

Oval Billiards on Surfaces of Constant Curvature

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Abstract

In this paper we prove that the billiard problem on surfaces of constant curvature defines a 2-dimensional conservative and reversible dynamical system, defined by a Twist diffeomorphism, if the boundary curve is a regular, simple, closed, strictly geodesically convex curve and at least C^2 curve.

1 Introduction

The plane billiard problem, originally defined by Birkhoff [3] in the beginning of the XX century, consists in the free motion of a point particle in a bounded plane region, reflecting elastically when it reaches the boundary.

In this work we extend this problem to bounded regions contained on geodesically convex subsets of surfaces of constant curvature. We will show that this new billiard defines a 2-dimensional conservative and reversible dynamical system, defined by a Twist diffeomorphism, if the boundary curve is an oval, i.e., a regular, simple, closed, strictly geodesically convex curve and at least C^2 curve. This is a classical result for the Euclidean case and is proved for instance in [13]. Nevertheless, and for the sake of completeness, we will present the proof for the three cases.

Billiards on the Euclidean plane were, and still are, extensively studied. Billiards on surfaces of constant curvature are much less studied and the papers focus on special properties. For instance, Veselov [16], Bolotin [5], Dragović, Jovanović and Radnović [7], Popov and Topalov [14], [15] and Bialy [2] deal with the question of integrability. B.Gutkin, Smilansky and E.Gutkin [9] looked at hyperbolic billiards on the sphere and the hyperbolic plane. E.Gutkin and Tabachnikov [10] studied geodesic polygonal billiards. Among them, only Bialy [2] looked more closely to oval billiards, but he starts from the variational formulation of the billiard problem, taking as granted that the geodesic distance defines a generating function, which is not our case here.

2 Surfaces of constant curvature

For the definition of the billiard, only the behaviour of the geodesics and the measure of angles will count. So, we can take as model one of the three surfaces:

- the Euclidean plane $\mathbb{E}^2$, given in $\mathbb{R}^3$ by $\{z = 1\}$ or $\mathcal{X}(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta, 1), \rho \geq 0, 0 \leq \theta < 2\pi$;

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- an open hemisphere of the unit sphere $\mathbb{S}_+^2$ given in $\mathbb{R}^3$ by

$$\langle (x, y, z), (x, y, z) \rangle = 1, \quad z > 0$$

where $\langle, \rangle$ is the usual inner product on $\mathbb{R}^3$ or by

$$\mathcal{X}(\rho, \theta) = (\sin \rho \cos \theta, \sin \rho \sin \theta, \cos \rho), \quad 0 < \rho < \pi, 0 \leq \theta < 2\pi;$$

- the upper sheet of the hyperbolic plane $\mathbb{H}_+^2$ given in $\mathbb{R}^3$ by

$$\langle\langle (x, y, z), (x, y, z) \rangle\rangle = 1, \quad z > 0$$

where $\langle\langle, \rangle\rangle$ is the inner product on $\mathbb{H}^2$ or by

$$\mathcal{X}(\rho, \theta) = (\sinh \rho \cos \theta, \sinh \rho \sin \theta, \cosh \rho), \quad \rho \geq 0, 0 \leq \theta < 2\pi$$

which will be called by S .

The geodesics on S are the intersections of the surface with the planes passing by the origin. S is geodesically convex. Then given a point $X \in S$ and unitary vector $v \in T_X S$ there is a unique minimising geodesic by X on the direction of v which is given by

$$\gamma(t) = \begin{cases} X + tv & \text{if } S = \mathbb{E}^2 \\ X \cos t + v \sin t & \text{if } S = \mathbb{S}_+^2 \\ X \cosh t + v \sinh t & \text{if } S = \mathbb{H}_+^2 \end{cases}$$

The distance between two points X and Y on S is so measured by

$$d_S(X, Y) = \begin{cases} \sqrt{\langle X - Y, X - Y \rangle} & \text{if } X, Y \in \mathbb{E}^2 \\ \arccos(\langle X, Y \rangle) & \text{if } X, Y \in \mathbb{S}_+^2 \\ \operatorname{arccosh}(-\langle\langle X, Y \rangle\rangle) & \text{if } X, Y \in \mathbb{H}_+^2 \end{cases}$$

Definition 1. A regular curve $\Gamma(t) \subset S$ is said to be geodesically strictly convex if the intersection of any geodesic tangent to Γ with the curve Γ has only one point.

It is proved on [12] for the spherical case, on [6] for the hyperbolic case and for instance on [1] for the Euclidean case that

Lemma 1. If a curve $\Gamma \subset S$ is closed, regular, simple, $C^n, n \geq 2$ and has strictly positive geodesic curvature then Γ is geodesically strictly convex.

So, by lemma 1, any oval is geodesically strictly convex. This property, together with the fact that S is geodesically convex will enable us to define oval billiard maps on surfaces of constant curvature.

3 Billiards on ovals

Let $\Omega \subset S$ be a region bounded by an oval Γ . Analogously to the plane case, we can define the billiard on Γ as the free motion of a point particle inside Ω , reflecting elastically at the impacts with Γ . Since the motion is free, the particle moves along a geodesic of S while it stays inside Ω and reflects making equal angles with the tangent at the impact points with Γ . The trajectory of the particle is a geodesic polygonal curve, with vertices at the impact points.

As Ω is a bounded subset of a geodesically convex surface, with strictly geodesically convex boundary, the motion is completely determined by the impact point and the direction of movement immediately after each reflection. So, a parameter which locates the point of impact, and the angle between the direction of motion and the tangent to the boundary at the impact point, may be used to describe the system.

Let l be the length of Γ , s the arclength parameter for Γ and $\psi \in (0, \pi)$ be angle that measures the direction of motion at the impact point. Let $\mathcal{C}$ be the cylinder $\mathbb{R}/l \times (0, \pi)$.

We can define the billiard map

$$T : \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C} \\ (s_0, \psi_0) & \longmapsto & (s_1, \psi_1) \end{array}$$

which associates to each impact point and direction of motion the next impact and direction. This billiard map defines a 2-dimensional dynamical system and the orbit of any initial point (s_0, ψ_0) is the set $\mathcal{O}(s_0, \psi_0) = \{T^i((s_0, \psi_0) = (s_i, \psi_i), i \in \mathbb{Z}\}$.

3.1 Properties of the generating function

Let S be $\mathbb{E}^2, \mathbb{S}_+^2$ or $\mathbb{H}_+^2$ and d_S be the geodesic distance on S . Let $\Gamma \subset S$ be an oval, parameterized by the arclength parameter s and Ω be the region bounded by Γ .

Lemma 2. *Let $T(s_0, \psi_0) = (s_1, \psi_1)$ be the billiard map on Γ and $g(s_0, s_1) = -d_S(\Gamma(s_0), \Gamma(s_1))$. Then g verifies*

$$\frac{\partial g}{\partial s_0}(s_0, s_1) = \cos \psi_0 \quad \text{and} \quad \frac{\partial g}{\partial s_1}(s_0, s_1) = -\cos \psi_1$$

Proof: Let $\tau_i, i = 0$ or 1 , be the unitary tangent vector to the oriented geodesic joining $\Gamma(s_0)$ to $\Gamma(s_1)$, at $\Gamma(s_i)$.

When $S = \mathbb{E}^2$ we have that $g^2(s_0, s_1) = \langle \Gamma(s_1) - \Gamma(s_0), \Gamma(s_1) - \Gamma(s_0) \rangle$ and then

$$\frac{\partial g}{\partial s_0}(s_0, s_1) = - \langle \Gamma'(s_0), \frac{\Gamma(s_1) - \Gamma(s_0)}{g((s_0), (s_1))} \rangle = \langle \Gamma'(s_0), \tau_0 \rangle = \cos \psi_0. \quad (1)$$

Analogously, $\frac{\partial g}{\partial s_1}(s_0, s_1) = \langle \Gamma'(s_1), \tau_1 \rangle = -\cos \psi_1$.

When $S = \mathbb{S}_+^2$ we have that $\cos g(s_0, s_1) = \langle \Gamma(s_0), \Gamma(s_1) \rangle$ and then

$$\begin{aligned} \frac{\partial g}{\partial s_0}(s_0, s_1) &= - \frac{\langle \Gamma'(s_0), \Gamma(s_1) \rangle}{\sin g(s_0, s_1)} \\ &= - \frac{\langle \Gamma'(s_0), \cos g(s_0, s_1) \Gamma(s_0) - \sin g(s_0, s_1) \tau_0 \rangle}{\sin g(s_0, s_1)} \\ &= \langle \Gamma'(s_0), \tau_0 \rangle = \cos \psi_0. \end{aligned} \quad (2)$$

Analogously $\frac{\partial g}{\partial s_1}(s_0, s_1) = \langle \Gamma'(s_1), -\tau_1 \rangle = -\cos \psi_1$.

When $S = \mathbb{H}_+^2$ we have that $\cosh g(s_0, s_1) = - \ll \Gamma(s_1), \Gamma(s_2) \gg$ and then

$$\begin{aligned} \frac{\partial g}{\partial s_0}(s_0, s_1) &= - \frac{\ll \Gamma'(s_0), \Gamma(s_1) \gg}{\sinh g(s_0, s_1)} \\ &= - \frac{\ll \Gamma'(s_0), \cosh g(s_0, s_1) \Gamma(s_0) - \sinh g(s_0, s_1) \tau_0 \gg}{\sinh g(s_0, s_1)} \\ &= \ll \Gamma'(s_0), \tau_0 \gg = \cos \psi_0. \end{aligned} \quad (3)$$

Analogously $\frac{\partial g}{\partial s_1}(s_0, s_1) = \ll \Gamma'(s_1), -\tau_1 \gg = -\cos \psi_1$. $\square$

Let $p = -\cos \psi \in (-1, 1)$. Lemma 2 implies that the arclength s and the tangent momentum p are conjugated coordinates with generating function g for the billiard map, or,

$$T(s_0, p_0) = (s_1, p_1) \iff \frac{\partial g}{\partial s_0} = -p_0, \quad \frac{\partial g}{\partial s_1} = p_1$$

leading to the variational definition of billiards.

Lemma 3. *Let k_i be the geodesic curvature of Γ at $s_i, i = 0, 1$. The second derivatives of g are*

$$\begin{aligned} \frac{\partial^2 g}{\partial s_i^2}(s_0, s_1) &= \begin{cases} \frac{\sin^2 \psi_i}{g(s_0, s_1)} - k_i \sin \psi_i & \text{in } \mathbb{E}^2 \\ \frac{\sin^2 \psi_i}{\tan g(s_0, s_1)} - k_i \sin \psi_i & \text{in } \mathbb{S}_+^2 \\ \frac{\sin^2 \psi_i}{\tanh g(s_0, s_1)} - k_i \sin \psi_i & \text{in } \mathbb{H}_+^2 \end{cases} \\ \frac{\partial^2 g}{\partial s_0 \partial s_1}(s_0, s_1) &= \begin{cases} \frac{\sin \psi_0 \sin \psi_1}{g(s_0, s_1)} & \text{in } \mathbb{E}^2 \\ \frac{\sin \psi_0 \sin \psi_1}{\sin g(s_0, s_1)} & \text{in } \mathbb{S}_+^2 \\ \frac{\sin \psi_0 \sin \psi_1}{\sinh g(s_0, s_1)} & \text{in } \mathbb{H}_+^2 \end{cases} \end{aligned}$$

Proof: Let τ_i , η_i and ν_i be the unitary tangent, normal and binormal vectors, respectively, to the oriented geodesic joining $\Gamma(s_0)$ to $\Gamma(s_1)$, at $\Gamma(s_i)$, seen as a curve in $\mathbb{R}^3$. When $S = \mathbb{S}_+^2$ or $\mathbb{H}_+^2$, since the geodesic is contained on a plane passing by the origin, $\eta_i = -\Gamma(s_i)$, ν_i is a constant unitary vector, normal to the plane, and $\{\tau_i, \nu_i\}$ is an orthonormal basis for $T_{\Gamma(s_i)}S$.

In the case $S = \mathbb{E}^2$ we differentiate equation (1) obtaining

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0^2} &= \frac{1 - \cos^2 \psi_0}{g} + \langle \Gamma''(s_0), \frac{\Gamma(s_1) - \Gamma(s_0)}{g} \rangle \\ &= \frac{\sin^2 \psi_0}{g} + \langle k_0 \sin \psi_0 \tau_0 + k_0 \cos \psi_0 \eta_0, -\tau_0 \rangle = \frac{\sin^2 \psi_0}{g} - k_0 \sin \psi_0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0 \partial s_1} &= \frac{-\langle \Gamma'(s_0), \Gamma'(s_1) \rangle + \cos \psi_0 \cos \psi_1}{g} \\ &= \frac{-\langle \cos \psi_0 \tau_0 - \sin \psi_0 \eta_0, \cos \psi_1 \tau_1 + \sin \psi_1 \eta_1 \rangle + \cos \psi_0 \cos \psi_1}{g} = \frac{\sin \psi_0 \sin \psi_1}{g} \end{aligned}$$

When $S = \mathbb{S}_+^2$ we differentiate (2) getting

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0^2} &= \frac{-\langle \Gamma''(s_0), \Gamma(s_1) \rangle - \cos^2 \psi_0 \cos g}{\sin g} \\ &= \frac{-\langle -\Gamma(s_0) + k_0 \Gamma(s_0) \times \Gamma'(s_0), \cos g \Gamma(s_0) + \sin g \tau_0 \rangle - \cos^2 \psi_0 \cos g}{\sin g} \\ &= \frac{\cos g - k_0 \sin \psi_0 \sin g - \cos^2 \psi_0 \cos g}{\sin g} = \frac{\cos g \sin^2 \psi_0}{\sin g} - k_0 \sin \psi_0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0 \partial s_1} &= \frac{-\langle \Gamma'(s_0), \Gamma'(s_1) \rangle + \cos \psi_0 \cos \psi_1 \cos g}{\sin g} \\ &= \frac{-\langle \cos \psi_0 \tau_0 - \sin \psi_0 \nu_0, \cos \psi_1 \tau_1 + \sin \psi_1 \nu_1 \rangle + \cos \psi_0 \cos \psi_1 \cos g}{\sin g} \\ &= \frac{\sin \psi_0 \sin \psi_1}{\sin g} \end{aligned}$$

The last case is analogous. When $S = \mathbb{H}_+^2$ we differentiate (3) getting

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0 \partial s_1} &= \frac{-\ll \Gamma'(s_0), \Gamma'(s_1) \gg + \cos \psi_0 \cos \psi_1 \cosh g}{\sinh g} \\ &= \frac{-\ll -\Gamma(s_0) + k_0 \Gamma(s_0) \times \Gamma'(s_0), \cosh g \Gamma(s_0) + \sinh g \tau_0 \gg - \cos^2 \psi_0 \cosh g}{\sinh g} \\ &= \frac{\cosh g \sin^2 \psi_0}{\sinh g} - k_0 \sin \psi_0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0 \partial s_1} &= \frac{-\ll \Gamma'(s_0), \Gamma'(s_1) \gg + \cos \psi_0 \cos \psi_1 \cosh g}{\sinh g} \\ &= \frac{-\ll \cos \psi_0 \tau_0 - \sin \psi_0 \nu_0, \cos \psi_1 \tau_1 + \sin \psi_1 \nu_1 \gg + \cos \psi_0 \cos \psi_1 \cosh g}{\sinh g} \\ &= \frac{\sin \psi_0 \sin \psi_1}{\sinh g} \end{aligned}$$

Finally, the calculation of $\frac{\partial^2 g}{\partial s_1^2}$ is analogous to $\frac{\partial^2 g}{\partial s_0^2}$. □

3.2 Properties of the billiard map

In this subsection we will prove that

Theorem 1. *Let S be $\mathbb{E}^2, \mathbb{S}_+^2$ or $\mathbb{H}_+^2$. Let $\Gamma \subset S$ be an oval, a closed, simple, regular C^n curve, $n \geq 2$, with strictly positive geodesic curvature. Then the billiard map T on Γ is a reversible, conservative, Twist, C^{n-1} -diffeomorphism.*

The proof will follow directly from the lemmas bellow. As above, s stands for the arclength parameter for Γ , k_i is the geodesic curvature of Γ at s_i , d_S is the distance on S and $g(s_0, s_1) = -d_S(\Gamma(s_0), \Gamma(s_1))$.

Lemma 4. *T is invertible and reversible.*

Proof: Any trajectory of the billiard problem can be travelled in both senses. So, if $T(s_i, \psi_i) = (s_{i+1}, \psi_{i+1})$ then $T^{-1}(s_i, \pi - \psi_i) = (s_{i-1}, \pi - \psi_{i-1})$.

Let I be the involution on $\mathcal{C}$ given by $I(s, \psi) = (s, \pi - \psi)$. Clearly $I^{-1} = I$.

We have then that $T^{-1} = I \circ T \circ I$ or $I \circ T^{-1} = T \circ I$, i.e., T is reversible. □

Lemma 5. *T is a C^{n-1} diffeomorphism.*

Proof: Let $T(\bar{s}_0, \bar{\psi}_0) = (\bar{s}_1, \bar{\psi}_1)$ and V_0 and V_1 be two disjoint open intervals containing $\bar{s}_0$ and $\bar{s}_1$ respectively. We define

$$G : V_0 \times V_1 \times (0, \pi) \mapsto \mathbb{R}, \quad G(s_0, s_1, \psi_0) = \frac{\partial g}{\partial s_0}(s_0, s_1) - \cos \psi_0.$$

G is a C^{n-1} function, since Γ and g are C^n . Then, by lemma 2, $G(\bar{s}_0, \bar{s}_1, \bar{\psi}_0) = 0$ and $\frac{\partial G}{\partial s_1}(s_0, s_1) = \frac{\partial^2 g}{\partial s_0 \partial s_1}(\bar{s}_0, \bar{s}_1) \neq 0$ by lemma 3, since $\bar{\psi}_0, \bar{\psi}_1 \in (0, \pi)$. So we can locally define a C^{n-1} function $s_1 = s_1(s_0, \psi_0)$ such that $G(s_0, s_1(s_0, \psi_0), \psi_0) = 0$.

Taking now $\psi_1(s_0, \psi_0) = \arccos(-\frac{\partial g}{\partial s_1}(s_0, s_1(s_0, \psi_0)))$ we conclude that $T(s_0, \psi_0) = (s_1(s_0, \psi_0), \psi_1(s_0, \psi_0))$ is a C^{n-1} function.

As T is invertible, with $T^{-1} = I \circ T \circ I$, we conclude that T is a C^{n-1} diffeomorphism. $\square$

Differentiating the expressions

$$\begin{aligned}\cos \psi_0 &= \frac{\partial g}{\partial s_0}(s_0, s_1(s_0, \psi_0)) \\ \cos \psi_1(s_0, \psi_0) &= -\frac{\partial g}{\partial s_1}(s_0, s_1(s_0, \psi_0))\end{aligned}$$

we obtain

Lemma 6.

$$\begin{aligned}\frac{\partial^2 g}{\partial s_0^2} + \frac{\partial^2 g}{\partial s_0 \partial s_1} \frac{\partial s_1}{\partial s_0} &= 0 \\ \frac{\partial^2 g}{\partial s_0 \partial s_1} \frac{\partial s_1}{\partial \psi_0} &= \sin \psi_0 \\ \frac{\partial^2 g}{\partial s_0 \partial s_1} + \frac{\partial^2 g}{\partial s_1^2} \frac{\partial s_1}{\partial s_0} &= \sin \psi_1 \frac{\partial \psi_1}{\partial s_0} \\ \frac{\partial^2 g}{\partial s_1^2} \frac{\partial s_1}{\partial \psi_0} &= \sin \psi_1 \frac{\partial \psi_1}{\partial \psi_0}\end{aligned} \tag{4}$$

Lemma 7. T is a Twist map.

Proof: By equation (4) together with lemma 3 and remembering that $\psi \in (0, \pi)$ we have that $\frac{\partial s_1}{\partial \psi_0} \neq 0$ and T has the Twist property. $\square$

Using the formulas of lemmas 6 and 3 we obtain the derivative of the billiard map as:

Lemma 8. $DT(s_0, \psi_0) = \begin{pmatrix} \frac{\partial s_1}{\partial s_0} & \frac{\partial s_1}{\partial \psi_0} \\ \frac{\partial \psi_1}{\partial s_0} & \frac{\partial \psi_1}{\partial \psi_0} \end{pmatrix}$ where

- in $\mathbb{E}^2$

$$\begin{aligned}\frac{\partial s_1}{\partial s_0} &= \frac{k_0 g - \sin \psi_0}{\sin \psi_1} \\ \frac{\partial s_1}{\partial \psi_0} &= \frac{g}{\sin \psi_1} \\ \frac{\partial \psi_1}{\partial s_0} &= \frac{k_0 \sin \psi_1 + k_1 \sin \psi_0 - k_1 k_0 g}{\sin \psi_1} \\ \frac{\partial \psi_1}{\partial \psi_0} &= \frac{\sin \psi_1 - k_1 g}{\sin \psi_1}\end{aligned}$$

- in $\mathbb{S}_+^2$

$$\begin{aligned}\frac{\partial s_1}{\partial s_0} &= \frac{k_0 \sin g - \sin \psi_0 \cos g}{\sin \psi_1} \\ \frac{\partial s_1}{\partial \psi_0} &= \frac{\sin g}{\sin \psi_1} \\ \frac{\partial \psi_1}{\partial s_0} &= \frac{k_0 \sin \psi_1 \cos g + \sin \psi_0 \sin \psi_1 \sin g - k_0 k_1 \sin g + k_1 \sin \psi_0 \cos g}{\sin \psi_1} \\ \frac{\partial \psi_1}{\partial \psi_0} &= \frac{\sin \psi_1 \cos g - k_1 \sin g}{\sin \psi_1}\end{aligned}$$

- in $\mathbb{H}_+^2$

$$\begin{aligned}
\frac{\partial s_1}{\partial s_0} &= \frac{k_0 \sinh g - \sin \psi_0 \cosh g}{\sin \psi_1} \\
\frac{\partial s_1}{\partial \psi_0} &= \frac{\sinh g}{\sin \psi_1} \\
\frac{\partial \psi_1}{\partial s_0} &= \frac{k_0 \sin \psi_1 \cosh g + \sin \psi_0 \sin \psi_1 \sinh g - k_0 k_1 \sinh g + k_1 \sin \psi_0 \cosh g}{\sin \psi_1} \\
\frac{\partial \psi_1}{\partial \psi_0} &= \frac{\sin \psi_1 \cosh g - k_1 \sinh g}{\sin \psi_1}
\end{aligned}$$

Calculating now the determinant of DT we prove that

Lemma 9. *T preserves the measure $d\mu = \sin \psi ds d\psi$.*

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