^{2n} dx - \frac{\text{Scal}_g(x_0)}{6n} \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}} z^{2n} r^2 dx + \varepsilon^2 \int_{B_{\frac{\delta}{2}}} z^{2n} r^2 dx + \varepsilon^3 \int_{B_{\frac{\delta}{2}}} z^{2n} O(r^3) dx.
$$

After a straightforward computation, we find for any $n \geq 5$ that

$$\|u_\varepsilon\|_{L^{2n}(M)}^2 = \|z\|_{L^{2n}(\mathbb{R}^n)}^2 - \frac{n}{n} \frac{4 \text{Scal}_g(x_0)}{6n} I_2 \varepsilon^2 + o(\varepsilon^2). \tag{4.25}$$

2. Estimate of $\|u_\varepsilon\|_{L^2(M)}^2$.

In this case, we write

$$\int_M u_\varepsilon^2 dv_g = O(1) \int_{B_\delta} z_\varepsilon^2 dx = O(\varepsilon^4) \int_{B_{\frac{\delta}{2}}} z^2 dx$$

and obtain by direct computation

$$\|u_\varepsilon\|_{L^2(M)}^2 = \begin{cases} 
O(\varepsilon^2) & \text{if } n = 6, \\
O(\varepsilon^3) & \text{if } n = 7, \\
O(\varepsilon^4 |\ln \varepsilon|) & \text{if } n = 8, \\
O(\varepsilon^4) & \text{if } n \geq 9. 
\end{cases} \tag{4.26}$$

3. Estimate of $\|\Delta_g u_\varepsilon\|_{L^2(M)}^2$.
First, write

\[
\int_M |\Delta_y u_\varepsilon|^2 \, dv_g = \int_{B_\varepsilon} |\eta \Delta_y z_\varepsilon + 2(\nabla_y \eta, \nabla_y z_\varepsilon) + (\Delta_y \eta)z_\varepsilon|^2 \, dv_g \\
= \int_{B_\varepsilon} |\Delta_y z_\varepsilon|^2 \, dv_g + \int_{B_\varepsilon \setminus B_\frac{\varepsilon}{2}} |\eta \Delta_y z_\varepsilon + 2(\nabla_y \eta, \nabla_y z_\varepsilon) + (\Delta_y \eta)z_\varepsilon|^2 \, dv_g. \quad (4.27)
\]

In order to compute the first term of the right-hand side of (4.27), we write the Laplacian in normal geodesic coordinates and, noticing that \(\Delta z_\varepsilon(r) < 0\) and \(z'_\varepsilon(r) < 0\) for \(r > 0\), one has

\[
|\Delta_y z_\varepsilon|^2 = |\Delta z_\varepsilon + z'_\varepsilon(r)\partial_r (\ln |\det g|)|^2 \\
= |\Delta z_\varepsilon|^2 + 2|\Delta z_\varepsilon||z'_\varepsilon(r)| \partial_r (\ln |\det g|) + |z'_\varepsilon(r)\partial_r (\ln |\det g|)|^2.
\]

From (4.24), there follows that

\[
\partial_r (\ln |\det g|) = -\frac{1}{\sqrt{|\det g|}} \frac{1}{3} \sum_{i,j=1}^n \text{Ric}_{ij}(x_0) \frac{x_i x_j}{r} + O(r^2).
\]

Therefore, through standard computations, we obtain for \(n \geq 7\)

\[
\int_{B_{\varepsilon/2}} |\Delta z_\varepsilon|^2 \, dv_g = \int_{\mathbb{R}^n} |\Delta z|^2 \, dx - \int_{\mathbb{R}^n \setminus B_{\varepsilon/2}} |\Delta z|^2 \, dx - \frac{\text{Scal}(x_0)}{6n} \varepsilon^2 \int_{\mathbb{R}^n} |\Delta z|^2 r^2 \, dx \\
+ \frac{\text{Scal}(x_0)}{6n} \varepsilon^2 \int_{\mathbb{R}^n \setminus B_{\varepsilon/2}} |\Delta z|^2 r^2 \, dx + \varepsilon^3 \int_{B_{\varepsilon/2}} |\Delta z|^2 O(r^3) \, dx \\
= \|\Delta z\|^2_{L^2(\mathbb{R}^n)} - \frac{\text{Scal}(x_0)}{6n} I_{L^2}^1 \varepsilon^2 + o(\varepsilon^2),
\]

\[
\int_{B_{\varepsilon/2}} |\Delta z_\varepsilon||z'_\varepsilon(r)| \partial_r (\ln |\det g|) \, dv_g \\
= -\frac{\text{Scal}(x_0)}{3n} \int_{B_{\varepsilon/2}} \Delta z_\varepsilon|z'_\varepsilon(r)| r \, dx + \int_{B_{\varepsilon/2}} |\Delta z_\varepsilon||z'_\varepsilon(r)| O(r^2) \, dx \\
= -\frac{\text{Scal}(x_0)}{3n} \varepsilon^2 \int_{\mathbb{R}^n} |\Delta z||z'(r)| r \, dx + \frac{\text{Scal}(x_0)}{3n} \varepsilon^2 \int_{\mathbb{R}^n \setminus B_{\varepsilon/2}} |\Delta z||z'(r)| r \, dx \\
+ \varepsilon^3 \int_{B_{\varepsilon/2}} |\Delta z||z'(r)| O(r^2) \, dx \\
= -\frac{\text{Scal}(x_0)}{3n} I_{L^2}^2 \varepsilon^2 + o(\varepsilon^2),
\]

and

\[
\int_{B_{\varepsilon/2}} \left|z'_\varepsilon(r)\partial_r (\ln |\det g|)\right|^2 \, dv_g = \varepsilon^4 \int_{B_{\varepsilon/2}} |z'(r)|^2 O(r^2) \, dx = o(\varepsilon^2).
\]
If $n = 6$, we have

\[
\int_{B_{\frac{2}{3}}} |\Delta z_\varepsilon|^2 \, dv_g = \int_{\mathbb{R}^n} |\Delta z|^2 \, dx - \int_{\mathbb{R}^n \setminus B_{\frac{2}{3}}} |\Delta z|^2 \, dx - \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 \int_{B_{\frac{2}{3}}} |\Delta z|^2 \, r^2 \, dx
\]
\[
+ \varepsilon^3 \int_{B_{\frac{2}{3}}} |\Delta z|^2 O(r^3) \, dx
\]
\[
= \|\Delta z\|_{L^2(\mathbb{R}^n)}^2 - \omega_{n-1} (n-4) \frac{\varepsilon^2}{3n} \int_0^{\frac{2}{\varepsilon}} \frac{(6 + 2s^2)^2}{(1 + s^2)^n} s^7 \, ds + O(\varepsilon^2)
\]
\[
= \|\Delta z\|_{L^2(\mathbb{R}^n)}^2 - \omega_{n-1} (n-4) \frac{2 \text{Scal}_g(x_0)}{3n} \frac{\varepsilon^2 |\ln \varepsilon|}{(\varepsilon^2 + 1)^3} + O(\varepsilon^2),
\]

\[
\int_{B_{\frac{2}{3}}} |\Delta z_\varepsilon| |z_\varepsilon'(r)| \partial_r (\ln \sqrt{\det g}) \, dv_g = -\frac{\text{Scal}_g(x_0)}{3n} \varepsilon^2 \int_{B_{\frac{2}{3}}} |\Delta z| |z_\varepsilon'(r)| \, dx + \varepsilon^3 \int_{B_{\frac{2}{3}}} |\Delta z| |z_\varepsilon'(r)| O(r^2) \, dx
\]
\[
= -\omega_{n-1} (n-4) \frac{\text{Scal}_g(x_0)}{3n} \varepsilon^2 \int_0^{\frac{2}{\varepsilon}} \frac{6 + 2s^2}{(1 + s^2)^3} s^7 \, ds + O(\varepsilon^2)
\]
\[
= O(\varepsilon^2),
\]

and

\[
\int_{B_{\frac{2}{3}}} \left| z_\varepsilon'(r) \partial_r (\ln \sqrt{\det g}) \right|^2 \, dv_g = \varepsilon^4 \int_{B_{\frac{2}{3}}} |z_\varepsilon'(r)|^2 O(r^2) \, dx = O(\varepsilon^2).
\]

Finally, we compute the second term of the right-hand side of (4.27). For $n \geq 7$, we have

\[
\int_{B_1 \setminus B_{\frac{1}{2}}} \eta \Delta_g z_\varepsilon + 2 \langle \nabla_g \eta, \nabla_g z_\varepsilon \rangle + (\Delta_g \eta) z_\varepsilon \, dv_g
\]
\[
= O(1) \left[ \int_{B_1 \setminus B_{\frac{1}{2}}} \left| \Delta_g z_\varepsilon \right|^2 \, dx + \int_{B_1 \setminus B_{\frac{1}{2}}} \left| \nabla_g z_\varepsilon \right|^2 \, dx + \int_{B_1 \setminus B_{\frac{1}{2}}} z_\varepsilon^2 \, dx \right]
\]
\[
= O(1) \left[ \int_{B_1 \setminus B_{\frac{1}{2}}} \left| \Delta z_\varepsilon \right|^2 \, dx + \int_{B_1 \setminus B_{\frac{1}{2}}} \left| z_\varepsilon'(r) \right|^2 r^2 \, dx + \int_{B_1 \setminus B_{\frac{1}{2}}} \left| z_\varepsilon'(r) \right|^2 \, dx + \int_{B_1 \setminus B_{\frac{1}{2}}} z_\varepsilon^2 \, dx \right]
\]
\[
= O(1) \left[ \int_{B_1 \setminus B_{\frac{1}{2}}} \left| \Delta z \right|^2 + \varepsilon^4 \int_{B_1 \setminus B_{\frac{1}{2}}} \left| z_\varepsilon'(r) \right|^2 r^2 + \varepsilon^2 \int_{B_1 \setminus B_{\frac{1}{2}}} \left| z_\varepsilon'(r) \right|^2 + \varepsilon^4 \int_{B_1 \setminus B_{\frac{1}{2}}} z_\varepsilon^2 \right]
\]
\[
= o(\varepsilon^2),
\]

while for $n = 6$ we get

\[
\int_{B_1 \setminus B_{\frac{1}{2}}} \left| \eta \Delta_g z_\varepsilon + 2 \langle \nabla_g \eta, \nabla_g z_\varepsilon \rangle + (\Delta_g \eta) z_\varepsilon \right|^2 \, dv_g = \delta^2 O(\varepsilon^2 |\ln \varepsilon|).
\]

Thus, we conclude for $n \geq 7$ that

\[
\|\Delta_g u_\varepsilon\|^2_{L^2(M)} = \|\Delta z\|^2_{L^2(\mathbb{R}^n)} - \frac{\text{Scal}_g(x_0)}{6n} I_4 \varepsilon^2 + o(\varepsilon^2)
\] (4.28)
and for \( n = 6 \), choosing \( \delta \) small enough, that

\[
\|\Delta_g u_\epsilon\|^2_{L^2(M)} = \|\Delta z\|^2_{L^2(\mathbb{R}^n)} - C(\epsilon) \text{Scal}_g(x_0) \epsilon^2 |\ln \epsilon| \tag{4.29}
\]

where \( C(\epsilon) \) approaches a positive number as \( \epsilon \to 0 \).

4. Conclusion

From (4.23) and estimates (4.25), (4.26) and (4.29), we obtain for \( n = 6 \)

\[
\frac{\|u_\epsilon\|^2_{L^2(\mathbb{R}^n)} - K^2_2 \|\Delta_g u_\epsilon\|^2_{L^2(M)}}{\|u_\epsilon\|^2_{L^2(M)}} = \frac{C(\epsilon) \text{Scal}_g(x_0) \epsilon^2 |\ln \epsilon| + O(\epsilon^2)}{O(\epsilon^2)} \to +\infty
\]
as \( \epsilon \to 0 \).

If \( n \geq 7 \), from (4.23), (4.25), (4.26) and (4.28), we have

\[
\frac{\|u_\epsilon\|^2_{L^2(\mathbb{R}^n)} - K^2_2 \|\Delta_g u_\epsilon\|^2_{L^2(M)}}{\|u_\epsilon\|^2_{L^2(M)}} = \frac{K^2_2 \text{Scal}_g(x_0) \left(I_4 - \frac{n-4}{n} I_2 I_3\right) \epsilon^2 + o(\epsilon^2)}{O(\epsilon^2)} \to +\infty
\]
as \( \epsilon \to 0 \), if and only if

\[
\frac{n}{n-4} I_1 I_4 > 1.
\]

By direct computation, we find

\[
I_1 = \int_{\mathbb{R}^n} \frac{z}{r^2} \, dx = \omega_{n-1} \int_0^\infty \frac{r^{n-2}}{(1 + r^2)^n} \, dr = \frac{\omega_n}{2n},
\]

\[
I_2 = \int_{\mathbb{R}^n} z |x|^2 \, dx = \omega_{n-1} \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^n} \, dr = \frac{\omega_n}{2n} \left(\frac{2n}{n-2}\right),
\]

\[
I_3 = \int_{\mathbb{R}^n} |\Delta z|^2 \, dx = \omega_{n-1}(n-4)^2 \int_0^\infty \frac{(n+2r^2)^2}{(1 + r^2)^n} \, r^{n-1} \, dr = \frac{\omega_n n(n-4)(n^2-4)}{2^n(n-2)},
\]

\[
I_4 = \int_{\mathbb{R}^n} |\Delta z| r^2 \, dx = \omega_{n-1}(n-4)^2 \int_0^\infty \frac{(n+2r^2)^2}{(1 + r^2)^n} \, r^{n+1} \, dr = \frac{\omega_n n(n-4)^2(n^2+4)}{2^n(n-6)},
\]

\[
I_5 = \int_{\mathbb{R}^n} |\Delta z| |z'| r \, dx = \omega_{n-1}(n-4)^2 \int_0^\infty \frac{n+2r^2}{(1 + r^2)^n} \, r^{n+1} \, dr = \frac{\omega_n n(n-4)(n-2)(n-4)}{2^n(n-6)}.
\]

Hence,

\[
I_4 = I_4^1 + 4I_4^2 = \frac{\omega_n n^2(n-4)(n^2+4n-20)}{2^n(n-6)}.
\]

Therefore,

\[
\frac{n}{n-4} \frac{I_1 I_4}{I_2 I_3} = \frac{n(n^2+4n-20)}{(n-4)(n-6)(n+2)} > 1 \tag{4.30}
\]

for \( n \geq 7 \), as wished.
4.4 Fourth order problems on compact manifolds

4.4.1 A Concentration-Compactness Principle

As a consequence of the asymptotically sharp Sobolev inequality of Theorem 4.1, we obtain the following version of the concentration-compactness principle which will be used in the proof of the existence part of Theorems 4.3A-C and 4.4A-C.

**Lemma 4.2.** [20] (Concentration-Compactness Principle) Let \((M, g)\) be a smooth compact Riemannian manifold, with or without boundary, of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\). Denote 
\[
p^*_1 = \frac{np}{n-p}\] 
and let \(K_1 = K_1(n, p)\) be the best constant in the first order Sobolev inequality, i.e.,
\[
\frac{1}{K_1(n, p)} = \inf_{u \in D^1_p(\mathbb{R}^n) \backslash \{0\}} \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*_1}(\mathbb{R}^n)}}.
\]
Assume that \(u_m \rightharpoonup u\) in \(E_i\) and
\[
|\Delta_g u_m|^p \, dv_g \rightharpoonup \mu,
\]
\[
|u_m|^{p^*_1} \, dv_g \rightharpoonup \nu,
\]
\[
|\nabla_g u_m|^{p^*_1} \, dv_g \rightharpoonup \pi,
\]
where \(\mu, \nu, \pi\) are bounded non-negative measures. Then, there exist at most a countable set \(J, \{x_j\}_{j \in J} \subset M\) and positive numbers \(\{\mu_j\}_{j \in J}, \{\nu_j\}_{j \in J}, \{\pi_j\}_{j \in J}\) such that
\[
\mu \geq |\Delta_g u|^p \, dv_g + \sum_{j \in J} \mu_j \delta_{x_j},
\]
\[
\nu = |u|^{p^*_1} \, dv_g + \sum_{j \in J} \nu_j \delta_{x_j},
\]
\[
\pi = |\nabla_g u|^{p^*_1} \, dv_g + \sum_{j \in J} \pi_j \delta_{x_j},
\]
with
\[
\nu_j^{1/p^*_1} \leq K_2 \mu_j^{1/p},
\]
\[
\pi_j^{1/p^*_1} \leq CK_1 \mu_j^{1/p},
\]
where \(C\) is a positive constant depending only on \((M, g)\).

**Proof.** Set \(v_m = u_m - u\), so that \(v_m \rightharpoonup 0\) in \(E_i\). Define
\[
\omega = \nu - |u|^{p^*_1} \, dv_g,
\]
\[
\theta = \pi - |u|^{p^*_1} \, dv_g,
\]
By the Brezis-Lieb lemma,
\[ |v_m|^{p^2} \, dv_g \to \omega, \]
\[ |\nabla g v_m|^{p^1} \, dv_g \to \theta. \]

Up to a subsequence we can assume that
\[ |\Delta_g v_m|^p \, dv_g \to \lambda \]
for some bounded nonnegative measure \( \lambda \). We have only to show that there hold reverse Hölder inequalities for each of the measures \( \omega \) and \( \theta \) with respect to \( \lambda \). The rest of the proof is standard (see Section 3.3 for details).

By Theorem 4.1, for each \( \varepsilon_1 > 0 \) there exists \( B_{\varepsilon_1} > 0 \) such that
\[
\|w\|_{L^p(M)}^p \leq (K_2^p + \varepsilon_1) \|\Delta_g w\|_{L^p(M)}^p + B_{\varepsilon_1} \|w\|_{L^p(M)}^p
\]
for every \( w \in E_1 \). Given \( \varepsilon > 0 \), choosing \( \varepsilon_1 \) small enough, it follows that for any \( \xi \in C^\infty(M) \) we have
\[
\left( \int_M |\xi|^{p^2} |v_m|^{p^2} \, dv_g \right)^{p/p^2} \leq (K_2^p + \varepsilon_1) \|\Delta_g (\xi v_m)\|_{L^p(M)}^p + B_{\varepsilon_1} \|\xi v_m\|_{L^p(M)}^p
\]
\[
\leq (K_2^p + \varepsilon_1)(1 + \varepsilon_1) \|\Delta_g v_m\|_{L^p(M)}^p + C_{\varepsilon_1} \|\langle \nabla g \xi, \nabla_g v_m \rangle \|_{L^p(M)}
\]
\[
+ C_{\varepsilon_1} \|\Delta_g (\xi v_m)\|_{L^p(M)}^p + B_{\varepsilon_1} \|\xi v_m\|_{L^p(M)}^p
\]
\[
\leq (K_2^p + \varepsilon) \int_M |\xi|^p |\Delta_g v_m|^p \, dv_g + C_{\varepsilon_1} \max_M |\nabla_g \xi|^p \|\nabla_g v_m\|_{L^p(M)}^p
\]
\[
+ C_{\varepsilon_1} \max_M (|\Delta_g \xi|^p + |\xi|^p) \|v_m\|_{L^p(M)}^p.
\]

Since, up to a subsequence \( v_m \to 0 \) and \( \nabla_g v_m \to 0 \) in \( L^p(M) \), taking the limit when \( m \to \infty \) we find
\[
\left( \int_M |\xi|^{p^2} \, d\omega \right)^{p/p^2} \leq (K_2^p + \varepsilon) \int_M |\xi|^p \, d\lambda
\]
for all \( \varepsilon > 0 \). Making \( \varepsilon \to 0 \), we obtain the first reverse Hölder inequality:
\[
\left( \int_M |\xi|^{p^2} \, d\omega \right)^{1/p^2} \leq K_2 \left( \int_M |\xi|^p \, d\lambda \right)^{1/p} \tag{4.31}
\]
for all \( \xi \in C^\infty(M) \).

Similarly, it is a well known result due to Aubin (see [7]) that for each \( \varepsilon_1 > 0 \) there exists \( B_{\varepsilon_1} = B(M, g, \varepsilon_1) > 0 \) such that
\[
\|\nabla_g w\|_{L^p(M)}^p \leq (K_2^p + \varepsilon_1) \|\nabla_g \nabla_g w\|_{L^p(M)}^p + B_{\varepsilon_1} \|\nabla_g w\|_{L^p(M)}^p
\]
\[
\leq (K_2^p + \varepsilon_1) \|\nabla_g^2 w\|_{L^p(M)}^p + B_{\varepsilon_1} \|\nabla_g w\|_{L^p(M)}^p.
\]
for all $w \in E_1$ or $w \in E_2$, while, according to [33], we have
\[
\|\nabla_g w\|^p_{L^p(M)} \leq (2^{p/n} K_1^p + \varepsilon_1) \|\nabla_g \nabla_g w\|^p_{L^p(M)} + B \varepsilon_1 \|\nabla_g w\|^p_{L^p(M)} \\
\leq (2^{p/n} K_1^p + \varepsilon_1) \|\nabla_g^2 w\|^p_{L^p(M)} + B \varepsilon_1 \|\nabla_g w\|^p_{L^p(M)}
\]
for all $w \in E_3$. On the other hand, as we saw in section 4.2, there exists a positive constant $\tilde{C} = \tilde{C}(M, g)$ such that
\[
\|\nabla_g w\|^p_{L^p(M)} \leq \tilde{C}^p \left( \|\Delta_g w\|^p_{L^p(M)} + \|\nabla_g w\|^p_{L^p(M)} \right)
\]
for all $w \in E_i$. Therefore, given $\varepsilon > 0$, with a convenient choice of $\varepsilon_1$, these inequalities imply
\[
\|\nabla_g w\|^p_{L^p(M)} \leq (C^p K_1^p + \varepsilon) \|\Delta_g w\|^p_{L^p(M)} + C \varepsilon \|\nabla_g w\|^p_{L^p(M)} + C \varepsilon \|w\|^p_{L^p(M)}
\]
for all $w \in E_i$. Proceeding as previously, for any $\xi \in C^\infty(M)$ we get
\[
\left( \int_M |\xi|^p \|\nabla_g v_m\|^p_{L^p(M)} dv_g \right)^{p/p_1} \leq (C^p K_1^p + \varepsilon) \int_M |\xi|^p \Delta_g v_m|^p dv_g + C \varepsilon \|\nabla_g v_m\|^p_{L^p(M)} + C \varepsilon \|v_m\|^p_{L^p(M)}.
\]
Again, taking the limit when $m \to \infty$ and then making $\varepsilon \to 0$, we find the second reverse Hölder inequality:
\[
\left( \int_M |\xi|^p d\theta \right)^{1/p_1} \leq CK \left( \int_M |\xi|^p d\lambda \right)^{1/p}
\]
for all $\xi \in C^\infty(M)$. ■

### 4.4.2 Proof of Theorems 4.3A-C

The proof of these theorems is done through a minimization argument involving Ekeland’s principle and the above version of the concentration-compactness principle (similar to the argument used in Section 3.4). In order to facilitate the reading, we will often omit the element of volume $dv_g$ in the notation of integrals.

The set $V_i$ defined in the introduction is the closed differentiable manifold $V_i = F^{-1}(1)$, where $F : E_i \to \mathbb{R}$ is the continuously differentiable functional
\[
F(u) = \int_M f(x) \|u\|^2 dv_g.
\]
Thus, by Ekeland’s variational principle, there exists a minimizing sequence $(u_m)$ for $J$ on $V_i$ such that $\|J'(u_m)\|_{(T_{u_m} V_i)} \to 0$. Since $J$ is coercive on $E_i$, $(u_m)$ is bounded. Thus, up to a subsequence, we may assume that $u_m \to u$ in $E_i$, $u_m \to u$ in $H^{1,p}(M)$ and that the conclusion of the concentration-compactness principle (Lemma 4.2) holds. Fix $k \in J$ and choose a cutoff function $\varphi_\varepsilon \in C_0^\infty(B_{2\varepsilon}(x_k))$ satisfying $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon \equiv 1$ in $B_\varepsilon(x_k)$ and
\[ |\nabla_g \varphi_\varepsilon| \leq \frac{C}{\varepsilon}, \quad |\Delta_g \varphi_\varepsilon| \leq \frac{C}{\varepsilon^2}, \]

for some constant \( C > 0 \) independent of \( \varepsilon \). Write

\[ \varphi_\varepsilon u_m = \zeta_m + \left( \int_M f(x) |u_m|^{p^*} \varphi_\varepsilon \, dv_g \right) u_m, \]

where

\[ \zeta_m := \left[ \varphi_\varepsilon - \left( \int_M f(x) |u_m|^{p^*} \varphi_\varepsilon \, dv_g \right) \right] u_m \in T_{u_m} V_1. \]

Since \( (\zeta_m) \) is a bounded sequence in \( E_1 \), it follows that

\[ \int_M |\Delta_g u_m|^{p-2} \Delta_g u_m \Delta_g \zeta_m + \int_M a(x) |\nabla_g u_m|^{p-2} \langle \nabla_g u_m, \nabla_g (\varphi_\varepsilon u_m) \rangle \]

\[ + \int_M b(x) |u_m|^{p-2} u_m \zeta_m \to 0, \]

and so

\[ \lim_{m \to \infty} \left( \int_M |\Delta_g u_m|^{p-2} \Delta_g u_m \Delta_g (\varphi_\varepsilon u_m) + \int_M a(x) |\nabla_g u_m|^{p-2} \langle \nabla_g u_m, \nabla_g (\varphi_\varepsilon u_m) \rangle \right) \]

\[ + \int_M b(x) |u_m|^{p-2} u_m (\varphi_\varepsilon u_m) \]

\[ = \lim_{m \to \infty} \left( \int_M f(x) |u_m|^{p^*} \varphi_\varepsilon \right) \left( \int_M |\Delta_g u_m|^p + \int_M a(x) |\nabla_g u_m|^p + \int_M b(x) |u_m|^p \right) \]

\[ = \left( \int_M f(x) \varphi_\varepsilon \, dv \right) \inf_{V_1} J. \]

On the other hand, we can also write

\[ \lim_{m \to \infty} \left( \int_M |\Delta_g u_m|^{p-2} \Delta_g u_m \Delta_g (\varphi_\varepsilon u_m) + \int_M a(x) |\nabla_g u_m|^{p-2} \langle \nabla_g u_m, \nabla_g (\varphi_\varepsilon u_m) \rangle \right) \]

\[ + \int_M b(x) |u_m|^{p-2} u_m (\varphi_\varepsilon u_m) \]

\[ = \lim_{m \to \infty} \left( \int_M \varphi_\varepsilon |\Delta_g u_m|^p + 2 \int_M |\Delta_g u_m|^{p-2} \Delta_g u_m \langle \nabla_g u_m, \nabla_g \varphi_\varepsilon \rangle \right) \]

\[ + \int_M (\Delta_g \varphi_\varepsilon) u_m |\Delta_g u_m|^{p-2} \Delta_g u_m + \int_M a(x) \varphi_\varepsilon |\nabla_g u_m|^p \]

\[ + \int_M a(x) u_m |\nabla_g u_m|^{p-2} \langle \nabla_g u, \nabla_g \varphi_\varepsilon \rangle + \int_M b(x) \varphi_\varepsilon |u_m|^p \]

\[ = \int_M \varphi_\varepsilon \, dv + \int_M a(x) \varphi_\varepsilon |\nabla_g u|^p + \int_M a(x) u |\nabla_g u|^{p-2} \langle \nabla_g u, \nabla_g \varphi_\varepsilon \rangle + \int_M b(x) \varphi_\varepsilon |u|^p \]

\[ + \lim_{m \to \infty} \left( 2 \int_M |\Delta_g u_m|^{p-2} \Delta_g u_m \langle \nabla_g \varphi_\varepsilon, \nabla_g u_m \rangle + \int_M (\Delta_g \varphi_\varepsilon) u_m |\Delta_g u_m|^{p-2} \Delta_g u_m \right). \]
We claim that
\[
\limsup_{m \to \infty} \left| \int_{M} \Delta_{g} u_{m} \right|^{p-2} \Delta_{g} u_{m} \left( \nabla_{g} \varphi_{\varepsilon}, \nabla_{g} u_{m} \right) \to 0,
\]
\[
\limsup_{m \to \infty} \left| \int_{M} (\Delta_{g} \varphi_{\varepsilon}) u_{m} \right| \left( \Delta_{g} u_{m} \right)^{p-2} \Delta_{g} u_{m} \to 0,
\]
\[
\int_{M} a(x) u \left| \nabla_{g} u \right|^{p-2} \left( \nabla_{g} u, \nabla_{g} \varphi_{\varepsilon} \right) \to 0
\]
as \( \varepsilon \to 0 \). This will follow from Hölder’s inequality and another application of the concentration-compactness principle. Indeed, as \( \varepsilon \to 0 \),
\[
\limsup_{m \to \infty} \left| \int_{M} \left( \Delta_{g} u_{m} \right)^{p-2} \Delta_{g} u_{m} \left( \nabla_{g} \varphi_{\varepsilon}, \nabla_{g} u_{m} \right) \right|
\]
\[
\leq \limsup_{m \to \infty} \left[ \left( \int_{M} \left| \Delta_{g} u_{m} \right|^{p} \right)^{\frac{p-1}{p}} \left( \int_{B_{2\varepsilon}(x_{k}) \setminus B_{\varepsilon}(x_{k})} \left| \nabla_{g} \varphi_{\varepsilon} \right|^{p} \right)^{\frac{1}{p}} \left( \int_{B_{2\varepsilon}(x_{k}) \setminus B_{\varepsilon}(x_{k})} \left| \nabla_{g} u_{m} \right|^{p} \right)^{\frac{1}{p}} \right]^{\frac{p}{p-1}}
\]
\[
\leq C \left[ \frac{1}{\varepsilon^{n}} \text{vol}_{g} \left( B_{2\varepsilon}(x_{k}) \setminus B_{\varepsilon}(x_{k}) \right) \right]^{\frac{p}{p-1}} \limsup_{m \to \infty} \left( \int_{B_{2\varepsilon}(x_{k}) \setminus B_{\varepsilon}(x_{k})} \left| \nabla_{g} u_{m} \right|^{p} \right)^{\frac{1}{p-1}}
\]
\[
\leq C \left( \int_{B_{2\varepsilon}(x_{k}) \setminus B_{\varepsilon}(x_{k})} \left| u \right|^{p} + \sum_{j \in J} \nu_{j} \delta_{x_{j}}(B_{2\varepsilon}(x_{k}) \setminus B_{\varepsilon}(x_{k})) \right)^{1/p} \to 0,
\]
and
\[
\int_{M} a(x) u \left| \nabla_{g} u \right|^{p-2} \left( \nabla_{g} u, \nabla_{g} \varphi_{\varepsilon} \right)
\]
\[
\leq \max_{M} |a| \left[ \left( \int_{M} \left| u \right|^{p} \right)^{\frac{1}{p}} \left( \int_{M} \left| \nabla_{g} u \right|^{p} \right)^{\frac{p-1}{p}} \left( \int_{B_{2\varepsilon}(x_{k}) \setminus B_{\varepsilon}(x_{k})} \left| \nabla_{g} \varphi_{\varepsilon} \right|^{p} \right)^{\frac{p}{p+1}} \right]
\]
\[
\leq C \left[ \frac{1}{\varepsilon^{n}} \text{vol}_{g} \left( (B_{2\varepsilon}(x_{k}) \setminus B_{\varepsilon}(x_{k})) \right) \right]^{\frac{p}{p+1}}
\]
\[
\leq C \left( \varepsilon^{n-p} \right)^{\frac{p}{p+1}} = C \varepsilon^{p} \to 0.
\]
Therefore, making \( \varepsilon \to 0 \) and using (4.32), we conclude that
\[
\mu_{k} = f(x_{k}) \nu_{k} \inf J.
\]
From the coercivity of \( J \), we know that \( \inf_{V_i} J > 0 \), whence we conclude that \( f(x_k) > 0 \). Using the concentration-compactness principle, we obtain

\[
\mu_k \geq \frac{1}{K_2^{n/2} (f(x_k) \inf_{V_i} J)^{n/2p_2}}.
\]

In particular, since \( \mu \) is a bounded measure, \( \mathcal{J} \) is a finite set. We assert that \( \mathcal{J} = \emptyset \). On the contrary, if there exists some \( k \in \mathcal{J} \), then, using the concentration-compactness principle and the coercivity of the functional \( J \), we obtain

\[
\inf_{V_i} J = \lim_{m \to \infty} \left( \int_M |\Delta_g u_m|^p \, dv_g + \int_M a(x) |\nabla_g u_m|^p \, dv_g + \int_M b(x) |u_m|^p \, dv_g \right)
\geq \int_M |\Delta_g u|^p \, dv_g + \int_M a(x) |\nabla_g u|^p \, dv_g + \int_M b(x) |u|^p \, dv_g + \sum \mu_j
\geq \mu_k \geq \frac{1}{K_2^{n/2} \left( f(x_k) \inf_{V_i} J \right)^{n/2p_2}} \geq \frac{1}{K_2^{n/2} \left( \max_M f \right)^{n/2p_2}} \left( \inf_{V_i} J \right)^{n/2p_2},
\]

which implies

\[
\inf_{V_i} J \geq \frac{1}{K_2^p \left( \max_M f \right)^{p/p_2}},
\]

contradicting (H\(_1\)). Brezis-Lieb lemma then implies \( u_m \to u \) in \( L^{p_2}(M) \), whence \( \int_M f(x) |u|^{p_2} \, dv_g = 1 \), i.e., \( u \in V_i \). As

\[
\int_M |\Delta_g u|^p + \int_M a(x) |\nabla_g u|^p + \int_M b(x) |u|^p \leq \liminf \int_M |\Delta_g u_m|^p + \int_M a(x) |\nabla_g u|^p + \int_M b(x) |u|^p
= \liminf \int_M |\Delta_g u_m|^p + \lim \int_M a(x) |\nabla_g u_m|^p + \lim \int_M b(x) |u_m|^p
= \liminf \left( \int_M |\Delta_g u_m|^p + \int_M a(x) |\nabla_g u_m|^p + \int_M b(x) |u_m|^p \right)
= \inf_{V_i} J,
\]

we conclude that \( u \) is a minimizer for \( J \) on \( V_i \).

The regularity part of Theorems 4.3A and 4.3B follows, respectively, from applying Lemmas 4.3 and 4.4 of the next subsection with \( c(x) = f(x) |u|^{2^*-2} \), and \( L^p \) and \( C^\gamma \) estimates for elliptic equations. The regularity part of Theorem 4.3C follows from applying Lemma 4.5 with \( c(x) = f(x) |u|^{p_2-2} \) and after some iterations of \( L^p \) and \( C^\gamma \) estimates to each equation of the system

\[
\begin{aligned}
-\Delta_g u &= |v|^{2^*-1} v \\
-\Delta_g v &= f(x) |u|^{p_2-2} u - b(x) |u|^{p-2} u \\
\end{aligned}
\text{ in } M,
\]

\[
\begin{aligned}
u = v = 0 \quad \text{on } \partial M.
\end{aligned}
\]
It remains to show the positivity of solutions in Theorems 4.3A and 4.3C; this follows from an adaptation of the arguments of van der Vorst [86].

Assume first that $p = 2$, $f \geq 0$, $a$ is a positive constant and that $b(x) \leq a^2/4$. Let $u$ be a minimizing solution of (P$_1$) or (P$_3$), according to the case considered, and let $v$ be the positive solution of the problem

$$-\Delta v + \frac{a}{2} v = \left| -\Delta u + \frac{a}{2} u \right| \quad \text{in } M,$$

satisfying $u = 0$ on $\partial M$, if $M$ has boundary. It follows from the maximum principle that $v \geq |u|$. Squaring both sides of the above equation and then integrating over $M$, we obtain:

$$\int_M |\Delta v|^2 + a \int_M ( -\Delta v)v + \frac{a^2}{4} \int_M v^2 = \int_M |\Delta u|^2 + a \int_M ( -\Delta u)u + \frac{a^2}{4} \int_M u^2,$$

whence

$$\int_M |\Delta v|^2 + a \int_M |\nabla v|^2 + \int_M b(x)v^2 + \int_M \left( \frac{a^2}{4} - b(x) \right) (v^2 - u^2)$$

$$= \int_M |\Delta u|^2 + a \int_M |\nabla u|^2 + \int_M b(x)u^2.$$

Since $b(x) \leq a^2/4$ and $f(x) \geq 0$, we conclude that $J(v) \leq J(u)$, and hence $v$ is a positive minimizing solution to (P$_1$).

Now assume $1 < p < \frac{n}{2}$, $f \geq 0$, $a = 0$ and $b \leq 0$. Let $v$ be a positive solution of

$$\begin{cases}
  -\Delta v = |\Delta u| & \text{in } M, \\
  v = 0 & \text{on } \partial M.
\end{cases}$$

By the maximum principle, $v \geq |u|$. Raising this equation to the power $p$ and integrating over $M$, we conclude that

$$J(v) = \frac{\int_M |\Delta v|^p + \int_M b(x)v^p}{\int_M f(x)v^p} \leq \frac{\int_M |\Delta u|^p + \int_M b(x)|u|^p}{\int_M f(x)|u|^p} = J(u).$$

\[\blacksquare\]

### 4.4.3 Regularity

**Lemma 4.3.** [20] Let $(M, g)$ be a smooth compact Riemannian manifold without boundary of dimension $n \geq 5$. Assume that $a \in C^1(M), b \in C^0(M), c \in L^n(M)$ and that the homogeneous equation

$$\Delta_g^2 u - \text{div}_g (a(x) \nabla_g u) + b(x)u = 0 \quad \text{in } M$$

admits in $H^{2,2}(M)$ only the trivial solution. If $u \in H^{2,2}(M)$ is a weak solution of

$$\Delta_g^2 u - \text{div}_g (a(x) \nabla_g u) + b(x)u = c(x)u \quad \text{in } M, \quad (4.33)$$
then \( u \in L^s(M) \) for all \( 1 \leq s < \infty \).

**Proof.** Given \( k > 0 \), define

\[
d_k(x) = \begin{cases}
c(x) & \text{if } |c(x)| > k \text{ or } |u(x)| > k, \\
0 & \text{if } |c(x)| \leq k \text{ and } |u(x)| \leq k,
\end{cases}
\]

and

\[
e_k(x) = (c(x) - d_k(x)) u.
\]

For each \( k > 0 \), we have \( d_k \in L^{\frac{4}{n+4}}(M) \) and \( e_k \in L^\infty(M) \). Moreover, given \( \varepsilon > 0 \), there exists \( k_\varepsilon \) such that \( \|d_k\|_{L^{\frac{4}{n+4}}(M)} \leq \varepsilon \) for all \( k \geq k_\varepsilon \). It follows from the hypothesis and standard elliptic \( L^p \)-theory that the operator \( L = \Delta^2_g - \text{div}_g (a(x)\nabla_g) + b(x) : H^{4,t}(M) \to L^t(M) \) is an isomorphism for any \( 1 < t < \infty \). Therefore, for each \( 1 < s < \infty \), we may define the bounded linear operator \( T_\varepsilon : L^s(M) \to H^{4,t}(M) \), where \( t = \frac{ns}{n+4s} \), by \( T_\varepsilon w = L^{-1}(d_\varepsilon w) \). In particular, if \( u \in H^{2,2}(M) \) is a weak solution of (4.33), then

\[
u - T_\varepsilon u = L^{-1}(e_\varepsilon).
\]

(4.34)

Using the critical Sobolev embedding \( H^{4,t}(M) \hookrightarrow L^s(M) \), we may consider \( T_\varepsilon \) as an operator from \( L^s(M) \) into \( L^s(M) \). We assert that

\[
\|T_\varepsilon\|_{L(L^s(M))} \leq C \varepsilon
\]

(4.35)

for some positive constant \( C = C(s) \) and, consequently, the operator \( I - T_\varepsilon \) is invertible for every \( \varepsilon \) sufficiently small. Indeed, by the Sobolev embedding and Hölder’s inequality,

\[
\|T_\varepsilon w\|_{L^s(M)} \leq C \|T_\varepsilon w\|_{H^{\frac{ns}{n+4s}}(M)} \\
\leq C \|d_\varepsilon \| L^{\frac{4}{n+4}}(M) \\
\leq C \|d_\varepsilon \| L^{\frac{4}{n+4}}(M) \|w\|_{L^s(M)} \\
\leq C \varepsilon \|w\|_{L^s(M)}.
\]

Taking \( \varepsilon \) sufficiently small, it follows from (4.34) and (4.35) that

\[
u = \sum_{n=0}^\infty T_\varepsilon^n(L^{-1}(e_\varepsilon)),
\]

which ends the proof of this lemma, since \( L^{-1}(e_\varepsilon) \in L^s(M) \) for all \( 1 \leq s < \infty \). ■

The same proof applies to the next lemma.
Lemma 4.4. [20] Let \((M,g)\) be a smooth compact Riemannian manifold with boundary of dimension \(n \geq 5\). Assume that \(a \in C^1(M), b \in C^0(M), c \in L^\frac{n}{2}(M)\) and that the homogeneous problem
\[
\begin{cases}
\Delta_g^2 u - \text{div}_g (a(x) \nabla_g u) + b(x) u = 0 & \text{in } M, \\
u = \nabla_g u = 0 & \text{on } \partial M,
\end{cases}
\]

admits in \(H^{2,2}_0(M)\) only the trivial solution. If \(u \in H^{2,2}_0(M)\) is a weak solution of
\[
\begin{cases}
\Delta_g^2 u - \text{div}_g (a(x) \nabla_g u) + b(x) u = c(x) u & \text{in } M, \\
u = \nabla_g u = 0 & \text{on } \partial M,
\end{cases}
\]
then \(u \in L^s(M)\) for all \(1 \leq s < \infty\).

Lemma 4.5. [20] Let \((M,g)\) be a smooth compact Riemannian manifold with boundary of dimension \(n \geq 3\) and \(1 < p < \frac{n}{2}\). Assume that \(b \in C^0(M), c \in L^\frac{n}{2}(M)\) and that either \(a = 0\), or \(p = 2\) and \(a\) is a nonnegative constant. If \(u \in H^{2,p}(M) \cap H^{1,p}_0(M)\) is a weak solution of
\[
\begin{cases}
\Delta_g \left(|\Delta_g u|^{p-2} \Delta_g u\right) - \text{div}_g \left(a |\nabla_g u|^{p-2} \nabla_g u\right) + b(x) |u|^{p-2} u = c(x) |u|^{p-2} u & \text{in } M, \\
u = \Delta_g u = 0 & \text{on } \partial M,
\end{cases}
\]
then \(u \in L^s(M)\) for all \(1 \leq s < \infty\).

Proof. Assume first \(a = 0\). Denoting \(c_0 = c - b \in L^\frac{n}{2}(M)\), (4.37) takes the form
\[
\begin{cases}
\Delta_g \left(|\Delta_g u|^{p-2} \Delta_g u\right) = c_0(x) |u|^{p-2} u & \text{in } M, \\
u = \Delta_g u = 0 & \text{on } \partial M.
\end{cases}
\]
In order to obtain regularity, it is convenient to write (4.38) as a coupled elliptic system with Dirichlet boundary condition. Define
\[v = -|\Delta_g u|^{p-2} \Delta_g u \in L^\frac{n}{2}(M)\].

We assert that \(v \in H^{2,q}(M) \cap H^{1,q}_0(M)\), with \(q = \frac{np}{(n+2)p-n} > 1\). Indeed, clearly
\[-\int_M v \Delta_g \varphi \, dv_g = \int_M c_0(x) |u|^{p-2} u \varphi \, dv_g\]
for every \(\varphi \in H^{2,p}(M) \cap H^{1,p}_0(M)\). Noticing that \(u \in L^\frac{n}{2}(M)\) implies \(c_0(x) |u|^{p-2} u \in L^q(M)\), let \(w \in H^{2,q}(M) \cap H^{1,q}_0(M)\) be a solution of the Dirichlet problem
\[
\begin{cases}
-\Delta_g w = c_0(x) |u|^{p-2} u & \text{in } M, \\
w = 0 & \text{on } \partial M.
\end{cases}
\]
It follows that
\[ \int_M (v - w) \Delta_g \varphi \, dv_g = 0 \]
for every \( \varphi \in H^{2,p}(M) \cap H^{1,p}_0(M) \). Hence, \( v = w \), proving our assertion. We can thus rewrite (4.38) as
\[
\begin{cases}
-\Delta_g u = |v|^{\frac{2}{p-2}} v \\
-\Delta_g v = c_0(x) |u|^{p-2} u \\
u = v = 0 \quad \text{on } \partial M.
\end{cases}
\]
(4.39)

Given \( k > 0 \), define
\[
d_k(x) = \begin{cases} 
  c_0(x) & \text{if } |c_0(x)| > k \text{ or } |u(x)| > k, \\
  0 & \text{if } |c_0(x)| \leq k \text{ and } |u(x)| \leq k,
\end{cases}
\]
and
\[
e_k(x) = (c_0(x) - d_k(x)) |u|^{p-2} u.
\]
Again, we have \( d_k \in L^\infty(M) \) and \( e_k \in L^\infty(M) \) for every \( k > 0 \). Furthermore, given \( \varepsilon > 0 \), there exists \( k_\varepsilon \) such that \( \|d_k\|_{L^\infty(M)} \leq \varepsilon \) for all \( k \geq k_\varepsilon \).

Since \( -\Delta_g : H^{2,t}(M) \cap H^{1,t}_0(M) \to L^t(M) \) is an isomorphism for each \( 1 < t < \infty \), given
\[
\max \left\{ \frac{n}{n-2}, \frac{1}{p-1} \right\} < t < \frac{n}{2(p-1)},
\]
we can write
\[ v - T_\varepsilon v = (-\Delta_g)^{-1}(e_k), \]
(4.40)
where \( T_\varepsilon : L^t(M) \to H^{2-\frac{n}{p+t}}(M) \cap H^{1-\frac{n}{p+t}}_0(M) \) is the homogeneous operator
\[
T_\varepsilon \omega = (-\Delta_g)^{-1} \left( d_{k_\varepsilon} \left| (-\Delta_g)^{-1} \left( |\omega|^{\frac{2}{p-2}} \omega \right) \right|^{p-2} \left( -\Delta_g \right)^{-1} \left( |\omega|^{\frac{2}{p-2}} \omega \right) \right).
\]
Thanks to the critical Sobolev embedding \( H^{2-\frac{n}{p+t}}(M) \hookrightarrow L^t(M) \), we may see \( T_\varepsilon \) as an operator from \( L^t(M) \) into \( L^t(M) \). Considering the usual norm on the space of homogeneous operators, we claim that
\[ \|T_\varepsilon\| \leq C \varepsilon \]
(4.41)
for some positive constant \( C = C(t) \). Indeed, by the boundedness of \( (-\Delta_g)^{-1} \) and Hölder’s
inequality, we have

\[ \|T_tw\|_{L^s(M)} \leq C \left\| d_{k_s} \left( (-\Delta_g)^{-1} \left( |w|^\frac{2}{p+2} w \right) \right)^{p-2} (-\Delta_g)^{-1} \left( |w|^\frac{2}{p+2} w \right) \right\|_{L^{\frac{n}{n-p+1}}(M)} \]

\[ \leq C \|d_{k_s}\|_{L^{\frac{n}{n-p+1}}(M)} \left\| (-\Delta_g)^{-1} \left( |w|^\frac{2}{p+2} w \right) \right\|_{L^{\frac{n}{n-2(p-1)}}(M)} \]

\[ \leq C \|w\|_{L^{\frac{n}{n-2(p-1)}}(M)} \]

which proves the assertion.

Choose \( t = \frac{p}{p-1} \). Noticing that \( v \in L'(^M) \) and the space of homogeneous operators under the standard norm is Banach, it follows from (4.40) and (4.41) that

\[ v = \sum_{n=0}^\infty T_n^s(L^{-1}(\epsilon_{k_s})) \]

if \( \varepsilon \) is sufficiently small. This implies that \( v \in L^t(M) \) for every \( \max \left\{ \frac{n}{n-2}, \frac{1}{p-1} \right\} < t < \frac{n}{2(p-1)} \). Let

\[ t = \frac{n s}{(p-1)(n+2)} \]

with \( s > \max \left\{ \frac{(p-1)n}{n-2p}, \frac{n}{n-2} \right\} \). Clearly \( t \) is in the admissible range. Then, from the critical Sobolev embedding \( H^{2, t(p-1)}(M) \hookrightarrow L^s(M) \), it follows that

\[ \|u\|_{L^s(M)} = \left\| (-\Delta_g)^{-1} \left( |v|^{\frac{2}{p+2}} v \right) \right\|_{L^s(M)} \leq C \left\| (-\Delta_g)^{-1} \left( |v|^{\frac{2}{p+2}} v \right) \right\|_{H^{2, t(p-1)}(M)} \]

\[ \leq C \left\| |v|^{\frac{2}{p+2}} v \right\|_{L^{t(p-1)}(M)} = C \|v\|_{L^t(M)} \]

and hence we conclude that \( u \in L^s(M) \) for all \( s > \max \left\{ \frac{(p-1)n}{n-2p}, \frac{n}{n-2} \right\} \). This finishes the proof in the case \( a = 0 \).

If \( p = 2 \) and \( a \) is a nonnegative constant, we consider instead the system

\[
\begin{align*}
-\Delta_g u &= v \\
-\Delta_g v + av &= c_0(x)u \\
u &= v = 0 & \text{on } \partial M,
\end{align*}
\]

and the proof is analogous. ■
4.4.4 Proof of Corollary 4.2

Proceeding as in the proof of Theorem 4.2, consider a geodesic ball $B_\delta(x_0) \subset (M\setminus\partial M)$, a radial cutoff function $\eta \in C^0_0(B_\delta)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{\frac{\delta}{2}}$ and $\eta = 0$ in $\mathbb{R}^n \setminus B_{\delta}$, and define, up to the exponential chart $\exp_{x_0}$,

$$u_\varepsilon(x) = \eta(x)z_\varepsilon(x),$$

where

$$z_\varepsilon(x) = \varepsilon^{-\frac{n}{p}}z\left(\frac{x}{\varepsilon}\right)$$

with $z$ being a positive radial minimizer for the Sobolev quotient (4.3). By Theorems 4.3A, 4.3B or 4.3C, according to which case we are dealing with, it is enough to show that for some sufficiently small $\varepsilon$ we have

$$\frac{\int_M |\Delta_g u_\varepsilon|^p dv_g + \int_M b(x) |u_\varepsilon|^p dv_g}{\left(\int_M f(x) |u_\varepsilon|^{p^*_2} dv_g\right)^{p/p^*_2}} \leq \frac{1}{K^p f(x_0)^{p/p^*_2}}.$$

Considering the expansions $\eta(x) = 1 + O(r^3)$, (4.24) and

$$f(x) = f(x_0) + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij} f(x_0) x_i x_j + O(r^3),$$

and noticing that $\Delta_g f(x_0) = \sum_{i=1}^n \partial_i f(x_0)$, we can write

$$\int_M f(x) |u_\varepsilon|^{p^*_2} dv_g$$

$$= f(x_0) \int_M |u_\varepsilon|^{p^*_2} dv_g + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij} f(x_0) \int_M |u_\varepsilon|^{p^*_2} x_i x_j dv_g + \int_M |u_\varepsilon|^{p^*_2} O(r^3) dv_g$$

$$= f(x_0) \int_{B_\delta} |u_\varepsilon|^{p^*_2} dx + \frac{3\Delta_g f(x_0) - f(x_0) \text{ Scal}_g(x_0)}{6n} \int_{B_\delta} |u_\varepsilon|^{p^*_2} \varepsilon^2 dx$$

$$+ \int_{B_\delta} |u_\varepsilon|^{p^*_2} O(r^3) dx$$

$$= f(x_0) \int_{\mathbb{R}^n} z^{p^*_2} dx - f(x_0) \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}} z^{p^*_2} dx + \frac{3\Delta_g f(x_0) - f(x_0) \text{ Scal}_g(x_0)}{6n} \varepsilon^2 \int_{\mathbb{R}^n} z^{p^*_2} \varepsilon^2 dx$$

$$- \frac{3\Delta_g f(x_0) - f(x_0) \text{ Scal}_g(x_0)}{6n} \varepsilon^2 \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{2}}} z^{p^*_2} \varepsilon^2 dx + \varepsilon^3 \int_{B_{\frac{\delta}{2}}} z^{p^*_2} O(r^3) dx.$$
Thus, we get

\[
\left( \int_M f(x) |u_\varepsilon|_{p_\varepsilon}^{p_2} \ dv_\varepsilon \right)^{p/p_\varepsilon} = f(x_0)^{p/p_\varepsilon} \|z\|_{L^{p_\varepsilon}(\mathbb{R}^n)}^p \left[ 1 + \frac{p}{np_\varepsilon} \frac{I_2}{T_1} \left( \frac{\Delta_g f(x_0)}{2f(x_0)} - \frac{\text{Scal}(x_0)}{6} \right) \varepsilon^2 + o(\varepsilon^2) \right].
\]  

(4.42)

For the next estimate, we write

\[
\int_M |\Delta_g u_\varepsilon|^p \ dv_\varepsilon = \int_{B_{\frac{1}{2}}} |\eta \Delta_g z_\varepsilon + 2 \langle \nabla_g \eta, \nabla_g z_\varepsilon \rangle + (\Delta_g \eta) z_\varepsilon|^p \ dv_\varepsilon
\]

(4.43)

\[
= \int_{B_{\frac{1}{2}}} |\Delta_g z_\varepsilon|^p \ dv_\varepsilon + \int_{B_{\frac{1}{2}} \setminus B_{\frac{1}{2}}} |\eta \Delta_g z_\varepsilon + 2 \langle \nabla_g \eta, \nabla_g z_\varepsilon \rangle + (\Delta_g \eta) z_\varepsilon|^p \ dv_\varepsilon.
\]

The estimate of the first term of the right-hand side of (4.43) requires the elementary inequality

\[|1 + t|^p \leq 1 + pt + C_{1,p} t^2 + C_{2,p} |t|^p,\]

valid for all \(t \in \mathbb{R}\), where \(C_{1,p}\) and \(C_{2,p}\) are some large constants depending only on \(p\), except in the case \(1 \leq p \leq 2\), when \(C_{1,p} = 0\). Writing the Laplacian in normal geodesic coordinates, since \(\Delta z_\varepsilon(r) < 0\) and \(z_\varepsilon'(r) < 0\) for all \(r > 0\), it follows that

\[
|\Delta_g z_\varepsilon|^p = |\Delta z_\varepsilon + z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|})|^p = |\Delta z_\varepsilon|^p \left| 1 + \frac{z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|})}{\Delta z_\varepsilon} \right|^p
\]

\[
\leq |\Delta z_\varepsilon|^p \left( 1 + p \frac{z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|})}{\Delta z_\varepsilon} + C_{1,p} \left| \frac{z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|})}{\Delta z_\varepsilon} \right|^2 + C_{2,p} \left| \frac{z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|})}{\Delta z_\varepsilon} \right|^p \right)
\]

\[
= \left| \Delta z_\varepsilon \right|^p + p \left| \Delta z_\varepsilon \right|^{p-1} \left| z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|}) \right|^2 + C_{1,p} \left| \Delta z_\varepsilon \right|^{p-2} \left| z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|}) \right|^2
\]

\[+ C_{2,p} \left| z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|}) \right|^p.
\]

Hence,

\[
\int_{B_{\frac{1}{2}}} |\Delta_g z_\varepsilon|^p \ dv_\varepsilon \leq \int_{B_{\frac{1}{2}}} |\Delta z_\varepsilon|^p \ dv_\varepsilon + p \int_{B_{\frac{1}{2}}} \left| \Delta z_\varepsilon \right|^{p-1} |z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|})|^2 \ dv_\varepsilon
\]

\[+ C_{1,p} \int_{B_{\frac{1}{2}}} \left| \Delta z_\varepsilon \right|^{p-2} \left| z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|}) \right|^2 \ dv_\varepsilon
\]

\[+ C_{2,p} \int_{B_{\frac{1}{2}}} \left| z_\varepsilon'(r) \partial_r (\ln \sqrt{|\det g|}) \right|^p \ dv_\varepsilon.
\]

Again using (4.24), it follows by straightforward computation from the asymptotic behaviors of
\[ \int_{B_{\frac{1}{2}}} |\Delta z|^{p} \, dv_g = \int_{\mathbb{R}^n} |\Delta z|^{p} \, dx - \int_{\mathbb{R}^n \setminus B_{\frac{1}{2}}} |\Delta z|^{p} \, dx \\
- \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 \int_{\mathbb{R}^n} |\Delta z|^{p} r^2 \, dx \\
+ \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 \int_{\mathbb{R}^n \setminus B_{\frac{1}{2}}} |\Delta z|^{p} r^2 \, dx \\
+ \varepsilon^3 \int_{B_{\frac{1}{2}}} |\Delta z|^{p} O(\varepsilon^3) \, dx \\
= \|\Delta z\|_{L^p(\mathbb{R}^n)}^{p} - I_4 \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 + o(\varepsilon^2), \] 
if \(1 < p < \frac{n+2}{4}\),

\[ \int_{B_{\frac{1}{2}}} |\Delta z|^{p-1} |z'_\varepsilon(r)| \partial_r (\ln \sqrt{\det g}) \, dv_g = \frac{-\text{Scal}_g(x_0)}{3n} \int_{B_{\frac{1}{2}}} |\Delta z|^{p-1} |z'_\varepsilon(r)| r \, dx \\
+ \int_{B_{\frac{1}{2}}} |\Delta z|^{p-1} |z'_\varepsilon(r)| O(r^2) \, dx \\
= \frac{-\text{Scal}_g(x_0)}{3n} \varepsilon^2 \int_{\mathbb{R}^n} |\Delta z|^{p-1} |z'(r)| r \, dx \\
+ \frac{\text{Scal}_g(x_0)}{3n} \varepsilon^2 \int_{\mathbb{R}^n \setminus B_{\frac{1}{2}}} |\Delta z|^{p-1} |z'(r)| r \, dx \\
+ \varepsilon^3 \int_{B_{\frac{1}{2}}} |\Delta z|^{p-1} |z'(r)| O(r^2) \, dx \\
= -I_4 \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 + o(\varepsilon^2) \]
if \(n \geq 5 \) and \( \frac{n(n+2)}{n^2+4} < p < \frac{n+2}{4}\),

\[ \int_{B_{\frac{1}{2}}} |\Delta z|^{p-2} |z'_\varepsilon(r)\partial_r (\ln \sqrt{\det g})|^2 \, dv_g = \varepsilon^4 \int_{B_{\frac{1}{2}}} |\Delta z|^{p-2} |z'(r)|^2 O(r^2) \, dx = o(\varepsilon^2) \]
if \(n \geq 5 \) and \( 2\frac{n-1}{n} < p < \frac{n+2}{4}\) (recall that for \(p \leq 2\) this term plays no role, since in this case \(C_{1,p} = 0\)), and

\[ \int_{B_{\frac{1}{2}}} |z'_\varepsilon(r)\partial_r (\ln \sqrt{\det g})|^p \, dv_g = \varepsilon^{2p} \int_{B_{\frac{1}{2}}} |z'(r)|^p O(r^p) \, dx = o(\varepsilon^2) \]
if \(n \geq 5 \) and \( \frac{n+2}{n} < p < \frac{n+2}{4}\).

Finally, we compute the second term of the right-hand side of (4.43). For \(n \geq 5\) and
Since this inequality is equivalent to (4.8), the proof is finished.

\[ \int_{B_\frac{1}{2}} |\Delta_g z_\varepsilon + 2 \langle \nabla_g \eta, \nabla_g z_\varepsilon \rangle + (\Delta_g \eta) z_\varepsilon|^p \, dv_g \]

\[ = O(1) \left[ \int_{B_\frac{1}{2}} |\Delta_g z_\varepsilon|^p \, dx + \int_{B_\frac{1}{2}} |\nabla_g z_\varepsilon|^p \, dx + \int_{B_\frac{1}{2}} z_\varepsilon^p \, dx \right] \]

\[ = O(1) \left[ \int_{B_\frac{1}{2}\setminus B_\frac{1}{4}} |\Delta z|^p \, dx + \int_{B_\frac{1}{2}\setminus B_\frac{1}{4}} |z'_e(r)|^p \, r^p \, dx + \int_{B_\frac{1}{2}\setminus B_\frac{1}{4}} |z'_e(r)|^p \, dx + \int_{B_\frac{1}{2}\setminus B_\frac{1}{4}} z^p \, dx \right] \]

\[ = O(1) \left[ \int_{B_\frac{1}{4}\setminus B_\frac{1}{8}} |\Delta z|^p + \varepsilon^{2p} \int_{B_\frac{1}{4}\setminus B_\frac{1}{8}} |z'_e(r)|^p \, r^p + \varepsilon^p \int_{B_\frac{1}{4}\setminus B_\frac{1}{8}} |z'_e(r)|^p + \varepsilon^{2p} \int_{B_\frac{1}{4}\setminus B_\frac{1}{8}} z^p \right] \]

\[ = o(\varepsilon^2). \]

Therefore,

\[ \|\Delta_g u\varepsilon\|_{L^p(M)}^p \leq \|\Delta z\|_{L^p(\mathbb{R}^n)}^p - I_4 \frac{\text{Scal}_g(x_0)}{6n} \varepsilon^2 - pI_1^2 \frac{\text{Scal}_g(x_0)}{3n} \varepsilon^2 + o(\varepsilon^2) \quad (4.44) \]

Finally, considering the expansions \( \eta(x) = 1 + O(r^3) \), \( dv_g = 1 + O(r^2) \), and

\[ b(x) = b(x_0) + \sum_{i=1}^n \partial_i b(x_0)x_i + O(r^2), \]

noticing that \( \int_{B_\frac{1}{2}} z^p x_i \, dx = 0 \), we obtain for \( n \geq 5 \) and \( \frac{n+2}{n} < p < \frac{n+2}{4} \) that

\[ \int_{M} b(x) |u\varepsilon|^p \, dv_g = b(x_0)\varepsilon^{2p} \int_{B_\frac{1}{2}} z^p \, dx + \varepsilon^{2p+2} \int_{B_\frac{1}{4}} z^p O(r^2) \, dx \]

\[ = o(\varepsilon^2). \]

Putting (4.42), (4.44) and (4.45) together, we get

\[ \int_{M} |\Delta_g u\varepsilon|^p \, dv_g + \int_{M} b(x) |u\varepsilon|^p \, dv_g \]

\[ \left( \int_{M} f(x) |u\varepsilon|^p \, dv_g \right)^{p/p^*_2} \]

\[ \leq \frac{\|\Delta z\|_{L^p(\mathbb{R}^n)}^p}{f(x_0)^{p/p^*_2} \|z\|_{L^{p^*_2}(\mathbb{R}^n)}} \frac{1}{1 + \frac{p \eta_2}{np^*_2 I_1} \left( \frac{\Delta_g f(x_0)(x_0) - \text{Scal}_g(x_0)}{2} \right) \varepsilon^2 + o(\varepsilon^2)} \]

\[ < \frac{1}{K_p^p f(x_0)^{p/p^*_2}} \]

if

\[ \frac{I_4 \text{Scal}_g(x_0)}{I_3} + \frac{p \eta_2}{np^*_2 I_1} \left( \frac{\Delta_g f(x_0)(x_0) - \text{Scal}_g(x_0)}{2} \right) > 0. \]

Since this inequality is equivalent to (4.8), the proof is finished. \[ \blacksquare \]
4.5 The role of the geometry on Brezis-Nirenberg type problems

Proof of Theorems 4.4A-C. Because $\lambda < \lambda_1$, the functional

$$J(u) = \int_M |\Delta_g u|^p \, dv_g - \lambda \int_M |u|^p \, dv_g$$

is coercive on $E_i$. Therefore, if $p = 2$, $n \geq 7$ and $\text{Scal}_g(x_0) > 0$, the proof follows from Corollary 4.2, taking $f \equiv 1$, and from (4.30). In the other cases, we apply Theorems 4.3A-C, according to the situation. Thus, we only need to show that $\inf_{V_t} J < 1/K^p_2$ or, equivalently,

$$\inf_{u \in E \setminus \{0\}} \frac{\int_M |\Delta_g u|^p \, dv_g - \lambda \int_M |u|^p \, dv_g}{\left(\int_M |u|^{p^*_2} \, dv_g\right)^{p/p^*_2}} < \frac{1}{K^p_2}. \quad (4.46)$$

We obtain (4.46) by proving that for all sufficiently small $\varepsilon$ there holds

$$\frac{\int_M |\Delta_g u_\varepsilon|^p \, dv_g - \lambda \int_M |u_\varepsilon|^p \, dv_g}{\left(\int_M |u_\varepsilon|^{p^*_2} \, dv_g\right)^{p/p^*_2}} < \frac{1}{K^p_2}, \quad (4.47)$$

where $u_\varepsilon \in C_0^\infty(M)$ are the functions defined in the proof of Corollary 4.2: here we choose $x_0$ to be any interior point of positive scalar curvature if there is one, or any interior point of a flat neighborhood of the manifold, when this is the case.

If $p = 2$, $n = 6$, $\text{Scal}_g(x_0) > 0$ and $\varepsilon$ is small enough, according to the estimates we did in the proof of Theorem 4.2 we have

$$\frac{\int_M |\Delta_g u_\varepsilon|^2 \, dv_g - \lambda \int_M |u_\varepsilon|^2 \, dv_g}{\left(\int_M |u_\varepsilon|^{2^*_2} \, dv_g\right)^{2/2^*_2}} = \frac{1}{K^2_2} \frac{1 - \text{Scal}_g(x_0) O(\varepsilon^2 \ln \varepsilon)}{1 - O(\varepsilon^2)} < \frac{1}{K^2_2}.$$

Now, let $M$ be flat in some neighborhood and $\frac{n}{n-2} < p < \sqrt{\frac{n}{2}}$. From Appendix B, it follows that

$$\int_M |\Delta u_\varepsilon|^p \, dx
= \int_{\mathbb{R}^n} |\Delta z|^p \, dx - \int_{\mathbb{R}^n \setminus B_{\frac{1}{2}}^+} |\Delta z|^p \, dx + \int_{B_{1/2}^+ \setminus B_{1/2}} |\eta \Delta z_\varepsilon + 2(\nabla \eta, \nabla z_\varepsilon) + (\Delta \eta) z_\varepsilon|^p \, dx
= \int_{\mathbb{R}^n} |\Delta z|^p \, dx - \int_{\mathbb{R}^n \setminus B_{\frac{1}{2}}^+} |\Delta z|^p \, dx
+ O(1) \left( \int_{B_{1/2}^+ \setminus B_{1/4}^+} |\Delta z|^p \, dx + \varepsilon^p \int_{B_{1/4}^+ \setminus B_{1/8}^+} |\nabla z|^p \, dx + \varepsilon^{2p} \int_{B_{1/8}^+ \setminus B_{1/16}^+} z^p \, dx \right)
= \|\Delta z\|_{L^p(\mathbb{R}^n)}^p + o(\varepsilon^{2p}),$$

$$\int_M |u_\varepsilon|^p \, dx = \varepsilon^{2p} \int_{B_{1/2}^+} z^p \, dx + \varepsilon^{2p+2} \int_{B_{1/8}^+ \setminus B_{1/16}^+} z^p O(r^2) \, dx = O(\varepsilon^{2p}),$$
and
\[ \int_M |u_\varepsilon|^2 \, dx = \int_{\mathbb{R}^n} z^{p^2} \, dx - \int_{\mathbb{R}^n \setminus B_{\frac{r}{2}}} z^{p^2} \, dx + \varepsilon^2 \int_{B_{\frac{r}{2}} \setminus B_{\frac{r}{2r}}} z^{p^2} O(r^2) \, dx \]
\[ = \|z\|^p_{L^p(\mathbb{R}^n)} - o(\varepsilon^{2p}). \]

Therefore, if \( \varepsilon \) is sufficiently small and \( \lambda > 0 \), we have
\[ \int_M |\Delta u_\varepsilon|^p \, dx - \lambda \int_M |u_\varepsilon|^p \, dx \leq \frac{\|\Delta z\|^p_{L^p(\mathbb{R}^n)} - \lambda O(\varepsilon^{2p})}{\|z\|^p_{L^p(\mathbb{R}^n)} - o(\varepsilon^{2p})} \]
\[ = \frac{1}{K_2} \frac{1 - \lambda O(\varepsilon^{2p})}{1 - o(\varepsilon^{2p})} < \frac{1}{K_2}. \]

If \( p = \sqrt{n/2} \), then
\[ \int_M |\Delta u_\varepsilon|^p \, dx = \frac{\|\Delta z\|^p_{L^p(\mathbb{R}^n)} - \lambda O(\varepsilon^{2p})}{\|z\|^p_{L^p(\mathbb{R}^n)} - o(\varepsilon^{2p})}, \]
\[ \int_M |u_\varepsilon|^p \, dx = O(\varepsilon^{2p} |\ln \varepsilon|), \]
and
\[ \int_M |u_\varepsilon|^2 \, dx = \|z\|^p_{L^p(\mathbb{R}^n)} - O(\varepsilon^{2p}), \]

hence (4.47) also holds in this case.

In order to show that there are positive solutions, consider any nontrivial solution \( u \in E_3 \) which minimizes the quotient in (4.46). Let \( w \in E_3 \) be the positive solution to the Dirichlet problem
\[ \begin{cases} -\Delta_g w = |\Delta_g u| & \text{in } M, \\ w = 0 & \text{on } \partial M. \end{cases} \]

By the maximum principle, \( w \geq |u| \) in \( M \). Therefore, if \( \lambda \geq 0 \),
\[ \frac{\int_M |\Delta_g w|^p \, dv_g - \lambda \int_M |u|^p \, dv_g}{\left(\int_M |w|^p \, dv_g\right)^{p/p^2}} \leq \frac{\int_M |\Delta_g w|^p \, dv_g - \lambda \int_M |u|^p \, dv_g}{\left(\int_M |u|^p \, dv_g\right)^{p/p^2}} < \frac{1}{K_2^p} \]
and thus \( w \) is a positive solution to the problem (BN3).

The nonexistence of positive solutions for \( \lambda \geq \lambda_1 \) in the case (BN1) follows immediately from direct integration and in the case (BN3) follows from Proposition 2.12 in [73], after reformulation in terms of elliptic systems. The positivity in Theorem 4.4C follows immediately from Theorem 4.3C taking \( b(x) = -\lambda. \) ■
Appendix A: Equivalence of Norms

Let \((M, g)\) be a smooth compact \(n\)-dimensional Riemannian manifold, with or without boundary. We assume this to mean that \(\partial M\) is a smooth manifold with the induced metric of \(M\). Choose a finite set of parametrizations \(\{\phi_k : \Omega_k \to U_k\}_{1 \leq k \leq N}\) such that \(\{U_k\}_{1 \leq k \leq N}\) is a covering of \(M\), and let \(\{\eta_k\}_{1 \leq k \leq N}\) be a partition of unity subordinated to this covering. Define the following norm in \(C^\infty(M)\):

\[
\|u\|_{2,p} = \sum_{k=1}^{N} \|\eta_k u \circ \phi_k\|_{H^2_p(\Omega_k)}.
\]

The change of coordinates theorem ensures that the above definition does not depend neither on the chosen parametrizations, nor on the partition of unity. Moreover, in \(C^\infty(M)\), the norm \(\|\cdot\|_{2,p}\) is equivalent to the norm \(\|\cdot\|_{H^2_p(M)}\). Indeed, writing \(\nabla_g\) and \(\nabla_g^2\) in local coordinates, for \(u \in C^\infty(M)\) we find

\[
\|u\|_{H^2_p(M)} = \left\|\sum \eta_k u \right\|_{H^2_p(M)} \leq \sum \|\eta_k u\|_{H^2_p(U_k)} \leq C \sum \left(\|\nabla^2 ((\eta_k u) \circ \phi_k)\|_{L^p(\Omega_k)} + \|\partial ((\eta_k u) \circ \phi_k)\|_{L^p(\Omega_k)} + \|\eta_k u \circ \phi_k\|_{L^p(\Omega_k)}\right)
\]

\[
\leq C \|u\|_{2,p},
\]

On the other hand, \(L^p\) theory applied to the elliptic operator \(\Delta_g\) (see Lemma 9.17 of [47]) gives:

\[
\|\eta_k u \circ \phi_k\|_{H^2_p(\Omega_k)} \leq C \left(\|\Delta_g ((\eta_k u) \circ \phi_k)\|_{L^p(\Omega_k)} + \|\eta_k u \circ \phi_k\|_{L^p(\Omega_k)}\right)
\]

\[
= C \left(\|\Delta_g \eta_k u\|_{L^p(U_k)} + \|\eta_k u\|_{L^p(U_k)}\right)
\]

\[
\leq C \left(\|\Delta_g u\|_{L^p(M)} + \|\nabla_g u\|_{L^p(M)} + \|u\|_{L^p(M)}\right)
\]

\[
\leq C \left(\|\nabla_g^2 u\|_{L^p(M)} + \|\nabla_g u\|_{L^p(M)} + \|u\|_{L^p(M)}\right),
\]

since \(|\Delta_g u|^2 \leq n |\nabla_g^2 u|^2\), whence the equivalence follows. Consequently, \(\|\cdot\|_{2,p}\) is a norm in \(H^2_p(M)\) equivalent to \(\|\cdot\|_{H^2_p(M)}\).

Now we are ready to prove the equivalence of the norms \(\|
\|_{H^2_p(M)}\) and

\[
\|u\|_{H^2_p(M)} = \left(\|\Delta_g u\|_{L^p(M)}^p + \|u\|_{L^p(M)}^p\right)^{1/p}.
\]

**Proposition A.1.** [20] Let \((M, g)\) be a smooth compact Riemannian manifold, with or without boundary. The norm \(\|\cdot\|_{H^2_p(M)}\) is equivalent to the norm \(\|\cdot\|_{H^2_p(M)}\).

**Proof.** According to the above discussion, it suffices to show the equivalence of the norms \(\|\cdot\|_{2,p}\) and \(\|\cdot\|_{H^2_p(M)}\). For convenience of notation, denote \(\eta_k u = (\eta_k u) \circ \phi_k\). By \(L^p\)-estimates for linear elliptic operators, for every \(u \in H^2_p(M)\) there exists a constant \(C > 0\) independent...
of $u$ such that

$$\|\partial^2(\eta_k u)\|_{L^p(\Omega_k)} + \|\partial(\eta_k u)\|_{L^p(\Omega_k)} \leq C \left( \|\Delta_g(\eta_k u)\|_{L^p(\Omega_k)} + \|\eta_k u\|_{L^p(\Omega_k)} \right).$$

In the following, $C$ will denote several possibly different constants independent of $u$. Expanding the right-hand side of this inequality, we get

$$\|\partial^2(\eta_k u)\|_{L^p(\Omega_k)} + \|\partial (\eta_k u)\|_{L^p(\Omega_k)} \leq C \left( \|\eta_k \Delta_g u\|_{L^p(\Omega_k)} + \|\nabla_g(\eta_k \cdot \nabla_g u)\|_{L^p(\Omega_k)} + \|\Delta_g(\eta_k u)\|_{L^p(\Omega_k)} + \|\eta_k u\|_{L^p(\Omega_k)} \right) \quad (4.48)$$

$$\leq C \left( \|\Delta_g u\|_{L^p(M)} + \|u\|_{L^p(M)} \right) + C \|\nabla_g u\|_{L^p(M)}.$$ 

On the other hand, by interpolation of lower derivatives norms in Sobolev spaces in open sets of the Euclidean space, we have

$$\|\nabla_g u\|_{L^p(M)}^p = \left( \sum_{\eta_k} \eta_k \nabla_g u \right)_{L^p(M)}^p \leq N^p \sum \|\eta_k \nabla_g u\|_{L^p(M)}^p \leq C \sum \int_{\Omega_k} |\eta_k \nabla_g u|^p \, dx$$

$$\leq C \sum \left( \int_{\Omega_k} |\nabla_g(\eta_k u)|^p \, dx + \int_{\Omega_k} |(\nabla_g) u|^p \, dx \right) \quad (4.49)$$

$$\leq C \sum \left( \epsilon \int_{\Omega_k} |\partial^2(\eta_k u)|^p \, dx + C \epsilon \int_{\Omega_k} |\eta_k u|^p \, dx \right) + C \int_M |u|^p \, dv_g$$

$$\leq C\epsilon \sum \int_{\Omega_k} |\partial^2(\eta_k u)|^p \, dx + C \int_M |u|^p \, dv_g.$$ 

Hence, choosing $\epsilon$ small enough and combining (4.48) and (4.49), we obtain

$$\|u\|_{2,p} \leq \sum \left( \|\Delta_g u\|_{L^p(M)} + \|u\|_{L^p(M)} \right) \leq C \left( \|\Delta_g u\|_{L^p(M)} + \|u\|_{L^p(M)} \right).$$

The inequality in the opposite direction follows immediately from a simple computation. ■

**Proposition A.2.** [20] Let $(M, g)$ be a compact Riemannian manifold with boundary. In the Sobolev spaces $H^2_0(M)$ and $H^2(M) \cap H^1_0(M)$, the norm $\|\cdot\|_{H^2_p(M)}$ is equivalent to the norm

$$\|u\|_{H^2_p(M)} = \|\Delta_g u\|_{L^p(M)}.$$ 

**Proof.** Using Theorem 3.72 of [7] and mimicking the proof of Lemma 9.17 of [47], we obtain that there exists a constant $C$ independent of $u$ such that

$$\|u\|_{H^2_p(M)} \leq C \|\Delta_g u\|_{L^p(M)}.$$ 

■
Appendix B: Estimates for integrals arising from second order mini-
mizers whose asymptotic behaviors are known

Let
\[ \frac{1}{K(n, p)} = \inf_{u \in D^{2,p}(\mathbb{R}^n) \setminus \{0\}} \frac{\|\Delta u\|_{L_p(\mathbb{R}^n)}}{\|u\|_{L_p^*(\mathbb{R}^n)}}. \]
where \( D^{2,p}(\mathbb{R}^n) \) denotes the completion of \( C_0^\infty(\mathbb{R}^n) \) under the norm
\[ \|u\|_{D^{2,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\Delta u|^p \, dx \right)^{1/p}. \]

Since Lions [69] (I.4, Corollary I.2, p. 165), it is known that minimizing sequences for the above Sobolev quotient have convergent subsequences in \( D^{2,p}(\mathbb{R}^n) \) if \( p > 1 \), and in \( L_p^*(\mathbb{R}^n) \), if \( p = 1 \). In particular, the minimum is achieved. Moreover, if \( p > 1 \), any minimizer \( z \) satisfies, up to translation and multiplication by a nonzero constant,
\[ z \text{ and } -\Delta z \text{ are spherically symmetric, nonnegative and decreasing in } |x|. \]

For \( p = 2 \), it was shown by Edmunds, Fortunato and Janelli [43] and Lieb [67] that
\[ K(n, 2) = \frac{16}{n(n - 4)(n^2 - 4)\omega_n^{4/n}} \]
where \( \omega_n \) denotes the volume of the unit \( n \)-dimensional sphere \( S^n \subset \mathbb{R}^{n+1} \), and the set of extremal functions is precisely
\[ z(x) = c \left( \frac{1}{\lambda + |x - x_0|^2} \right)^{\frac{n-4}{2}} \]
where \( \lambda > 0 \), \( c \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^n \). Although the explicit value of \( K(n, p) \) and the exact shape of minimizers are not known for \( p \neq 2 \), the asymptotic behaviors of the extremal functions up to second order were determined by Hulshof and van der Vorst [57] for any \( 1 < p < \frac{n}{2} \):

\[ \begin{align*}
\lim_{r \to \infty} r^{n-2}z(r) &= C_1 & \text{if } & 1 < p < \frac{2n-1}{n}, \\
\lim_{r \to \infty} \frac{r^{n-2}}{\ln r} z(r) &= C_2 & \text{if } & p = \frac{2n-1}{n}, \\
\lim_{r \to \infty} r^{n-2} (\Delta z(r)) &= C_3 & \text{if } & 2\frac{n-1}{n} < p < \frac{n}{2},
\end{align*} \]  

(4.50)

\[ \begin{align*}
\lim_{r \to \infty} r^{\frac{n-2}{n}} (\Delta z(r)) &= C_4 & \text{if } & 1 < p < \frac{2n(n-1)}{n^2 + 2n - 4}, \\
\lim_{r \to \infty} r^{\frac{n-2}{n}} (\ln r) \frac{1}{r^{\frac{1}{p}}} (\Delta z(r)) &= C_5 & \text{if } & p = \frac{2n(n-1)}{n^2 + 2n - 4}, \\
\lim_{r \to \infty} r^{\frac{n-2}{n}} (\Delta z(r)) &= C_6 & \text{if } & 2\frac{n(n-1)}{n^2 + 2n - 4} < p < \frac{n}{2}.
\end{align*} \]  

(4.51)
The constants $C_i$ in these estimates depend only on $n$ and $p$. In particular, from these estimates we immediately obtain estimates for the growth of the radial derivative of $z$:

\[
\begin{align*}
\lim_{r \to \infty} r^{n-1}(-z'(r)) &= C_7 & \text{if } 1 < p < \frac{2n - 1}{n}, \\
\lim_{r \to \infty} \frac{r^{n-1}}{\ln r} (-z'(r)) &= C_8 & \text{if } p = \frac{n - 1}{n}, \\
\lim_{r \to \infty} \frac{a_n r^{n-1}}{r^n} (-z'(r)) &= C_9 & \text{if } \frac{2n - 1}{n} < p < \frac{n}{2}.
\end{align*}
\]

Indeed, since

\[
\Delta z(r) = z''(r) + \frac{n - 1}{r} z'(r) = \frac{[r^{n-1}z'(r)]'}{r^{n-1}},
\]

write

\[
r^{n-1}z'(r) = \int_0^r \Delta z(s)s^{n-1} \, ds.
\]

and use (4.51) in order to get (4.52).

**B1. Estimates used in the proof of the nonvalidity of the optimal inequality**

In this section, we present the estimates of the integrals used in order to establish the nonvalidity of the optimal inequality for $p = 2$ on manifolds with positive scalar curvature somewhere. Recall that in order for the critical Sobolev embedding to be defined in this case, we must have a priori $n \geq 5$. In the following, we somewhat abuse the language for integrals which lead to $\ln$ and, in addition, drop all constants.

**Estimate of $\|u_\varepsilon\|_{L^{2^*}(M)}^2$.**

\[
\begin{align*}
\int_{\mathbb{R}^n \setminus B_\varepsilon} z^{2^*} \, dx &= O(\varepsilon^n), \\
\int_{\mathbb{R}^n} z^{2^*} |x|^2 \, dx &= O(1), \\
\int_{\mathbb{R}^n \setminus B_\varepsilon} z^{2^*} |x|^2 \, dx &= O(\varepsilon^{n-2}), \\
\int_{B_\varepsilon^+} z^{2^*} |x|^3 \, dx &= O(1).
\end{align*}
\]
Estimate of $\|u_\varepsilon\|_{L^2(M)}^2$.

$$\int_{B_1^\varepsilon} z^2 \, dx = \begin{cases} O(\varepsilon^{n-8}) & \text{if } 5 \leq n \leq 7, \\ O(\ln|\varepsilon|) & \text{if } n = 8, \\ O(1) & \text{if } n \geq 9, \end{cases}$$

$$\int_{B_1^\varepsilon} z^2|x|^2 \, dx = \begin{cases} O(\varepsilon^{n-10}) & \text{if } 5 \leq n \leq 9, \\ O(|\ln|\varepsilon||) & \text{if } n = 10, \\ O(1) & \text{if } n \geq 11. \end{cases}$$

Estimate of $\|\Delta u_\varepsilon\|_{L^2(M)}^2$.

$$\int_{\mathbb{R}^n \setminus B_1^\varepsilon} |\Delta z|^2 \, dx = O(\varepsilon^{n-4}),$$

$$\int_{\mathbb{R}^n} |\Delta z|^2 |x|^2 \, dx = O(1) \quad \text{if } n \geq 7,$$

$$\int_{\mathbb{R}^n \setminus B_1^\varepsilon} |\Delta z|^2 |x|^2 \, dx = \begin{cases} O(\varepsilon^{-1}) & \text{if } n = 5, \\ O(|\ln|\varepsilon||) & \text{if } n = 6, \\ O(\varepsilon^{n-6}) & \text{if } n \geq 7, \end{cases}$$

$$\int_{B_1^\varepsilon} |\Delta z|^2 |x|^2 \, dx = 4\omega_{n-1}(n-4)^2 |\ln|\varepsilon|| + O(1) \quad \text{if } n = 6,$$

$$\int_{B_1^\varepsilon} |\Delta z|^2 |z'| \, dx = \begin{cases} O(\varepsilon^{-2}) & \text{if } n = 5, \\ O(\varepsilon^{-1}) & \text{if } n = 6, \\ O(|\ln|\varepsilon||) & \text{if } n = 7, \\ O(\varepsilon^{n-7}) & \text{if } n \geq 8. \end{cases}$$
\[
\int_{B_\frac{1}{n}} |\Delta z|^2 \, dx = O(\varepsilon^{n-4}),
\]
\[
\int_{B_\frac{1}{n}} |z'(r)|^2 r^2 \, dx = \begin{cases} 
O(\varepsilon^{-3}) & \text{if } n = 5, \\
O(\varepsilon^{-2}) & \text{if } n = 6, \\
O(\varepsilon^{-1}) & \text{if } n = 7, \\
O(\|\ln \varepsilon\|) & \text{if } n = 8, \\
O(\varepsilon^{-8}) & \text{if } n \geq 9.
\end{cases}
\]
\[
\int_{B_\frac{1}{n}} |z'(r)|^2 \, dx = \begin{cases} 
O(\varepsilon^{-1}) & \text{if } n = 5, \\
O(\|\ln \varepsilon\|) & \text{if } n = 6, \\
O(\varepsilon^{-8}) & \text{if } n \geq 7,
\end{cases}
\]
\[
\int_{B_\frac{1}{n}} z^2 \, dx = \begin{cases} 
O(\varepsilon^{-3}) & \text{if } n = 5, \\
O(\varepsilon^{-2}) & \text{if } n = 6, \\
O(\varepsilon^{-1}) & \text{if } n = 7, \\
O(\|\ln \varepsilon\|) & \text{if } n = 8, \\
O(\varepsilon^{-8}) & \text{if } n \geq 9.
\end{cases}
\]

**B2. Estimates used in the proof of Corollary 2 and in the proof of the role of the geometry on Brezis-Nirenberg type problems**

**Estimate of \( \int_M |u_\varepsilon|^p \, d\nu \)**

\[
\int_{\mathbb{R}_n \setminus B_\frac{1}{n}} z^p \, dx = \begin{cases} 
O(\varepsilon^{\frac{n^2(p-1)}{n^2 + 2p}}) & \text{if } 1 < p < 2 \frac{n-1}{n}, \\
O(\varepsilon^{\frac{n^2(p-1)}{n^2 + 2p}} \|\ln \varepsilon\|^{\frac{n^2}{n^2 + 2p}}) & \text{if } p = 2 \frac{n-1}{n}, \\
O(\varepsilon^{\frac{n^2}{n^2 - 2p}}) & \text{if } 2 \frac{n-1}{n} < p < \frac{n}{2},
\end{cases}
\]
\[
\int_{\mathbb{R}_n \setminus B_\frac{1}{n}} |x|^2 \, dx = \begin{cases} 
O(\varepsilon^{\frac{n^2(p-1)}{n^2 + 2p} - 2}) & \text{if } 1 < p < \frac{n(n+2)}{n^2 + 4}, \\
O(\|\ln \varepsilon\|) & \text{if } p = \frac{n(n+2)}{n^2 + 4}, \\
O(1) & \text{if } \frac{n(n+2)}{n^2 + 4} < p < \frac{n}{2},
\end{cases}
\]
\[
\int_{\mathbb{R}_n \setminus B_\frac{1}{n}} |x|^2 \, dx = \begin{cases} 
O(\varepsilon^{\frac{n^2(p-1)}{n^2 + 2p} - 2}) & \text{if } 1 < p < 2 \frac{n-1}{n}, \\
O(\varepsilon^{\frac{n^2(p-1)}{n^2 + 2p} - 2} \|\ln \varepsilon\|^{\frac{n^2}{n^2 - 2p}}) & \text{if } p = 2 \frac{n-1}{n}, \\
O(\varepsilon^{\frac{n}{n^2 - 2p}}) & \text{if } 2 \frac{n-1}{n} < p < \frac{n}{2}.
\end{cases}
\]
\[
\int_{B^*_{\frac{1}{2}}} z^p |x|^3 dx = \begin{cases} 
O\left(\varepsilon^{\frac{n^2(n+1)}{n+2p-3}}\right) & \text{if } 1 < p < \frac{n(n+3)}{n^2+6}, \\
O(|\ln \varepsilon|) & \text{if } p = \frac{n(n+3)}{n^2+6}, \\
O(1) & \text{if } \frac{n(n+3)}{n^2+6} < p < \frac{n+3}{3}, \\
O(|\ln \varepsilon|) & \text{if } p = \frac{n+3}{3}, \\
O(\varepsilon^{\frac{n-2p}{n+3}}) & \text{if } \frac{n+3}{3} < p < \frac{n}{2}.
\end{cases}
\]

Estimate of \( \int_M |\Delta_y u_\varepsilon|^p dv_y \)

\[
\int_{\mathbb{R}^n \setminus B^*_{\frac{1}{2}}} |\Delta z|^p |x|^2 dx = \begin{cases} 
O(1) & \text{if } 1 < p < \frac{2n(n-1)}{n^2+2n-4}, \\
O(|\ln \varepsilon|) & \text{if } p = \frac{2n(n-1)}{n^2+2n-4}, \\
O(\varepsilon^{\frac{n-2p}{n+3}}) & \text{if } \frac{2n(n-1)}{n^2+2n-4} < p < \frac{n}{2}.
\end{cases}
\]

\[
\int_{\mathbb{R}^n \setminus B^*_{\frac{1}{2}}} |\Delta z|^p |x|^2 dx = \begin{cases} 
O(1) & \text{if } 1 < p < \frac{n+2}{4}, \\
O(|\ln \varepsilon|) & \text{if } p = \frac{n+2}{4}, \\
O(\varepsilon^{\frac{n-2p}{n+3}}) & \text{if } \frac{n+2}{4} < p < \frac{n}{2}.
\end{cases}
\]

\[
\int_{\mathbb{R}^n \setminus B^*_{\frac{1}{2}}} |\Delta z|^p |x|^2 dx = \begin{cases} 
O(1) & \text{if } 1 < p < \frac{2n(n-1)}{n^2+2n-4}, \\
O(|\ln \varepsilon|) & \text{if } p = \frac{2n(n-1)}{n^2+2n-4}, \\
O(\varepsilon^{\frac{n-2p}{n+3}}) & \text{if } \frac{2n(n-1)}{n^2+2n-4} < p < \frac{n+2}{4}, \\
O(|\ln \varepsilon|) & \text{if } p = \frac{n+2}{4}, \\
O(\varepsilon^{\frac{n-2p}{n+3}}) & \text{if } \frac{n+2}{4} < p < \frac{n}{2}.
\end{cases}
\]

\[
\int_{B^*_{\frac{1}{2}}} |\Delta z|^p |x|^3 dx = \begin{cases} 
O(1) & \text{if } 1 < p < \frac{n+3}{5}, \\
O(|\ln \varepsilon|) & \text{if } p = \frac{n+3}{5}, \\
O(\varepsilon^{\frac{n-2p}{n+3}}) & \text{if } \frac{n+3}{5} < p < \frac{n}{2}.
\end{cases}
\]
\[ \int_{\mathbb{R}^n} |\Delta z|^{p-1} |z'(r)| \, r \, dx = \begin{cases} 
O(1) & \text{if } 1 < p < \frac{2(n-1)}{n^2 + 2n - 4}, \\
O(\varepsilon^{-1} |\ln \varepsilon|) & \text{if } p = \frac{2(n-1)}{n^2 + 2n - 4}, \\
O(\varepsilon^{-1}) & \text{if } \frac{2(n-1)}{n^2 + 2n - 4} < p < \frac{2n-1}{n}, \text{ and } n = 3, \\
O(\varepsilon^{-1} |\ln \varepsilon|) & \text{if } p = \frac{2n-1}{n}, \\
O\left(\varepsilon^{\frac{n-2p}{n+2p}}-1\right) & \text{if } \frac{2n-1}{n} < p < \frac{n}{2}, \\
O(1) & \text{if } 1 < p < \frac{2n(n-1)}{n^2 + 2n - 4}, \\
O(\|\ln \varepsilon\|^2) & \text{if } p = \frac{2n(n-1)}{n^2 + 2n - 4}, \\
O(\|\ln \varepsilon\|) & \text{if } \frac{2n(n-1)}{n^2 + 2n - 4} < p < \frac{2n-1}{n}, \text{ and } n = 4, \\
O(\|\ln \varepsilon\|^2) & \text{if } p = \frac{2n-1}{n}, \\
O\left(\varepsilon^{\frac{n-2p}{n+2p}}-1\right) & \text{if } \frac{2n-1}{n} < p < \frac{n}{2}, \\
O\left(\varepsilon^{\frac{n^2(n-1)}{n^2 - 2n^2 - 4n - 4}}-1\right) & \text{if } \text{if } 1 < p < \frac{n(n+2)}{n^2 + 4}, \\
O(\|\ln \varepsilon\|) & \text{if } p = \frac{n(n+2)}{n^2 + 4}, \\
O(1) & \text{if } \frac{n(n+2)}{n^2 + 4} < p < \frac{n+2}{4}, \text{ and } n \geq 5, \\
O(\|\ln \varepsilon\|) & \text{if } p = \frac{n+2}{4}, \\
O\left(\varepsilon^{\frac{n-2p}{n+2p}}-1\right) & \text{if } \frac{n+2}{4} < p < \frac{n}{2}. 
\end{cases} \]
\[
\int_{B_{\frac{1}{2}}} |\Delta z|^{p-1} |z'(r)|^2 r^{2} dx = \begin{cases} 
O\left(e^{\frac{n^2}{n+2} - 3}\right) & \text{if } 1 < p < \frac{2n(n-1)}{n^2 + 2n - 4}, \\
O\left(e^{p-1} \ln \varepsilon\right) & \text{if } p = \frac{n^2 + 2n - 4}{n^2 + 2n - 4}, \\
O\left(e^{-2}\right) & \text{if } \frac{2n(n-1)}{n^2 + 2n - 4} < p < \frac{n-1}{n}, \text{ and } n = 3, \\
O\left(e^{-2} \ln \varepsilon\right) & \text{if } p = \frac{n-1}{n}, \\
O\left(e^{\frac{n-2p}{p-1} - 3}\right) & \text{if } 2 - \frac{n-1}{n} < p < \frac{n}{2}, \\
O\left(e^{\frac{n^2(p-1)}{n+2p} - 3}\right) & \text{if } 1 < p < \frac{n^2 + 2n - 4}{2n(n-1)}, \\
O\left(e^{-1} \ln \varepsilon\right) & \text{if } p = \frac{n^2 + 2n - 4}{2n(n-1)}, \\
O\left(e^{-1}\right) & \frac{2n(n-1)}{n^2 + 2n - 4} < p < \frac{n-1}{n}, \text{ and } n = 4, \\
O\left(e^{-1} \ln \varepsilon\right) & \text{if } p = \frac{n-1}{n}, \\
O\left(e^{\frac{n-2p}{p-1} - 3}\right) & \frac{2n-1}{n} < p < \frac{n}{2}, \\
O\left(e^{\frac{n^2(p-1)}{n+2p} - 3}\right) & \frac{2n(n-1)}{n^2 + 2n - 4} < p < \frac{n-1}{n}, \text{ and } n = 5, \\
O\left(e^{-1} \ln \varepsilon\right) & \text{if } p = \frac{n-1}{n}, \\
O\left(e^{-1}\right) & \frac{2n-1}{n} < p < \frac{n}{2}, \\
O\left(e^{\frac{n^2(p-1)}{n+2p} - 3}\right) & \frac{n(n+3)}{n^2 + 6} < p < \frac{n^2 + 6}{n(n+3)}, \\
O\left(e^{-1} \ln \varepsilon\right) & \text{if } p = \frac{n+3}{5}, \\
O\left(e^{-1}\right) & \frac{n+3}{5} < p < \frac{n}{2}.
\end{cases}
\]

\[
\int_{B_{\frac{1}{2}}} |\Delta z|^{p-2} |z'(r)|^2 r^{2} dx = \begin{cases} 
O\left(e^{\frac{n-2p}{p-1} - 4}\right) & \text{if } 2n-1 < p < \frac{n}{2}, \text{ and } n = 3, 4, 5, 6, \\
O\left(1\right) & \text{if } \frac{n-1}{n} < p < \frac{n+4}{6}, \\
O\left(\ln \varepsilon\right) & \text{if } p = \frac{n+4}{6}, \text{ and } n \geq 7, \\
O\left(e^{\frac{n-2p}{p-1} - 4}\right) & \frac{n+4}{6} < p < \frac{n}{2}.
\end{cases}
\]

\[
\int_{B_{\frac{1}{2}}} |z'(r)|^p r^p dx = \begin{cases} 
O\left(e^{n(p-1)-2p}\right) & \text{if } 1 < p \leq \frac{n}{2} - \frac{n-1}{n}, \text{ and } n = 3, 4, 5, \\
O\left(e^{\frac{n-2p}{p-1} - 2p}\right) & \frac{n-1}{n} < p < \frac{n}{2}, \\
O\left(e^{n(p-1)-2p}\right) & \text{if } 1 < p < \frac{n}{2}, \\
O\left(\ln \varepsilon\right) & \text{if } p = \frac{n}{n-2}, \\
O\left(1\right) & \frac{n}{n-2} < p < \sqrt{n}, \text{ and } n \geq 6, \\
O\left(\ln \varepsilon\right) & \text{if } p = \sqrt{n}, \\
O\left(e^{\frac{n-2p}{p-1} - 2p}\right) & \sqrt{n} < p < \frac{n}{2}.
\end{cases}
\]
\[ \int_{B_1^+ \setminus B_{1/2}^+} |\Delta z|^p \, dx = \begin{cases} 
 O\left(\varepsilon^{\frac{p(n-1) - n}{n-p}}\right) & \text{if } 1 < p < \frac{2n(n-1)}{n^2 + 2n - 4}, \\
 O\left(\varepsilon^{\frac{n-2p}{n-p}}|\ln \varepsilon|^{\frac{n}{n-p}}\right) & \text{if } p = \frac{2n(n-1)}{n^2 + 2n - 4}, \\
 O\left(\varepsilon^{\frac{n-2p}{n-p}}\right) & \text{if } \frac{2n(n-1)}{n^2 + 2n - 4} < p < \frac{n}{2}. 
\end{cases} \]

\[ \int_{B_1^+ \setminus B_{1/2}^+} |z|^p r^p \, dx = \begin{cases} 
 O\left(\varepsilon^{n(1-2p)}\right) & \text{if } 1 < p < \frac{n-1}{n}, \\
 O\left(\varepsilon^{n(1-2p)}|\ln \varepsilon|^p\right) & \text{if } p = \frac{n-1}{n}, \\
 O\left(\varepsilon^{\frac{n-2p}{n-p}} - 2p\right) & \text{if } \frac{n-1}{n} < p < \frac{n}{2}. 
\end{cases} \]

\[ \int_{B_1^+ \setminus B_{1/2}^+} |z|^p \, dx = \begin{cases} 
 O\left(\varepsilon^{n(1-2p)}\right) & \text{if } 1 < p < \frac{n-1}{n}, \\
 O\left(\varepsilon^{n(1-2p)}|\ln \varepsilon|^p\right) & \text{if } p = \frac{n-1}{n}, \\
 O\left(\varepsilon^{\frac{n-2p}{n-p}} - 2p\right) & \text{if } \frac{n-1}{n} < p < \frac{n}{2}. 
\end{cases} \]

Estimate of \( \int_M |u_\varepsilon|^p \, dv_g \)

\[ \int_{B_1^+} z^p \, dx = \begin{cases} 
 O\left(\varepsilon^{n(1-2p)}\right) & \text{if } 1 < p < \frac{n-1}{n}, \\
 O\left(\varepsilon^{n(1-2p)}|\ln \varepsilon|^p\right) & \text{if } p = \frac{n-1}{n}, \\
 O\left(\varepsilon^{\frac{n-2p}{n-p}} - 2p\right) & \text{if } \frac{n-1}{n} < p < \frac{n}{2}. 
\end{cases} \]

\[ \int_{B_1^+} z^p r^2 \, dx = \begin{cases} 
 O\left(\varepsilon^{n(1-2p-2)}\right) & \text{if } 1 < p < \frac{n+1}{n-2}, \\
 O\left(\varepsilon^{n(1-2p-2)}|\ln \varepsilon|^p\right) & \text{if } p = \frac{n+1}{n-2}, \\
 O\left(\varepsilon^{\frac{n-2p}{n-p}} - 2p - 2\right) & \text{if } \frac{n+1}{n-2} < p < \frac{n}{2}. 
\end{cases} \]
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