CHAPTER 3

# Dynkin Systems

Let $\Omega$ be any non-empty set. We denote by $2^\Omega$ the family of all subsets of $\Omega$, including the emptyset.

DEFINITION 3.0.1. *A collection $\mathcal{D} \subset 2^\Omega$ is called a* Dynkin System *(or simply* D-system*) if the following conditions hold:*

(1) $\Omega \in \mathcal{D}$.
(2) *If $A, B \in \mathcal{D}$, $A \subset B$, then $B \backslash A \in \mathcal{D}$.*
(3) *If $A_n \in \mathcal{D}$ for all $n \geq 1$, $A_n \nearrow A$, then $A \in \mathcal{D}$*

Observe that D-systems are stable by complementation since $A \in \mathcal{D}$ implies $A^c = \Omega \backslash A \in \mathcal{D}$. Since $B \backslash A = B \cap A^c$, $\sigma$-algebras are D-systems, but D-systems are not necessarily stable under intersections.

LEMMA 3.0.1. *A collection $\mathcal{F} \subset 2^\Omega$ is a $\sigma$-algebra if and only if it is a D-system stable under intersection.*

PROOF. The "only if" part is trivial. Then, assume $\mathcal{F}$ is a D-system stable under intersection. Let $A, B \in \mathcal{F}$. We have $A \cup B = (A^c \cap B^c)^c = \Omega \backslash (A^c \cap B^c) \in \mathcal{F}$. Let $A_n \in \mathcal{F}$, $B_n := \bigcup_{k=1}^n A_k$. Since $B_n \in \mathcal{F}$ and $B_n \nearrow \bigcup_{n \geq 1} B_n$, we have that $\bigcup_{n \geq 1} B_n \in \mathcal{F}$. This shows that $\mathcal{F}$ is a $\sigma$-algebra. □

As can be easily verified, the intersection of an arbitrary family of D-systems is a D-system. Therefore, given any collection $\mathcal{C} \subset 2^\Omega$, one can define the smallest D-system containing $\mathcal{C}$, called the D-system generated by $\mathcal{C}$, denoted $\mathcal{D}(\mathcal{C})$. In practice, it is interesting to compare the D-system $\mathcal{D}(\mathcal{C})$ with the $\sigma$-algebra $\sigma(\mathcal{C})$. One clearly has $\mathcal{D}(\mathcal{C}) \subset \sigma(\mathcal{C})$.

THEOREM 3.0.1. *If $\mathcal{C} \subset 2^\Omega$ is stable under intersection, then $\mathcal{D}(\mathcal{C}) = \sigma(\mathcal{C})$.*

PROOF. To simplify the notations, denote $\mathcal{D}(\mathcal{C})$ by $\mathcal{D}$ and $\sigma(\mathcal{C})$ by $\mathcal{F}$. We already saw that $\mathcal{D} \subset \mathcal{F}$. To show that $\mathcal{D} \supset \mathcal{F}$, it suffices to verify that $\mathcal{D}$ is a $\sigma$-algebra. By Lemma 3.0.1, it suffices to verify that $\mathcal{D}$ is stable under intersection. Define $\mathcal{D}_1 := \{B \in \mathcal{D} : B \cap C \in \mathcal{D} \, \forall C \in \mathcal{C}\}$. We verify that $\mathcal{D}_1 = \mathcal{D}$. By definition, $\mathcal{D}_1 \subset \mathcal{D}$. To verify that $\mathcal{D}_1 \supset \mathcal{D}$, it suffices to see that $\mathcal{D}_1$ is a D-system containing $\mathcal{C}$. Now $\mathcal{D}_1 \supset \mathcal{C}$ follows from the fact that $\mathcal{C}$ is closed under intersection. This also implies that $\Omega \in \mathcal{D}_1$. Let $B_1, B_2 \in \mathcal{D}_1$, $B_1 \subset B_2$, $C \in \mathcal{C}$. Then

$$(B_2 \backslash B_1) \cap C = B_2 \cap C \cap (B_1^c \cup C^c) = (B_2 \cap C) \backslash (B_1 \cap C) \in \mathcal{D}$$

Then, if $B_n \in \mathcal{D}_1$, $B_n \nearrow B$, then $B \cap C = \bigcup_n (B_n \cap C) \in \mathcal{D}$, logo $B \in \mathcal{D}_1$. This proves that $\mathcal{D}$ is a D-system.
Define $\mathcal{D}_2 := \{A \in \mathcal{D} : A \cap B \in \mathcal{D} \forall B \in \mathcal{D}\}$. We verify that $\mathcal{D}_2 = \mathcal{D}$, which will show that $\mathcal{D}$ is stable under intersection. By the first step, $\mathcal{D}_2$ contains $\mathcal{C}$. As before, one can show that $\mathcal{D}_2 = \mathcal{D}$. This shows that $\mathcal{D}$ is stable under intersection, and finishes the proof of the theorem. □

The previous result is usually used in the following form:

COROLLARY 3.0.1. *Let $\mathcal{C} \subset 2^\Omega$ be stable under intersection. If $\mathcal{D}$ is a D-system containing $\mathcal{C}$, then $\mathcal{D} \supset \sigma(\mathcal{C})$.*

The last result is useful to show that the measurable sets of some $\sigma$-algebra $\mathcal{F}$ satisfy particular property. An example of application is given in the following proposition and its corollary.

PROPOSITION 3.0.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ ($\mathcal{A}_k \subset \mathcal{F}$) be independent collections [1], each of which is stable under intersection. Then the $\sigma$-algebras $\sigma(\mathcal{A}_1), \dots, \sigma(\mathcal{A}_n)$ are independent.*

PROOF. Without loss of generality, we can suppose that each $\mathcal{A}_k$ contains $\Omega$. We will show that if $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are independent and stable under intersection, then $\sigma(\mathcal{A}_1), \mathcal{A}_2, \dots, \mathcal{A}_n$ are independent (and stable under intersection). The proof then follows by induction. Fix $A_2 \in \mathcal{A}_2, \dots, A_n \in \mathcal{A}_n$, set $F := A_2 \cap \dots \cap A_n$ and let $\mathcal{D}_F := \{A \in \mathcal{A}_1 : P(A \cap F) = P(A)P(F)\}$. We have $\mathcal{D}_F \ni \Omega$. Then, let $A, B \in \mathcal{D}_F$ with $A \subset B$: $P((B \backslash A) \cap F) = P(B \backslash A)P(F)$, and so $B \backslash A \in \mathcal{D}_F$. Finally, if $A_n \in \mathcal{D}_F$, $A_n \nearrow A$, then $P(A \cap F) = \lim_n P(A_n \cap F) = \lim_n P(A_n)P(F) = P(A)P(F)$, and so $A \in \mathcal{D}_F$. This shows that $\mathcal{D}_F$ is a D-system. Since $\mathcal{D}_F \supset \mathcal{A}_1$, Corollary 3.0.1 gives $\mathcal{D}_F \supset \sigma(\mathcal{A}_1)$. Since this holds for all choice of $F$, we have shown that $\sigma(\mathcal{A}_1), \mathcal{A}_2, \dots, \mathcal{A}_n$ are independent. □

COROLLARY 3.0.2. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be independent sub-$\sigma$-algebras ($\mathcal{F}_k \subset \mathcal{F}$ for all $k$). Then, for any $1 \leq k \leq n$, $\sigma(\mathcal{F}_1, \dots, \mathcal{F}_k)$ and $\sigma(\mathcal{F}_{k+1}, \dots, \mathcal{F}_n)$ are independent.*

PROOF. Let $\mathcal{A}$ be the collection of all intersections $\bigcap_{j=1}^k A_j$ with $A_j \in \mathcal{F}_j$, and $\mathcal{B}$ be the collection of all intersections $\bigcap_{j=k+1}^n B_j$ with $B_j \in \mathcal{F}_j$. Clearly, $\mathcal{A}$ and $\mathcal{B}$ are stable under intersection. By Proposition 3.0.1, $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ are independent. But $\sigma(\mathcal{A}) = \sigma(\mathcal{F}_1, \dots, \mathcal{F}_k)$ and $\sigma(\mathcal{B}) = \sigma(\mathcal{F}_{k+1}, \dots, \mathcal{F}_n)$. □

COROLLARY 3.0.3. *Assume the variables $(X_n)_{n \geq 1}$ are independent. Then for all $k \geq 1$, $\sigma(X_1, \dots, X_k)$ and $\sigma(X_{k+1}, \dots)$ are independent.*

PROOF. Let $\mathcal{A} := \sigma(X_1, \dots, X_k)$, $\mathcal{B} := \bigcup_{j \geq 1} \sigma(X_{k+1}, \dots, X_{k+j})$. Clearly, both $\mathcal{A}$ and $\mathcal{B}$ are stable under intersection. Now by Corollary 3.0.2, $\sigma(X_1, \dots, X_k)$ and $\sigma(X_{k+1}, \dots, X_{k+j})$ are independent for all $j \geq 1$. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are independent. By Proposition 3.0.1, $\sigma(\mathcal{A}) (\equiv \mathcal{A})$ and $\sigma(\mathcal{B})$ are independent. But $\sigma(\mathcal{B}) = \sigma(X_{k+1}, \dots)$, which proves the lemma. □

[1] Remember that $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent if for all $I \subset \{1, 2, \dots, n\}$, any family $A_i, i \in I$ is independent: $P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$.