Periodic Orbits of Oval Billiards on Surfaces of Constant Curvature

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Abstract

In this paper we define and study the billiard problem on bounded regions on surfaces of constant curvature. We show that this problem defines a 2-dimensional conservative and reversible dynamical system, defined by a Twist diffeomorphism, if the boundary of the region is an oval. Using these properties and defining good perturbations for billiards, we show that having only a finite number of nondegenerate periodic orbits for each fixed period is an open property for billiards on surfaces of constant curvature and a dense one on the hyperbolic plane. We finish this paper studying the stability of these nondegenerate orbits.

1 Introduction

The planar billiard problem, originally defined by Birkhoff [2] in the beginning of the XX century, consists in the free motion of a point particle in a bounded planar region, reflecting elastically when it reaches the boundary.

In this work we extend this problem to bounded regions on geodesically convex subsets of surfaces of constant curvature. We will show that this new billiard also defines a 2-dimensional conservative and reversible dynamical system, defined by a Twist diffeomorphism, if the boundary of the region is an oval, i.e., a regular, simple, closed, oriented, strictly geodesically convex and at least C^2 curve. This is a classical result for the Euclidean case, proved, for instance, in [12].

Once we have proved that we have a very special dynamical system, we address the question of how many n-periodic orbits such a billiard can have. Bolotin [4] proved that the geodesic circular billiard on surfaces of constant curvature is integrable and then has infinitely many orbits of any period. A classical result for Twist maps (see, for instance, [12]), proved for planar oval billiards in [2], applied to our billiards states that the oval billiard map T has at least two 2-periodic orbits and at least four n-periodic orbits, for each fixed $n \neq 2$. Generic C^1 planar billiards have only a finite number of nondegenerate periodic orbits, for each fixed period, as proved by Dias Carneiro, Oliffson Kamphorst and Pinto-de-Carvalho in [7]. In this new context we get a less general result and show that having only a finite number of nondegenerate periodic orbits, for each fixed period, is an open and dense property for C^∞ oval billiards on the Hyperbolic Plane and is only open on a hemisphere of the unit sphere. We finish this paper studying the stability of these nondegenerate orbits using the MacKay-Meiss Criterium [15].

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Billiards on the Euclidean plane were, and still are, extensively studied. Billiards on surfaces of constant curvature are much less studied and the papers focus on special properties. For instance, Veselov [18] , Bolotin [4], Dragovíc, Jovanovíc and Radnovíc [8], Popov and Topalov [16], [17] and Bialy [1] deal with the question of integrability. B.Gutkin, Smilansky and E.Gutkin [10] looked at hyperbolic billiards on the sphere and the hyperbolic plane. E.Gutkin and Tabachnikov [11] studied geodesic polygonal billiards. Blumen, Kim, Nance and Zarnitsky [3] studied periodic orbits of billiards on surfaces of constant curvature, using the tools of geometric optics. Among them, only Bialy [1] and Zhang [19] looked more closely to oval billiards.

2 Ovals on surfaces of constant curvature

For the study of billiards, we will only be interested in the behavior of the geodesics and the measure of angles. Excluding the Euclidean plane, we can then take as model of surface of constant curvature, denoted by S, one of the surfaces: an open hemisphere of the unit sphere $\mathbb{S}^2_+$, given in $\mathbb{R}^3$ by $\{z = \sqrt{1-x^2-y^2}, z > 0\}$ or the upper sheet of the hyperbolic plane $\mathbb{H}^2_+$, given in $\mathbb{R}^{2,1}$ by $\{z = \sqrt{1+x^2+y^2}\}$.
The geodesics on S are the intersections of the surface with the planes passing by the origin. S is geodesically convex and the distance between two points X and Y on S is measured by

$$d_S(X,Y) = \left\{ \begin{array}{cl} \arccos\langle X,Y\rangle & \text{if} \quad X,Y \in \mathbb{S}^2_+ \\ \operatorname{arccosh}(-\langle\langle X,Y\rangle\rangle) & \text{if} \quad X,Y \in \mathbb{H}^2_+ \end{array} \right.$$

where $\langle,\rangle$ is the usual inner product on $\mathbb{R}^3$ and $\langle\langle,\rangle\rangle$ is the inner product on $\mathbb{R}^{2,1}$.
Given $X,Y \in S$, the geodesic from X to Y is

$$Y \ = \left\{ \begin{array}{rcl} X\cos d + \tau \sin d & \text{in} & \mathbb{S}^2_+ \\ X\cosh d + \tau \sinh d & \text{in} & \mathbb{H}^2_+ \end{array} \right.$$

where $d := d_S(X,Y)$ and τ is the unitary tangent vector to the geodesic at X.

Definition 1. *A regular curve $\Gamma(t) \subset S$ is said to be geodesically strictly convex if the intersection of any geodesic tangent to Γ with the curve Γ has only one point.*

It is proved on [14] for the spherical case and on [5] for the hyperbolic case that

Lemma 1. *If a curve $\Gamma \subset S$ is closed, regular, simple, $C^j, j \geq 2$ and has strictly positive geodesic curvature then Γ is geodesically strictly convex.*

Definition 2. *An oval is a regular, simple, closed, oriented, C^j curve, $j \geq 2$, with strictly positive geodesic curvature .*

By lemma 1, any oval is geodesically strictly convex.

3 Billiards on ovals

Let $\Gamma \subset S$ be an oval and Ω the region bounded by Γ. Analogously to the planar case, we can define the billiard on Γ as the free motion of a point particle inside Ω, reflecting elastically at the impacts with Γ. Since the motion is free, the particle moves along a geodesic of S while it stays inside Ω and reflects making equal angles with the tangent at the impact with Γ. The trajectory of the particle is a geodesic polygonal line, with vertices at the impact points.

As Ω is a bounded subset of a geodesically convex surface, with strictly geodesically convex boundary, the motion is completely determined by the impact point and the direction of movement immediately after each reflection. So, a parameter which locates the point of impact, and the angle between the direction of motion and the tangent to the boundary at the impact point, may be used to describe the system.

Let l be the length of Γ, s the arclength parameter for Γ and $\psi \in (0,\pi)$ be the angle that measures the direction of motion at the impact point. Let $\mathcal{C}$ be the cylinder $\mathbb{R}/l\mathbb{Z} \times (0,\pi)$.

We can define the billiard map T_Γ on $\mathcal{C}$ which associates to each impact point and direction of motion (s_0,ψ_0) the next impact and direction $(s_1,\psi_1) = T_\Gamma(s_0,\psi_0)$.

This billiard map defines a 2-dimensional dynamical system and the orbit of any initial point (s_0,ψ_0) is the set $\mathcal{O}(s_0,\psi_0) = \{T_\Gamma^i(s_0,\psi_0) = (s_i,\psi_i), i \in \mathbb{Z}\}$.

3.1 Properties of the generating function

As above, let d_S be the geodesic distance on S and $\Gamma \subset S$ be an oval, parameterized by the arclength parameter s.

Lemma 2. *Let $T_\Gamma(s_0,\psi_0) = (s_1,\psi_1)$ be the billiard map on Γ and $g(s_0,s_1) = -d_S(\Gamma(s_0),\Gamma(s_1))$. Then g verifies*

$$\frac{\partial g}{\partial s_0}(s_0,s_1) = \cos\psi_0 \qquad \textit{and} \qquad \frac{\partial g}{\partial s_1}(s_0,s_1) = -\cos\psi_1$$

Proof. Let $\tau_i, i = 0$ or 1, be the unitary tangent vector to the oriented geodesic joining $\Gamma(s_0)$ to $\Gamma(s_1)$, at $\Gamma(s_i)$.

When $S = \mathbb{S}^2_+$ we have that $\cos g(s_0,s_1) = \langle \Gamma(s_0), \Gamma(s_1)\rangle$ and then

$$\begin{aligned} \frac{\partial g}{\partial s_0}(s_0,s_1) &= \frac{\langle \Gamma'(s_0), -\Gamma(s_1)\rangle}{\sin g(s_0,s_1)} = \frac{\langle \Gamma'(s_0), \sin g(s_0,s_1)\,\tau_0 - \cos g(s_0,s_1)\,\Gamma(s_0)\rangle}{\sin g(s_0,s_1)} \\ &= \langle \Gamma'(s_0), \tau_0\rangle = \cos\psi_0. \end{aligned} \tag{1}$$

Analogously $\frac{\partial g}{\partial s_1}(s_0,s_1) = \langle \Gamma'(s_1), -\tau_1\rangle = -\cos\psi_1$.

When $S = \mathbb{H}^2_+$ we have that $\cosh g(s_0,s_1) = -\langle\langle \Gamma(s_0), \Gamma(s_1)\rangle\rangle$ and then

$$\begin{aligned} \frac{\partial g}{\partial s_0}(s_0,s_1) &= \frac{\langle\langle \Gamma'(s_0), -\Gamma(s_1)\rangle\rangle}{\sinh g(s_0,s_1)} = \frac{\langle\langle \Gamma'(s_0), \sinh g(s_0,s_1)\,\tau_0 - \cosh g(s_0,s_1)\,\Gamma(s_0)\rangle\rangle}{\sinh g(s_0,s_1)} \\ &= \langle\langle \Gamma'(s_0), \tau_0\rangle\rangle = \cos\psi_0. \end{aligned} \tag{2}$$

Analogously $\frac{\partial g}{\partial s_1}(s_0,s_1) = \langle\langle \Gamma'(s_1), -\tau_1\rangle\rangle = -\cos\psi_1$. □

Let $p = -\cos\psi \in (-1,1)$. Lemma 2 implies that the arclength s and the tangent momentum p are conjugated coordinates with generating function g for the billiard map, or,

$$T_\Gamma(s_0,p_0) = (s_1,p_1) \Longleftrightarrow \frac{\partial g}{\partial s_0} = -p_0, \;\; \frac{\partial g}{\partial s_1} = p_1$$

leading to the variational definition of billiards.

Lemma 3. *Let k_i be the geodesic curvature of Γ at $s_i, i = 0, 1$. The second derivatives of g are*

$$\frac{\partial^2 g}{\partial s_i^2}(s_0, s_1) = \begin{cases} \frac{\sin^2\psi_i}{\tan g(s_0,s_1)} + k_i \sin\psi_i & in \quad \mathbb{S}^2_+ \\ \frac{\sin^2\psi_i}{\tanh g(s_0,s_1)} + k_i \sin\psi_i & in \quad \mathbb{H}^2_+ \end{cases}$$

$$\frac{\partial^2 g}{\partial s_0 \partial s_1}(s_0, s_1) = \begin{cases} \frac{\sin\psi_0 \sin\psi_1}{\sin g(s_0,s_1)} & in \quad \mathbb{S}^2_+ \\ \frac{\sin\psi_0 \sin\psi_1}{\sinh g(s_0,s_1)} & in \quad \mathbb{H}^2_+ \end{cases}$$

Proof. Let τ_i , η_i and ν_i be the unitary tangent, normal and binormal vectors, respectively, to the oriented geodesic joining $\Gamma(s_0)$ to $\Gamma(s_1)$, at $\Gamma(s_i)$, seen as a curve in $\mathbb{R}^3$. When $S = \mathbb{S}^2_+$ or $\mathbb{H}^2_+$, since the geodesic is contained on a plane passing by the origin, $\eta_i = -\Gamma(s_i)$, ν_i is a constant unitary vector, normal to the plane, and $\{\tau_i, \nu_i\}$ is an orthonormal basis for the tangent plane of S at $\Gamma(s_i)$. For simplicity of notation we write just g for $g(s_0, s_1)$.
In the case $S = \mathbb{S}^2_+$ we differentiate (1) getting

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0^2} &= \frac{-\langle \Gamma''(s_0), \Gamma(s_1)\rangle - \cos^2\psi_0 \cos g}{\sin g} \\ &= \frac{-\langle -\Gamma(s_0) + k_0\,\Gamma(s_0) \times \Gamma'(s_0), \cos g\,\Gamma(s_0) - \sin g\,\tau_0\rangle - \cos^2\psi_0 \cos g}{\sin g} \\ &= \frac{\cos g + k_0 \sin\psi_0 \sin g - \cos^2\psi_0 \cos g}{\sin g} = \frac{\cos g \sin^2\psi_0}{\sin g} + k_0 \sin\psi_0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0 \partial s_1} &= \frac{-\langle \Gamma'(s_0), \Gamma'(s_1)\rangle + \cos\psi_0 \cos\psi_1 \cos g}{\sin g} \\ &= \frac{-\langle \cos\psi_0 \tau_0 - \sin\psi_0 \nu_0, \cos\psi_1 \tau_1 + \sin\psi_1 \nu_1\rangle + \cos\psi_0 \cos\psi_1 \cos g}{\sin g} \\ &= \frac{\sin\psi_0 \sin\psi_1}{\sin g} \end{aligned}$$

When $S = \mathbb{H}^2_+$ we differentiate (2) getting

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0 \partial s_1} &= \frac{-\langle\langle \Gamma'(s_0), \Gamma'(s_1)\rangle\rangle + \cos\psi_0 \cos\psi_1 \cosh g}{\sinh g} \\ &= \frac{-\langle\langle -\Gamma(s_0) + k_0\,\Gamma(s_0) \times \Gamma'(s_0), \cosh g\,\Gamma(s_0) - \sinh g\,\tau_0\rangle\rangle - \cos^2\psi_0 \cosh g}{\sinh g} \\ &= \frac{\cosh g \sin^2\psi_0}{\sinh g} + k_0 \sin\psi_0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 g}{\partial s_0 \partial s_1} &= \frac{-\langle\langle \Gamma'(s_0), \Gamma'(s_1)\rangle\rangle + \cos\psi_0 \cos\psi_1 \cosh g}{\sinh g} \\ &= \frac{-\langle\langle \cos\psi_0 \tau_0 - \sin\psi_0 \nu_0, \cos\psi_1 \tau_1 + \sin\psi_1 \nu_1\rangle\rangle + \cos\psi_0 \cos\psi_1 \cosh g}{\sinh g} \\ &= \frac{\sin\psi_0 \sin\psi_1}{\sinh g} \end{aligned}$$

In both cases, the calculation of $\frac{\partial^2 g}{\partial s_1^2}$ is analogous to $\frac{\partial^2 g}{\partial s_0^2}$. □

3.2 Properties of the billiard map

In this subsection we will prove that

Theorem 1. *Let $\Gamma \subset S$ be a C^j oval, $j \geq 2$. The associated billiard map T_Γ is a reversible, conservative, Twist, C^{j-1}-diffeomorphism.*

The proof will follow directly from the lemmas below. As above, s stands for the arclength parameter for Γ, k_i is the geodesic curvature of Γ at s_i, d_S is the distance on S and $g(s_0, s_1) = -d_S(\Gamma(s_0), \Gamma(s_1))$.

Lemma 4. *T_Γ is invertible and reversible.*

Proof. Any trajectory of the billiard problem can be travelled in both senses. So, if $T_\Gamma(s_i, \psi_i) = (s_{i+1}, \psi_{i+1})$ then $T_\Gamma^{-1}(s_i, \pi - \psi_i) = (s_{i-1}, \pi - \psi_{i-1})$.
Let I be the involution on $\mathcal{C}$ given by $I(s, \psi) = (s, \pi - \psi)$. Clearly $I^{-1} = I$.
We have then that $T_\Gamma^{-1} = I \circ T_\Gamma \circ I$ or $I \circ T_\Gamma^{-1} = T_\Gamma \circ I$, i.e., T_Γ is reversible. □

Lemma 5. *T_Γ is a C^{j-1} diffeomorphism.*

Proof. Let $T_\Gamma(\overline{s}_0, \overline{\psi}_0) = (\overline{s}_1, \overline{\psi}_1)$ and V_0 and V_1 be two disjoint open intervals containing $\overline{s}_0$ and $\overline{s}_1$ respectively. We define

$$G : V_0 \times V_1 \times (0, \pi) \mapsto \mathbb{R}, \quad G(s_0, s_1, \psi_0) = \frac{\partial g}{\partial s_0}(s_0, s_1) - \cos\psi_0.$$

G is a C^{j-1} function, since Γ and g are C^j. Then, by lemma 2, $G(\overline{s}_0, \overline{s}_1, \overline{\psi}_0) = 0$ and $\frac{\partial G}{\partial s_1}(s_0, s_1) = \frac{\partial^2 g}{\partial s_0 \partial s_1}(\overline{s}_0, \overline{s}_1) \neq 0$ by lemma 3, since $\overline{\psi}_0, \overline{\psi}_1 \in (0, \pi)$. So we can locally define a C^{j-1} function $s_1 = s_1(s_0, \psi_0)$ such that $G(s_0, s_1(s_0, \psi_0), \psi_0) = 0$.
Taking now $\psi_1(s_0, \psi_0) = \arccos(-\frac{\partial g}{\partial s_1}(s_0, s_1(s_0, \psi_0)))$ we conclude that $T_\Gamma(s_0, \psi_0) = (s_1(s_0, \psi_0), \psi_1(s_0, \psi_0))$ is a C^{j-1} function.
As T_Γ is invertible, with $T_\Gamma^{-1} = I \circ T_\Gamma \circ I$, we conclude that T_Γ is a C^{j-1} diffeomorphism. □

Differentiating the expressions $\cos\psi_0 = \frac{\partial g}{\partial s_0}(s_0, s_1(s_0, \psi_0))$ and $\cos\psi_1(s_0, \psi_0) = -\frac{\partial g}{\partial s_1}(s_0, s_1(s_0, \psi_0))$ we obtain

Lemma 6.

$$\frac{\partial^2 g}{\partial s_0^2} + \frac{\partial^2 g}{\partial s_0 \partial s_1}\frac{\partial s_1}{\partial s_0} = 0 \qquad \frac{\partial^2 g}{\partial s_0 \partial s_1}\frac{\partial s_1}{\partial \psi_0} = -\sin\psi_0$$

$$\frac{\partial^2 g}{\partial s_0 \partial s_1} + \frac{\partial^2 g}{\partial s_1^2}\frac{\partial s_1}{\partial s_0} = \sin\psi_1 \frac{\partial \psi_1}{\partial s_0} \qquad \frac{\partial^2 g}{\partial s_1^2}\frac{\partial s_1}{\partial \psi_0} = \sin\psi_1 \frac{\partial \psi_1}{\partial \psi_0}$$

Lemma 7. *T_Γ is a Twist map.*

Proof. By lemmas 3 and 6 and remembering that $\psi \in (0, \pi)$ and $g < 0$, we have that $\frac{\partial s_1}{\partial \psi_0} > 0$ and T_Γ has the Twist property. □

Using the formulas of lemmas 6 and 3, we obtain the derivative of the billiard map as:

Lemma 8. $DT_\Gamma(s_0, \psi_0) = \begin{pmatrix} \frac{\partial s_1}{\partial s_0} & \frac{\partial s_1}{\partial \psi_0} \\ \frac{\partial \psi_1}{\partial s_0} & \frac{\partial \psi_1}{\partial \psi_0} \end{pmatrix}$ *where*

- *in* $\mathbb{S}^2_+$

$$\frac{\partial s_1}{\partial s_0} = \frac{-k_0 \sin g - \sin\psi_0 \cos g}{\sin\psi_1}, \quad \frac{\partial s_1}{\partial \psi_0} = \frac{-\sin g}{\sin\psi_1}, \quad \frac{\partial \psi_1}{\partial \psi_0} = \frac{-\sin\psi_1 \cos g - k_1 \sin g}{\sin\psi_1}$$

$$\frac{\partial \psi_1}{\partial s_0} = \frac{-k_0 \sin\psi_1 \cos g + \sin\psi_0 \sin\psi_1 \sin g - k_0 k_1 \sin g - k_1 \sin\psi_0 \cos g}{\sin\psi_1}$$

- *in* $\mathbb{H}^2_+$

$$\frac{\partial s_1}{\partial s_0} = \frac{-k_0 \sinh g - \sin\psi_0 \cosh g}{\sin\psi_1}, \quad \frac{\partial s_1}{\partial \psi_0} = -\frac{\sinh g}{\sin\psi_1}, \quad \frac{\partial \psi_1}{\partial \psi_0} = \frac{-\sin\psi_1 \cosh g - k_1 \sinh g}{\sin\psi_1}$$

$$\frac{\partial \psi_1}{\partial s_0} = \frac{-k_0 \sin\psi_1 \cosh g - \sin\psi_0 \sin\psi_1 \sinh g - k_0 k_1 \sinh g - k_1 \sin\psi_0 \cosh g}{\sin\psi_1}$$

Calculating now the determinant of DT_Γ it is easy to see that

Lemma 9. *T_Γ preserves the measure $d\mu = \sin\psi ds d\psi$.*

4 Periodic Orbits

The oval billiard map T_Γ is then a conservative reversible discrete 2-dimensional dynamical system, defined by a C^{j-1}-Twist map, $j \geq 2$. To each n-periodic orbit of T_Γ is associated a closed geodesic polygon with vertices on the oval Γ. Among them we distinguish the Birkhoff periodic orbits of type (m, n), the n-periodic orbits such that the corresponding trajectory winds m times around Γ before closing, meaning that the orbit has rotation number m/n. The classical result for Birkhoff periodic orbits of Twist maps (see, for instance, [12], page 356) applied to our billiards[1] states that:

Theorem 2. *Given relatively primes m and n, $n \geq 2$ and $0 < m < n$, there exist at least two Birkhoff orbits of type (m, n) for the oval billiard map T_Γ.*

At least two does not necessarily mean *in a finite number*. As was proved by Bolotin in [4], the geodesic circular billiard on S is integrable and then has infinitely many Birkhoff orbits of any period. On the other side, generic C^1-diffeomorphisms defined on compact sets have a finite number of nondegenerate periodic orbits of each period. This will be the case also for oval billiards on S, although encountering here two main differences: the domain of the billiard map is an open cylinder and perturbations of billiards as diffeomorphisms may not be billiards. To circumvent these problems we will perturb the boundary curve Γ, instead of the map itself, and will find compact sets on the open cylinder where the periodic orbits lay, analogously to the way it was done by Dias Carneiro, Oliffson Kamphorst and Pinto-de-Carvalho [7] for planar billiards. For technical reasons that will be clear below, we will only consider C^∞ boundary ovals. Using those facts we will be able to prove that

Theorem 3. *For each fixed period $n \geq 2$, having only a finite number of n-periodic orbits, all nondegenerate, is an open and dense property for C^∞ oval billiards on $\mathbb{H}^2_+$. For C^∞ oval billiards on $\mathbb{S}^2_+$ it is only an open property.*

[1] For oval planar billiards, this result was proven by Birkhoff in [2].

4.1 Normal perturbations of ovals

Let $\Gamma : I \mapsto S$ be a C^∞ oval parameterized by the arclength parameter s and $\eta(s)$ be the unitary normal vector such that $\{\Gamma'(s), \eta(s)\}$ is an oriented positively and orthonormal basis of the tangent plane of S at $\Gamma(s)$.

Definition 3. *β is ϵ-C^2-close to Γ if β can be written as*

$$\beta(s) = \begin{cases} \dfrac{\Gamma(s) + \lambda(s)\eta(s)}{\sqrt{1+\lambda^2(s)}} & in \;\; \mathbb{S}^2_+ \\ \dfrac{\Gamma(s) + \lambda(s)\eta(s)}{\sqrt{1-\lambda^2(s)}} & in \;\; \mathbb{H}^2_+ \end{cases} \tag{3}$$

where $\lambda : I \to \mathbb{R}$ is $C^j, j \geq 2$ with $||\lambda||_2 < \epsilon$.

Remark that if β is ϵ-C^2-close to Γ then the trace of β is contained on the tubular neighbourhood $V_\epsilon(\Gamma)$, given by the radial projection of the set $\{\Gamma(s)+\lambda\eta(s),\ s \in I,\ -\epsilon < \lambda < \epsilon\}$ onto S, which, as Γ is an oval, is an open subset of S for ϵ sufficiently small.

Lemma 10. *If ϵ is sufficiently small, λ is C^∞ and β is ϵ-C^2-close to Γ, then β is a C^∞ oval.*

Proof. As λ is a C^∞ function, β is a C^∞ curve. As Γ is closed, β is a closed curve.
Moreover, $\beta'(s) = \Gamma'(s) + R_1(s, \lambda(s), \lambda'(s))$ and $\beta''(s) = \Gamma''(s) + R_2(s, \lambda(s), \lambda'(s), \lambda''(s))$ with $||R_i|| \to 0$ if $||\lambda||_2 \to 0$, uniformly on s, which implies that, if ϵ is sufficiently small, β is regular and has strictly positive geodesic curvature. $\square$

Lemma 11. *If ϵ is sufficiently small and β is ϵ-C^2-close to Γ, then their associated generating functions g_Γ and g_β are close in the C^2 topology.*

Proof. Let Γ be a C^∞ oval on S, parameterized by the arclength parameter s and β be a normal perturbation of Γ as in (3), parameterized by the arclength parameter σ. The generating functions are $g_\Gamma(s_0, s_1) = -d_S(\Gamma(s_0), \Gamma(s_1))$ and $g_\beta(s_0, s_1) = -d_S(\beta(\sigma(s_0)), \beta(\sigma(s_1)))$.
The result will follow immediately from the following remarks:
$\beta(s) = \Gamma(s)+R_0(s, \lambda(s))$, $\beta'(s) = \Gamma'(s)+R_1(s, \lambda(s), \lambda'(s))$, $\beta''(s) = \Gamma''(s)+R_2(s, \lambda(s), \lambda'(s), \lambda''(s))$ and $\sigma = s + R_3(s, \lambda, \lambda')$, with $\|R_i\| \to 0$ if $\|\lambda\|_2 \to 0$, uniformly on s. $\square$

Proposition 1. *If ϵ is sufficiently small and β is ϵ-C^2-close to Γ then their associated billiard maps T_Γ and T_β are close in the C^1 topology.*

Proof. By the construction of the billiard map from the generating function, as was done for instance in the proof of lemma 5, billiard maps will be close in the C^1 topology if their generating functions are close in the C^2 topology. So the result follows directly from the lemma 11. $\square$

4.2 Finite number of nondegenerate n-periodic orbits

4.2.1 Openness

Lemma 12. *Let $\Gamma : I \to S$ be an oval. There exists a positive real number δ_n such that any n-periodic orbit of T_Γ has at least one point on the compact strip $I \times [\delta_n, \pi - \delta_n]$.*

Proof. Let $\{(\overline{s}_0,\overline{\psi}_0),...,(\overline{s}_{n-1},\overline{\psi}_{n-1})\}$ be an n-periodic orbit of T_Γ. Then the points $\Gamma(\overline{s}_i)$ are the vertices of a geodesic polygon P inscribed on Γ. Let us suppose that this geodesic polygon is simple and let ζ_i be the internal angles.
In $\mathbb{H}^2_+$, by Gauss-Bonnet Theorem, $\sum\zeta_i \leq (n-2)\pi$. Then $\sum\overline{\psi}_i \geq \pi$ and there exists i_0 such that $\overline{\psi}_{i_0} \geq \frac{\pi}{n}$. By the reversibility of T_Γ, $\pi - \overline{\psi}_{i_0} \geq \frac{\pi}{n}$ and then $\frac{\pi}{n} \leq \overline{\psi}_{i_0} \leq \pi - \frac{\pi}{n}$.
In $\mathbb{S}^2_+$, Gauss-Bonnet Theorem gives $\sum\zeta_i > (n-2)\pi$. So we have to work in a slightly different way. Since $\Gamma(I) \subset \mathbb{S}^2_+$ there exists $m_0 > n$ such that the area A_Γ enclosed by Γ satisfies $A_\Gamma < 2\pi - \frac{\pi n}{m_0}$. Suppose now that there is an n-periodic orbit $\{(\overline{s}_0,\overline{\psi}_0),...,(\overline{s}_{n-1},\overline{\psi}_{n-1})\}$, associated to a simple geodesic polygon P and such that $\overline{\psi}_i < \frac{\pi}{m_0}$ for all i. The area A_P enclosed by P satisfies $A_P < A_\Gamma < 2\pi - \frac{\pi n}{m_0}$. But, again by Gauss-Bonnet Theorem, $A_P \geq 2\pi - \frac{1}{2}\sum(\pi - \zeta_i) = 2\pi - \sum\overline{\psi}_i > 2\pi - \frac{\pi n}{m_0} > A_\Gamma$ and then there is at least one i_0 such that $\frac{\pi}{m_0} < \overline{\psi}_i$. Once more, by the reversibility of T_Γ, $\frac{\pi}{m_0} \leq \overline{\psi}_{i_0} \leq \pi - \frac{\pi}{m_0}$.
If the geodesic polygon P is not simple, we can take a new simple polygon $\tilde{P}$ with the same vertices on Γ as P and with internal angles $\tilde{\zeta}_i$. Taking only the internal angles ζ_i of P at the vertices on Γ we have that $\sum\zeta_i < \sum\tilde{\zeta}_i$ and the result follows. □

Proposition 2. *For a fixed period $n \geq 2$, the set of C^∞-ovals on S such that its associated billiard map has only a finite number of n-periodic orbits, all nondegenerate, is an open set.*

Proof. Suppose that the C^∞ diffeomorphism T_Γ has only nondegenerate n-periodic orbits. To each one of these orbits corresponds a fixed point of T^n_Γ on the compact strip $I \times [\delta_n, \pi - \delta_n]$. So they must be in a finite number and then T_Γ has only a finite number of nondegenerate n-periodic orbits. Taking ϵ sufficiently small, any perturbation β ϵ-C^2-close to Γ corresponds to a billiard map T_β C^1-close to T_Γ and will have also only a finite number of nondegenerate n-periodic orbits. □

4.2.2 Density

Suppose $\{(\overline{s}_0,\overline{\psi}_0),(\overline{s}_1,\overline{\psi}_1),...,(\overline{s}_{n-1},\overline{\psi}_{n-1})\}$ a degenerate n-periodic orbit for T_Γ. As $\det(DT^n_\Gamma|_{(\overline{s}_0,\overline{\psi}_0)}) = 1$, being degenerate translates as $\mathrm{tr}(DT^n_\gamma|_{(\overline{s}_0,\overline{\psi}_0)}) = \pm 2$.
By lemma 8

$$\begin{aligned} DT^n_{(\overline{s}_0,\overline{\psi}_0)} &= DT_{(\overline{s}_{n-1},\overline{\psi}_{n-1})}DT_{(\overline{s}_{n-2},\overline{\psi}_{n-2})}...DT_{(\overline{s}_0,\overline{\psi}_0)} \\ &= \frac{1}{\sin\overline{\psi}_0...\sin\overline{\psi}_{n-1}}A_{n-1}....A_1A_0 \end{aligned}$$

Each matrix $A_i = k_{i+1}k_iB_i + k_{i+1}C_i + k_iD_i + E_i$, where k_i is the geodesic curvature of Γ at s_i, and the entries of the matrices B_i, C_i, D_i, E_i depend only on the angles $\overline{\psi}_i$ and $\overline{\psi}_{i+1}$ and on the geodesic distance between $\Gamma(s_i)$ and $\Gamma(s_{i+i})$.
Let us fix our attention on one impact point of the degenerate n-periodic trajectory, say $\Gamma(s_1)$. If it impacts m_1 times at $\Gamma(s_1)$ then

$$\mathrm{tr}(DT^n_{(\overline{s}_0,\overline{\psi}_0)}) = p_1(k_1) = b_{m_1}k_1^{m_1} + ... + b_1k_1 + c_1 = \pm 2$$

where the coefficients b_j and c_1 do not depend on k_1.
If any of the $b_j \neq 0$, we can take a sufficiently small normal perturbation β, as in (3), with a function λ satisfying $\lambda(s_1) = 0, \lambda'(s_1) = 0, \lambda''(s_1) \neq 0$, and $\lambda \equiv 0$ outside an interval containing s_1 and no other point of the trajectory. So we preserve the periodic orbit but change the

geodesic curvature at the vertex $\Gamma(s_1)$, which implies $\mathrm{tr}(DT^n_\beta) \neq \mathrm{tr}(DT^n_\Gamma)$, i.e, on β the orbit is nondegenerate.
If all the $b_j = 0$ we take the next impact s_2 (as a billiard has no fixed point, $s_2 \neq s_1$) and then:

$$\mathrm{tr}(DT^n_{(\bar{s}_0,\bar{\psi}_0)}) = p_2(k_2) = b_{m_2}k_2^{m_2} + ... + b_1k_2 + c_2 = c_1 = \pm 2$$

where the coefficients b_j and c_2 do not depend on k_1 and k_2.
If any of the $b_j \neq 0$ we can take the small perturbation β at s_2 as above. Otherwise, we continue the process till the last impact, say s_0. We have then

$$\mathrm{tr}(DT^n_{(\bar{s}_0,\bar{\psi}_0)}) = p_0(k_0) = b_{m_0}k_0^{m_0} + ... + b_1k_0 + c_0 = \pm 2$$

where the coefficients b_j and c_0 do not depend any more on any of the geodesic curvatures k_i.
For $\mathbb{H}^2_+$ we have calculated c_0 obtaining $c_0 = (-1)^n 2\cosh L \neq \pm 2$ where $L \neq 0$ is the perimeter of the geodesic polygonal trajectory.
Then, in this case, there is a j such that $b_j \neq 0$ and we can approach T_Γ by billiards with a nondegenerate n-periodic orbit.
As nondegenerate periodic orbits are isolated, with a finite number of perturbations we can construct an oval β as close as we want to Γ such that the associated billiard map has only a finite number of nondegenerate n-periodic orbits.
We have then

Proposition 3. *For a fixed period $n \geq 2$, the set of C^∞-ovals on $\mathbb{H}^2_+$ such that its associated billiard map has only a finite number of n-periodic orbits, all nondegenerate, is a dense set.*

Unfortunately, the same techniques do not work at $\mathbb{S}^2_+$. If all the b_j are zero except for the last impact s_0 we get $\mathrm{tr}(DT^n_{(\bar{s}_0,\bar{\psi}_0)}) = p_0(k_0) = b_{m_0}k_0^{m_0} + ... + b_1k_0 + c_0 = \pm 2$ where, taking L as the perimeter of the trajectory, $c_0 = (-1)^n\, 2\, cos\, L$ and all the coefficients b_j are multiples of $\sin L$. So, if $L = \mu\pi$, $c_0 = \pm 2$ and all the $b_j = 0$ and our normal perturbation do not destroy the degenerescence of the orbit.
Summarizing we have

Theorem 4. *For each fixed period $n \geq 2$, having only a finite number of n-periodic orbits, all nondegenerate, is an open and dense property for C^∞ oval billiards on $\mathbb{H}^2_+$. For C^∞ oval billiards on $\mathbb{S}^2_+$ it is only an open property.*

4.2.3 A remark about the planar case

Dias Carneiro, Oliffson Kamphorst and Pinto-de-Carvalho [7] proved that having only a finite number of n-periodic orbits, all nondegenerate, is a generic property for C^2 planar oval billiards. Although their result is true, they did not analyze the trajectories with multiple impact points. Using our techniques we can fill in this gap remarking that if all the b_j are zero we get in the final step

$$\mathrm{tr}(DT^n_{(\bar{s}_0,\bar{\psi}_0)}) = p_0(k_0) = b_{m_0}k_0^{m_0} + ... + b_1k_0 + (-1)^n 2L = \pm 2$$

where L the perimeter of the trajectory.
If $L \neq 1$, $c_0 \neq \pm 2$ and then there is a $b_j \neq 0$ and the special normal perturbation on the plane destroys the degenerescence.
If $L = 1$, the first coefficient $b_1 = (-1)^{n+1}2(\frac{1}{\sin\bar{\psi}_0} + ... + \frac{1}{\sin\bar{\psi}_{m_0}}) \neq 0$ and, again, the special normal perturbation solves the problem.

4.3 Stability of periodic orbits

Let $\mathcal{O}(s_0,\psi_0) = \{(s_0,\psi_0),(s_1,\psi_1),...,(s_{n-1},\psi_{n-1})\}$ be an n-periodic orbit for T_Γ.
As $\det(DT_\Gamma^n|_{(s_0,\psi_0)}) = 1$, $\mathcal{O}(s_0,\psi_0)$ is hyperbolic if $|\mathrm{tr}(DT_\gamma^n|_{(s_0,\psi_0)})| > 2$, elliptic if $|\mathrm{tr}(DT_\gamma^n|_{(s_0,\psi_0)})| < 2$ and parabolic if $|\mathrm{tr}(DT_\gamma^n|_{(s_0,\psi_0)})| = 2$.
Let $W(s_0,s_1,...,s_{n-1}) = \sum_{i=0}^{n-1} g(s_i,s_{i+1})$, where $g(s_0,s_1) = -d_s(\Gamma(s_0),\Gamma(s_1))$ is the generating function and $s_n = s_0$, be the action defined on the n-torus $\mathbb{R}/l\mathbb{Z}\times...\times\mathbb{R}/l\mathbb{Z}$ minus the set $\{(s_0,s_1,...,s_{n-1})\,\text{s.t}\,\exists i\neq j\,\text{with}\,s_i = s_j\}$.
The critical points of W are the coordinates of the vertices on Γ of the n-periodic trajectories, since $\frac{\partial g}{\partial s_i}(s_{i-1},s_i) = -\frac{\partial g}{\partial s_i}(s_i,s_{i+1})$. Remark that $W<0$ and it will always have a global minimum (perhaps degenerate), but not a global maximum.
The MacKay-Meiss formula [15] relates the derivative of T_Γ^n with the Hessian matrix H of W by

$$2 - \mathrm{tr}DT_\Gamma^n|_{(s_0,\psi_0)} = (-1)^{n+1}\frac{\det H}{b_0b_1...b_{n-1}}$$

where $b_i = \frac{\partial^2 g}{\partial s_i\partial s_{i+1}}(s_i,s_{i+1})$.
It implies that the nondegenerate critical points of the action W are the nondegenerate n-periodic orbits of T_Γ.
So, if T_Γ has only nondegenerate n-periodic orbits, the nondegenerate minima of W will always be hyperbolic. The other n-periodic orbits can be either hyperbolic or elliptic.

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