Fermi acceleration in time dependent billiards

S Oliffson Kamphorst, S Pinto de Carvalho, Jafferson K L da Silva (UFMG)
E D Leonel (UNESP)

CSF - May-June 2006

- Natural generalization of the one dimensional Fermi accelerator model: particle bouncing between two moving walls.
- Simple example of time dependent Hamiltonian systems.

Fundamental question: a particle moving inside a bounded region and undergoing elastic collisions with a moving boundary can be given unlimited energy?

Exact and numerical results indicate that the answer depends on the nature of the phase space of the static model and on the shape of the boundary.

References

[1] Results from: (+ J. Koiller, R. Markarian)
Time-dependent billiards (Nonlinearity 1995)
Static and time-dependent perturbations of the classical elliptical billiard (J Stat Phys 1996)
Bounded Gain of Energy on the Breathing Circle Billiard (Nonlinearity 1999)
Fermi acceleration in time-dependent oval shape billiards (preprint 2006)

[2] R. Douady: *Applications du théorème des tores invariants* (Thèse U. Paris VII 1982)

[3] V. Zhanitsky (Nonlinearity 2000)

[4] - A.J. Lichtenberg, M.A. Liebermann: in *Regular and Stochastic Motion*, (Springer-Verlag 1983)
- T. Krüger, L.D. Pustyl'nikov, S.E. Troubetzkoy (Nonlinearity 1995)
- L.D. Pustyl'nikov (Soviet Math. Dokl. 1987)
- A. Loskutov, A. Ryabov (J. Stat Phys. 2002)

1 General description: ingredients

- A time-dependent plane region whose boundary moves transversally as a periodic function of time. The billiard problem on it consists on the free motion of a point particle inside this region, colliding elastically with the pulsating boundary.
- A time dependent Hamiltonian with 2 degrees of freedom.
- Use energy (and its canonical conjugate variable time) as variables, introducing an autonomous system in 6 dimensions.
- The continuous time model can be translated (using the usual billiard coordinates) into a 4 dimensional map describing successive impacts with the moving boundary.

Coordinates:

- Geometrical coordinates $\begin{cases} s & \text{position on the instantaneous boundary} \\ \alpha & \text{direction of motion} \end{cases}$
- Energy coordinates $\begin{cases} E & \text{energy} \\ t & \text{instant (phase) of the impact} \end{cases}$

$$(s_0, \alpha_0, v_0, t_0) \to (s_1, \alpha_1, v_1, t_1)$$

Is an implicitly defined diffeomorphism preserving the measure

$$d\omega = (v \sin\alpha + u)\, ds\, d\alpha\, dE\, dt$$

u: outward normal velocity of the boundary, $v = \sqrt{2E}$: particle speed

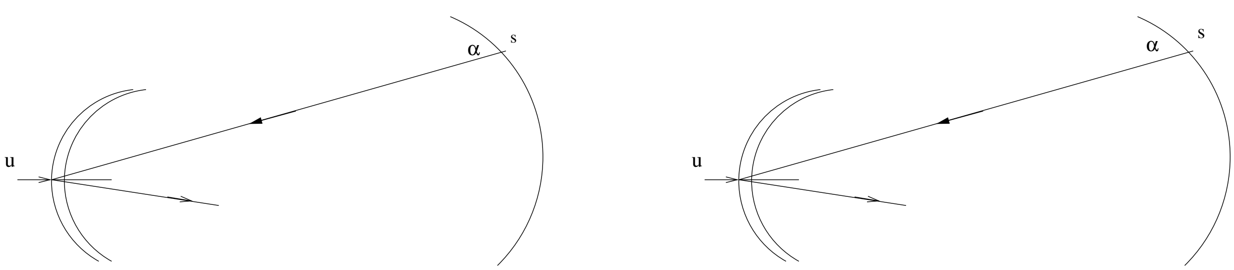


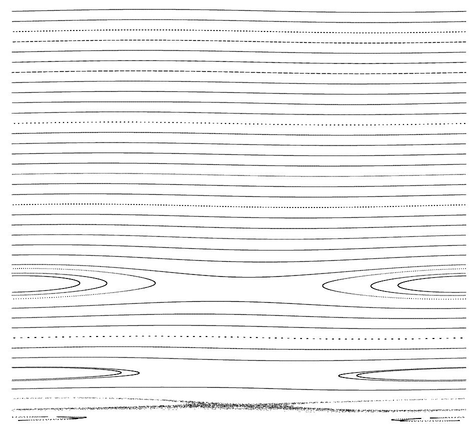
Figure 1: a) Moving out wall: decrease (normal) velocity; b) Moving in wall: increase velocity

2 The Breathing Circle: integrable models

The breathing circle is the region bounded instantaneously by a circle whose radius is a periodic function of time. In this special model, the angular momentum is conserved, implying a reduction to a 2-dimensional diffeomorphism.

Theorem 1 *Given the billiard on the breathing circle $x^2 + y^2 = R(t)^2$, with $R(t)$ a strictly positive T-periodic C^k function, $k > 7$, then, for any admissible initial condition, a particle will move with bounded velocity.*

Proof: For each fixed value of the angular momentum and v sufficiently high, we demonstrate the existence of invariant spanning curves, using a corollary of Herman's *Théorème des Courbes Transladées* by R. Douady. These spanning curves are invariant curves around the elliptic fixed point at infinity.



$v \times t (\mathrm{mod}\ 2\pi)$

Remark 1 *This result may probably be extended for quasi periodic motion of the boundary applying the results of [3].*

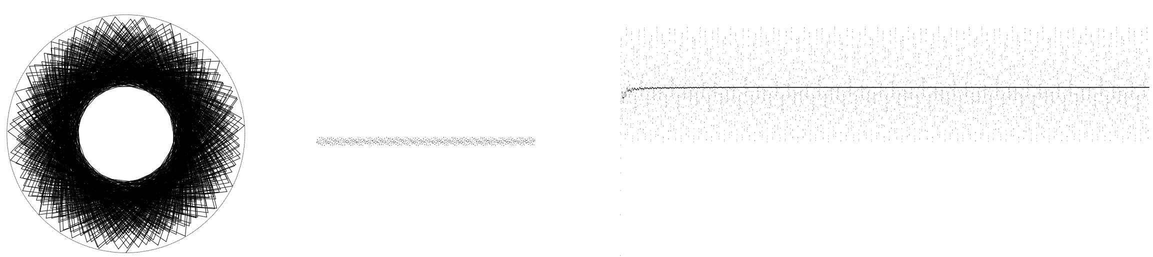


Figure 2: Quasi periodic breathing circle $R(t) = 1 + \epsilon_1 \cos \omega_1 t + \epsilon_2 \cos \omega_2 t$ with $\epsilon_1 = \epsilon_2 = 0.1$, $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$. Trajectory, phase space and mean velocity for $\alpha = 0.3\pi$, $v = 5$:

Remark 2 *Some numerical results indicate that the oscillating elliptical billiard, also has bounded energy.*

3 Fermi acceleration

Unbounded velocity occurs for:

- One dimension with periodic non smooth movement of the walls (Lichtenberg and Lieberman, Douady)
- 2 dimensional billiards with random exchange of energy with the walls. (In this case, the evolution of the velocity may be described by a *random walk* implying the exponent 1/2.)

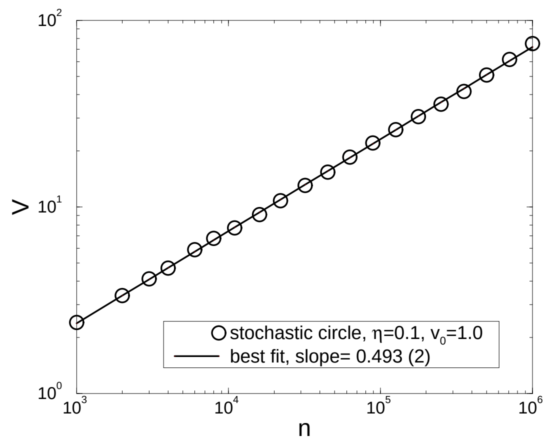


Figure 3: Stochastic circle - Mean velocity after 10^6 iterations, averaged over 100 initial conditions

- Lorentz gas (Loskutov et al)
- *Chaotic orbits* in stadium-like (Loskutov et al) or oval (Leonel) billiards.

Old conjecture: If the particle undergoes a *chaotic motion* it can be given unlimited energy by a regular and periodic motion of the boundary. *Regular trajectories* have bounded energy.

We present numerical evidences that this is not generally true.

4 A family of non integrable billiards

Consider curves defined in polar coordinates by

$$R(\theta) = 1 + \epsilon \cos 2\theta$$

It is strictly convex if $\epsilon < 0.2$. If $\epsilon > 0.2$ it has nonconvex pieces and for $\epsilon \geq 1$ the curve has self intersection and is no more simple.

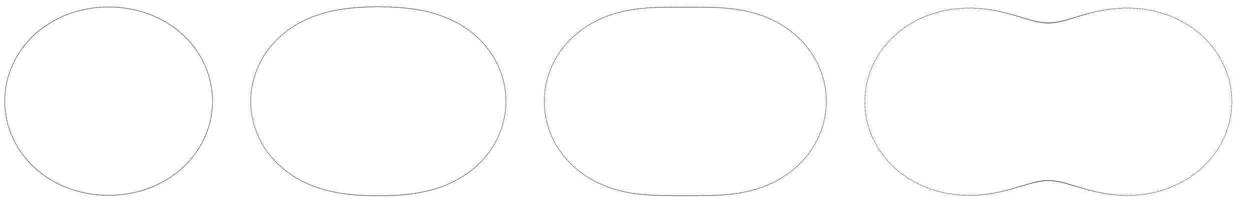


Figure 4: Curves for $\epsilon = 0.05$, 0.15, 0.2 and 0.4

The billiard map has a period 2 orbit for $\theta = \pm\pi/2$. It is elliptic for $\epsilon < 0.2$ and hyperbolic for $\epsilon > 0.2$. As long as the billiard is strictly convex, there are rotational invariant curves close to the boundary, which disappear for $\epsilon > 0.2$.

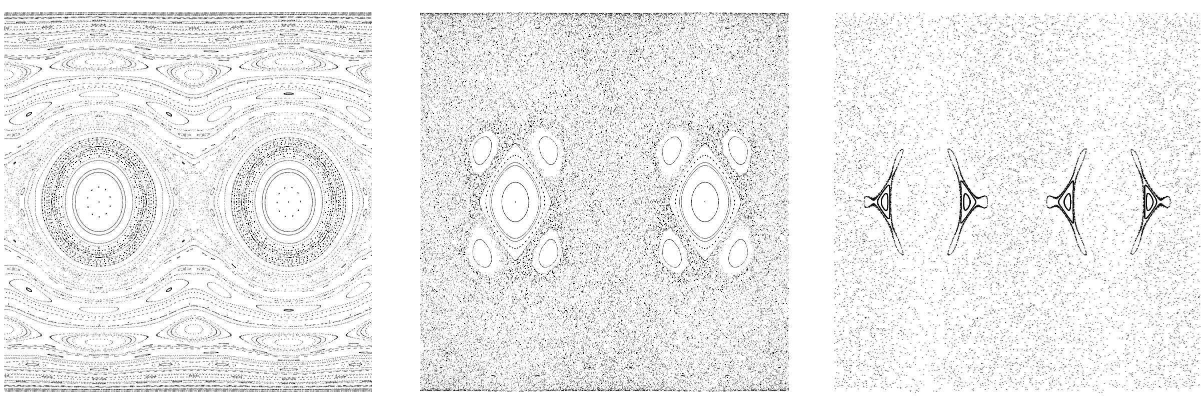


Figure 5: Phase space for $\epsilon = 0.05$, 0.15 and 0.4

The system is non integrable and, at least numerically, has both regular and irregular orbits.

Time dependent perturbations are defined by

$$R(\theta, t) = (1 + \eta_2 \cos t) + \epsilon(1 + \eta_1 \cos t) \cos 2\theta$$

If $\eta_1 = \eta_2$ shape is preserved, we call it breathing mode.

Although we are investigating a lot of different situations we will present here the perturbation of the curve corresponding to $\epsilon = 0.4$ because it has a large *chaotic region* where Fermi Acceleration may occur.

Notation

- ${V_n}$: modulus of the velocity after n collisions (iterations)
  E_n : energy after n iterations.

- ${V_n}^* = \dfrac{1}{n+1}\sum_{j=0}^{n} V_j$: orbit average of V after n iterations
  ${E_n}^* = \dfrac{1}{n+1}\sum_{j=0}^{n} E_j$: orbit average of E after n iterations.

- $< . >$: ensemble average (over initial conditions).

5 No Fermi Acceleration in the chaotic region for the breathing mode with $\epsilon = 0.4$

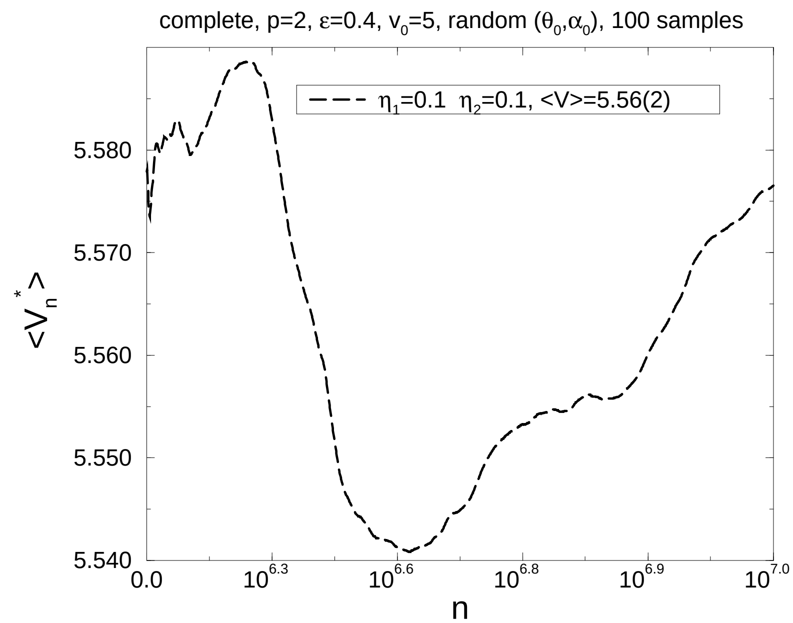
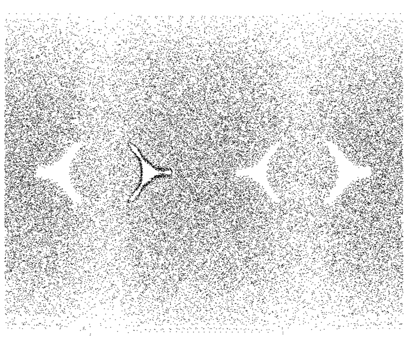


Figure 6: Averaged mean velocity for breathing mode



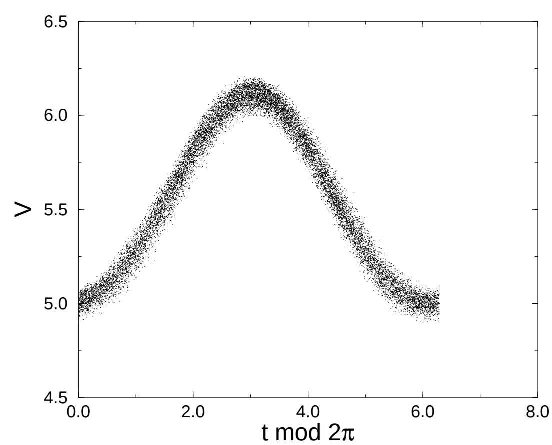


Figure 7: Orbit generated by a single initial condition

6 Fermi acceleration with an exponent larger than $1/2$ for other modes (changing shape) with $\epsilon = 0.4$

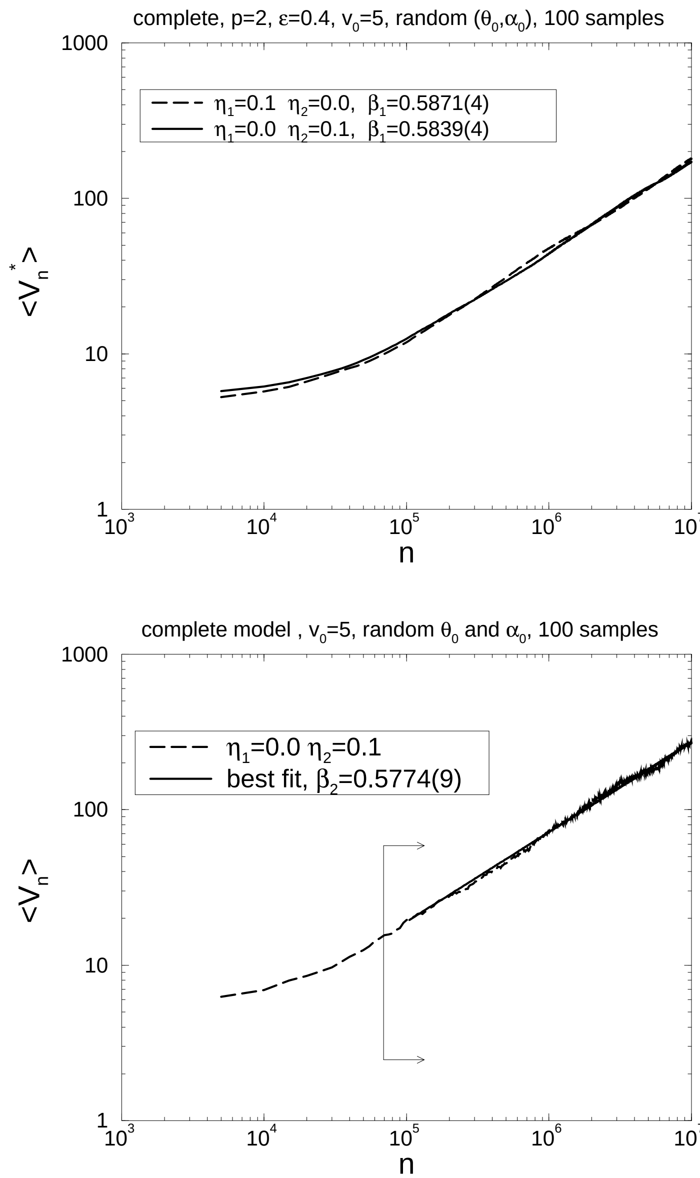


Figure 8:

7 Fermi acceleration for all modes, including the breathing, for the simplified model

The the simplified model is an approximation widely used in the literature, in particle all the results obtained by Loskutov are for simplified models.

One allows the boundary to interact with the particle through momentum exchange, but assuming that the boundary does not change in time and so

$$R(\theta, t) \sim R(\theta, 0)$$

There is no fundamental difference between a model changing shape or not.

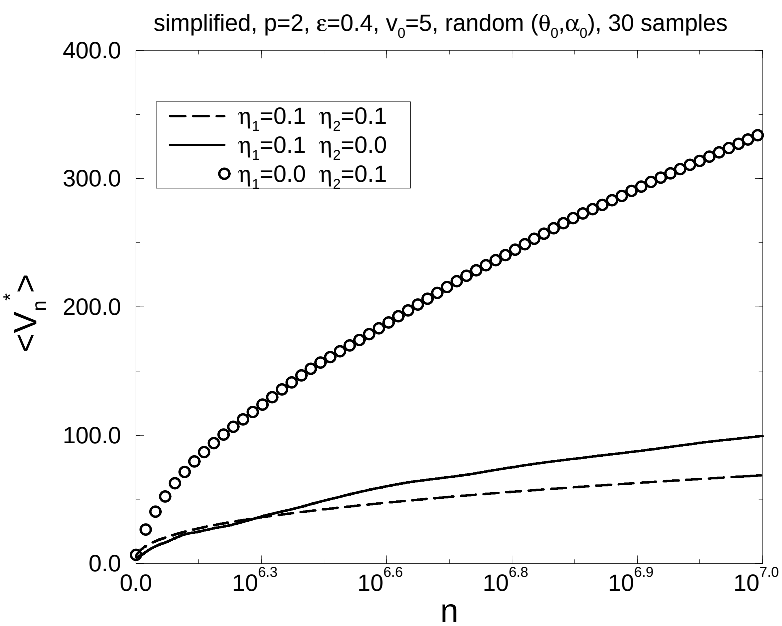


Figure 9: Simplified models

8 Stability/Instability × Fermi Acceleration

For small ϵ the billiard has many regular (stable) regions, corresponding to invariant curves and elliptical islands. Is also has instability regions, corresponding to hyperbolic behaviour.

All the results bellow are for $R(\theta, t) = 1 + \epsilon(1 + \eta_1 \cos t) \cos 2\theta$;

8.1 $\epsilon = 0.05$ and $\eta_1 = 0.5$

Fermi acceleration for an orbit spanning unstable regions of the geometrical phase space:

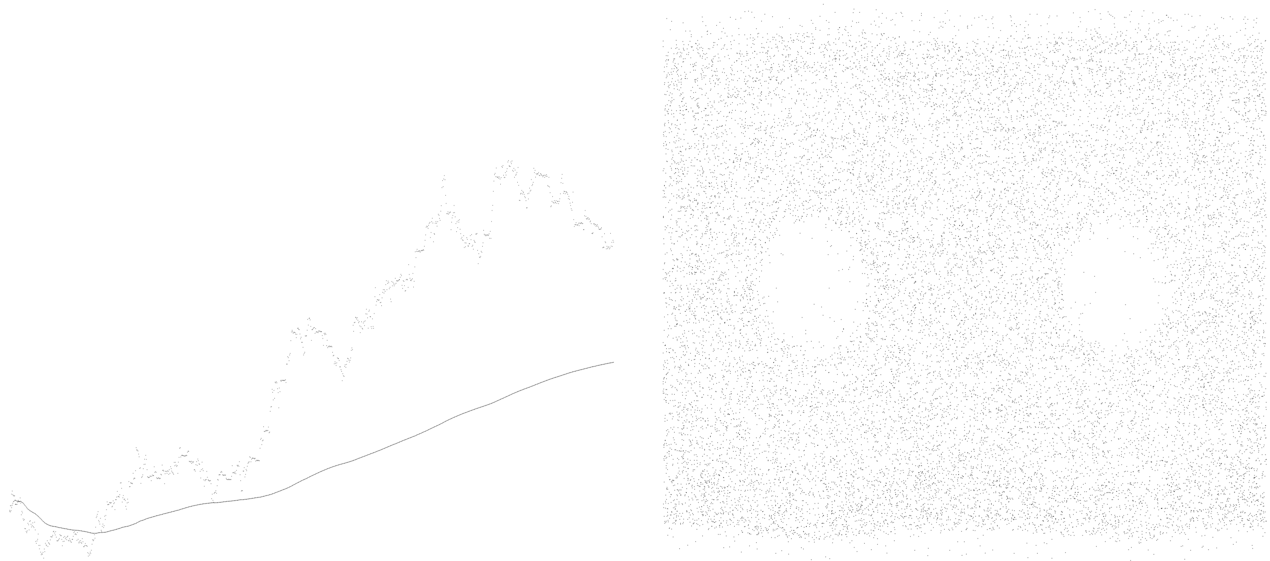


Figure 10: V_n, V_n^* and sampled geometrical phase space for 2×10^7 iterations of an initial condition in an instability region.

No Fermi acceleration for an orbit staying in a stable region of the geometrical phase space:

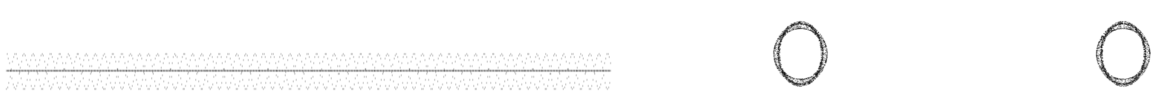


Figure 11: V_n, V_n^* and sampled geometrical phase space for 2×10^7 iterations of an initial condition in a stable region.

8.2 $\epsilon = 0.15$ and $\eta_1 = 0.25$

Fermi acceleration for an orbit spanning unstable regions of the geometrical phase space:

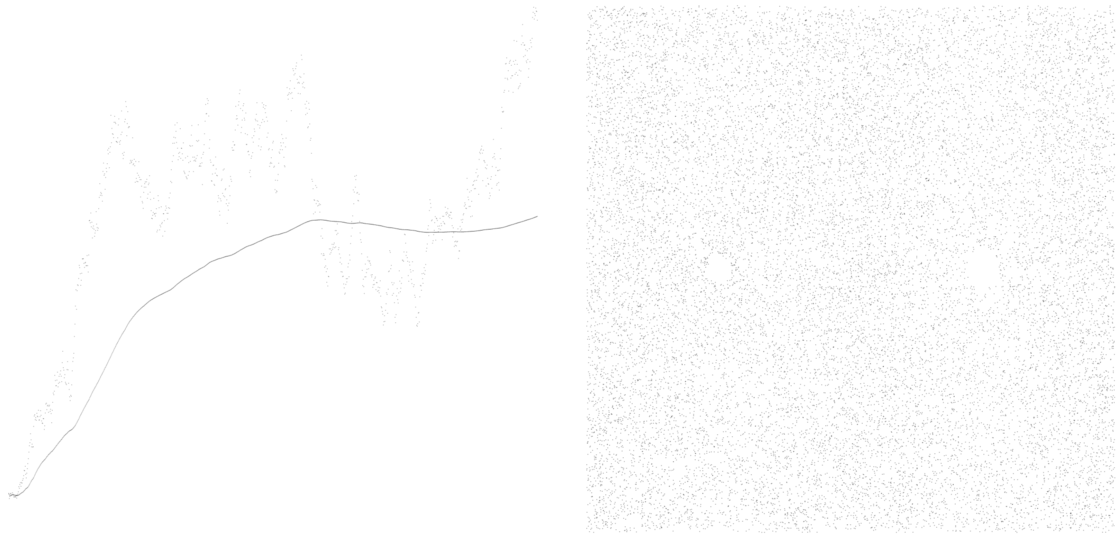


Figure 12: V_n, V_n^* and sampled geometrical phase space for 2×10^6 iterations of an initial condition in an instability region.

Fermi acceleration for an orbit in the regular region of the phase space after crossover to unstable regions.

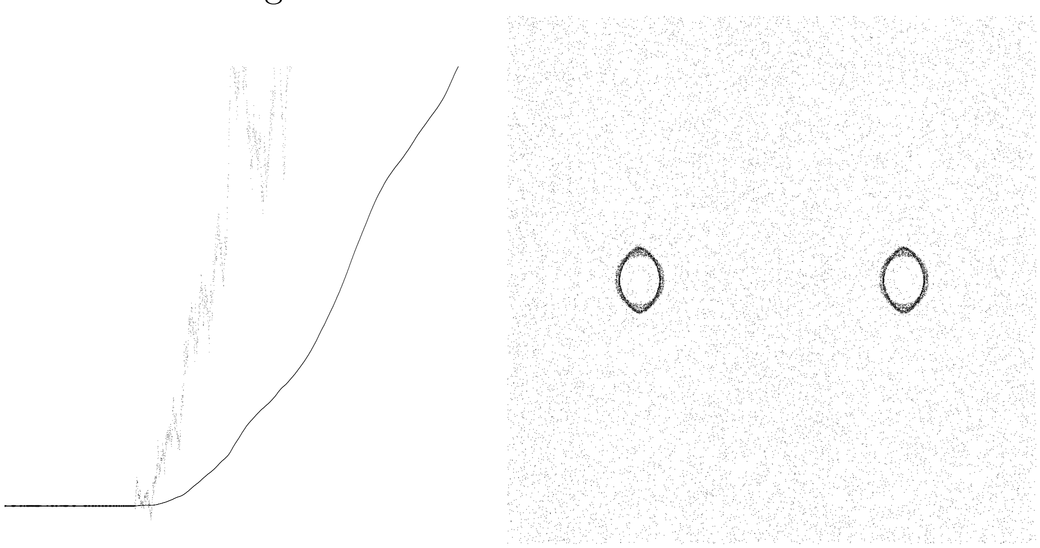


Figure 13: V_n, V_n^* and sampled geometrical phase space for 2×10^7 iterations of an initial condition in a stable region.

9 Questions

- Is there or not Fermi acceleration for the breathing models? If not, why?
- What is the difference between the simplified and the complete model?
- Can one prove the existence of Fermi acceleration for a given model?
- Why is the exponent of the Fermi acceleration bigger than 1/2?
- Is there always a crossover between regions of the geometrical phase space induced by the time dependent perturbation of the moving boundary or are there invariant regions?