Limit Sets of Convex non Elastic Billiards

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September 22, 2010

Abstract

Inspired by the work of Pujals and Sambarino on dominated splitting, we present billiards with a modified reflection law which constitute simple examples of dynamical systems with limit sets with dominated splitting and where the dynamics is a rational or irrational rotation.

1 Introduction

Pujals and Sambarino [12] studied C^2 surface diffeomorphisms having a weak form of hyperbolicity, called dominated splitting, on its limit set. They proved that, in this case, the limit set can be decomposed into, roughly speaking, two parts: one where the dynamics consists of periodic and almost periodic motions and another, where the dynamics is expansive.

Our purpose, in this work, is to construct simple examples of dynamical systems with attractors admitting a dominated splitting and where the dynamics is of the first type. We will follow the ideas developed in [11] and [2]. In [11], non conservative billiards were introduced by a modification of the reflection rule and the existence of attractors was demonstrated for a wide class of dispersing and semi-dispersing billiards and of billiards with focusing components. In [2] models were studied numerically and different attractors, periodic and chaotic are presented.

We concentrate on billiard tables with boundary formed by a unique focusing component. Strictly convex billiards present structures like KAM stability islands and invariant rotational curves non homotopic to a point. We will investigate how, in the presence of non conservative perturbations, an invariant curve will give rise to an attractor. The maps we consider here are more general than the pinball billiards introduced in [11], as the perturbation of the angle is not necessarily biased to the normal direction.

In section 2 we present the main tools needed for working with dominated splitting. Section 3 deals with the basic properties of classical billiards on ovals.

In section 4 we introduce our *non elastic billiards*. They are defined as a composition of a classical billiard followed by a change of the reflection angle, corresponding to a contraction in the vertical fibers of an invariant rotational curve. We will prove that under certain bounds on the contraction, there exists a compact strip in the phase space, such that the non elastic billiard map is a C^2-diffeomorphism from that strip onto its image. Its limit set contains the invariant curve and has a dominated splitting. Moreover, the non elastic dynamics on the invariant curve is determined by its rotation number with respect to the original classical billiard map.

This result will guide us to construct the examples of non elastic billiards on ovals with dominated splitting and attractors supporting a rational or an irrational rotation. They are presented in section 5, where we explore their properties theoretically and numerically.

2 Dominated Splitting, Cone Fields and Quadratic Forms

Let M be a compact Riemannian manifold and $f : M \to M' \subset M$ a diffeomorphism. An f-invariant set Λ is said to have a dominated splitting if we can decompose its tangent bundle into two non trivial invariant continuous subbundles $T_\Lambda M = E \oplus F$, such that, for constants $C > 0$ and $0 < \gamma < 1$:

$$\|Df^n_{|E(x)}\| \; \|Df^{-n}_{|F(f^n(x))}\| \leq C\gamma^n, \text{ for all } x \in \Lambda, n \geq 0. \tag{1}$$

It is assumed that neither of the subbundles is trivial. We remark that any hyperbolic splitting is a dominated one.

This definition says that, for n large, every direction not belonging to E must converge exponentially fast under iteration of Df to the direction F.

As usual, let the *limit set* be $L(f) = \overline{\bigcup_{x \in M} (\omega(x) \cup \alpha(x))}$ where $\omega(x)$ and $\alpha(x)$ are the ω and α-limit sets of x, respectively. A spectral decomposition theorem can be obtained for C^2 surface diffeomorphisms having dominated splitting over the limit set $L(f)$:

Theorem ([12]): *Let M be a compact 2-manifold and $f : M \to M' \subset M$ a C^2-diffeomorphism. Assume that $L(f)$ has a dominated splitting. Then $L(f)$ can be decomposed into $L(f) = \mathcal{I} \cup \tilde{\mathcal{L}}(f) \cup \mathcal{R}$ such that*

1. *$\mathcal{I}$ is a set of periodic points with bounded periods contained in a disjoint union of finitely many normally hyperbolic periodic arcs or simple closed curves.*
2. *$\mathcal{R}$ is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.*
3. *$\tilde{\mathcal{L}}(f)$ can be decomposed into a disjoint union of finitely many compact invariant and transitive sets (called basic sets). The periodic points are dense in $\tilde{\mathcal{L}}(f)$ and at most finitely many of them are non-hyperbolic periodic points. The (basic) sets above are the union of finitely many (nontrivial) homoclinic classes. Furthermore $f|\tilde{\mathcal{L}}(f)$ is expansive.*

Let $u, v : M \mapsto \mathcal{T}M$ be two vector fields such that for each $x \in M$, $u(x) = u_x$ and $v(x) = v_x$ are two linearly independent vectors in the tangent space $\mathcal{T}_xM$. They induce a nondegenerate quadratic form Q on $\mathcal{T}M$ by $Q_x(au_x + bv_x) = ab$ and a cone field given at each x by $\mathcal{C}(x) = \{w \in \mathcal{T}_xM : Q_x(w) > 0\} \cup \{0\}$ and whose boundaries, at each point, are given by $\mathcal{C}_0(x) = \{w \in \mathcal{T}_xM : Q_x(w) = 0\}$. If the vector fields are continuous, the quadratic form and the cone field are also continuous.

Given $x \in M$ and a vector $w = au_x + bv_x \in \mathcal{T}_xM$, let $Df_xw = a_1u_{f(x)} + b_1v_{f(x)} \in \mathcal{T}_{f(x)}M$ denote the image of w under the derivative Df_x. Then we have

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = [Df_x]_U \begin{pmatrix} a \\ b \end{pmatrix} \tag{2}$$

where $[Df_x]_U$ is the matrix representation of the derivative at x, with the choice of $\{u_x, v_x\}$ and $\{u_{f(x)}, v_{f(x)}\}$ as bases of $\mathcal{T}_xM$ and $\mathcal{T}_{f(x)}M$ respectively.

We can now state a very useful tool:

Lemma 1 *Let Λ be a compact f-invariant subset of M. If there is a choice of vector fields u, v such that the entries of $[Df_x]_U$ are strictly positive for every $x \in \Lambda$, then Λ has a dominated splitting.*

Proof: If the entries of $[Df_x]_U$ are strictly positive for every $x \in \Lambda$ then for every $w = au_x + bv_x$, $x \in \Lambda$, $ab \geq 0, a^2 + b^2 > 0$ we have $a_1b_1 > 0$ where $Df_xw = a_1u_{f(x)} + b_1v_{f(x)}$. This implies that, for every $x \in \Lambda$, $Df_x(\mathcal{C}(x) \cup \mathcal{C}_0) \subset \mathcal{C}(f(x))$. From [14], section 1, we have then that $Q_{f(x)}(Df_xw) > Q_x(w)$ $\forall x \in \Lambda, w \neq 0$. And so, the weaker contraction is smaller then the minimal expansion of the derivative at x. By Proposition 4.1, also in [14], it follows that Λ has dominated splitting. ■

3 Classical Billiards on Ovals

Let Γ be an oval, i.e., a plane, simple, closed, $C^k, k \geq 3$, curve, with strictly positive curvature, parameterized counterclockwise by φ, the angle between the tangent vector and an horizontal axis. Let $R(\varphi)$ be its radius of curvature at φ.

The classical billiard problem on Γ consists on the free motion of a point particle inside Γ, making elastic reflections at the impacts with the boundary. The motion is then determined by the point of reflection at Γ and the direction of motion immediately after each reflection.

They can be given by the parameter $\varphi \in [0, 2\pi)$, that will locate the point of reflection and by the angle $\alpha \in (0, \pi)$ between the the tangent vector and the outgoing trajectory, measured counterclockwise.

The classical billiard defines a map T from the open cylinder $[0, 2\pi) \times (0, \pi)$ into itself, $T(\varphi_0, \alpha_0) = (\varphi_1, \alpha_1)$, which has some very well known properties: it is a C^{k-1}-diffeomorphism, preserving the measure $d\nu = R(\varphi) \sin \alpha \, d\alpha d\varphi$, has the monotone Twist property and is reversible with respect to the reversing symmetry $H(\varphi, \alpha) = (\varphi, \pi - \alpha)$. See, for instance, [7] for the properties of billiards and twist maps listed in this section.

Its derivative at (φ_0, α_0) is

$$DT_{(\varphi_0,\alpha_0)} = \frac{1}{R_1 \sin \alpha_1} \begin{pmatrix} L - R_0 \sin \alpha_0 & L \\ L - R_0 \sin \alpha_0 - R_1 \sin \alpha_1 & L - R_1 \sin \alpha_1 \end{pmatrix} \tag{3}$$

where $R_i = R(\varphi_i)$ and L is the distance between $\Gamma(\varphi_0)$ and $\Gamma(\varphi_1)$.

A continuous closed curve γ on the cylinder $[0, 2\pi) \times (0, \pi)$ which is not homotopic to a point is called a rotational curve. It is invariant if $T(\gamma) = \gamma$. As T is a monotone twist map, by Birkhoff's Theorem any invariant rotational curve γ is the graph of a Lipschitz function $g : [0, 2\pi) \mapsto (0, \pi)$. Moreover, as T preserves the uniformly continuous measure with respect to Lebesgue, $d\nu$, two distinct invariant rotational curves do not intersect. This, together with the reversibility of T and the compactness of γ, imply that either $g(\varphi) \equiv \frac{\pi}{2}$ or there exist constants b and B such that $0 < b \leq g(\varphi) \leq B < \frac{\pi}{2}$ or $\frac{\pi}{2} < B \leq g(\varphi) \leq b < \pi$.

The reversibility also implies that $T|_\gamma$ preserves (or reverses) the orientation of $S^1 \equiv [0, 2\pi)$. Then, as $T|_\gamma$ is a homeomorphism, the dynamics is given by its rotation number ρ and by Poincaré's Classification Theorem for homeomorphisms of the circle [7]

If γ is C^1 or more, then $T|_\gamma$ is a diffeomorphism and only three cases are possible: either $\rho \in \mathbb{R} \setminus \mathbb{Q}$ and $T|_\gamma$ is conjugated to a rotation with angle $2\pi\rho$ or $\rho = \frac{p}{q} \in \mathbb{Q}$ and then either $T|_\gamma$ has only periodic points of period q or has a finite number of them connected by heteroclinic orbits.

The C^1 or more character of γ also implies that a tangent vector $(1, g'(\varphi_0))$ is sent by $DT_{(\varphi_0,\alpha_0)}$ on a tangent vector to γ at (φ_1, α_1). The preservation of orientation implies that the first coordinate of $DT_{(\varphi_0,\alpha_0)}(1, g'(\varphi_0))$ must be strictly positive. So $L[1 + g'(\varphi_0)] - R_0 \sin \alpha_0 > 0$ and $1 - g'(\varphi_0) > \frac{R_0 \sin \alpha_0}{L} > 0$. This implies that $g'(\varphi) > -1$ for every φ. As the billiard is reversible, the graph of $\tilde{g}(\varphi) = \pi - g(\varphi)$ is also a rotational invariant curve and then $\tilde{g}'(\varphi) = -g'(\varphi) > -1$ for every φ. So, for any C^1 invariant γ =graph(g), $-1 < g'(\varphi) < 1$.

To each invariant rotational curve γ is associated a caustic [13], a curve lying inside the billiard table and tangent to every segment of billiard trajectory between two consecutive impacts. If (φ_0, α_0), and so (φ_1, α_1), belong to γ, then $\alpha_0 = g(\varphi_0)$, $\alpha_1 = g(\varphi_1)$ and the quantities $\dfrac{R_0 \sin \alpha_0}{1 + g'(\varphi_0)}$ and $\dfrac{R_1 \sin \alpha_1}{1 - g'(\varphi_1)}$ measure the distance to the tangency point of the segment of the trajectory with the caustic from, respectively, the initial $\Gamma(\varphi_0)$ and the final point $\Gamma(\varphi_1)$ ([10]). They are strictly positive and

$$\frac{R_0 \sin \alpha_0}{1 + g'_0} + \frac{R_1 \sin \alpha_1}{1 - g'_1} = L \tag{4}$$

where $g'(\varphi_i) = g'_i$.

4 Non Elastic Billiards

Let $T(\varphi_0,\alpha_0)=(\varphi_1,\alpha_1)$ be a C^2 classical billiard map on an oval, with a C^2 invariant rotational curve γ_0, given by the graph of $\alpha=g(\varphi)$.

A compact subset of the phase space $[0,2\pi)\times(0,\pi)$ with non-empty interior and whose boundaries are two distinct rotational curves (not necessarily invariant nor graphs) will be called a compact strip.

Let $I\in\mathbb{R}$ be a closed interval containing 0. Given $h:I\to I$, a C^2 strictly increasing contraction with $h(0)=0$, we can define a non elastic billiard map P on a compact strip Σ containing γ_0 by

$$P(\varphi_0,\alpha_0)=(\varphi_1,\alpha_1-h(\alpha_1-g(\varphi_1)))$$

with Σ chosen such that if $(\varphi,\alpha)\in\Sigma$ then $\alpha-g(\varphi)\in I$. P is the composition of a classical billiard followed by a change at the reflection angle, corresponding to a contraction in the vertical fibers of the invariant rotational curve γ_0.

Observe that $h(t)\equiv 0$ corresponds to the classical billiard and that $h(t)=t$ corresponds to a map that sends all the points in Σ on the invariant curve (called *slap billiard* in [11]).

The derivative of P is given by

$$DP_{(\varphi_0,\alpha_0)}=\frac{1}{R_1\sin\alpha_1}\begin{pmatrix}1 & 0\\ h_1'g_1' & 1-h_1'\end{pmatrix}\begin{pmatrix}L-R_0\sin\alpha_0 & L\\ L-R_0\sin\alpha_0-R_1\sin\alpha_1 & L-R_1\sin\alpha_1\end{pmatrix} \tag{5}$$

where $h_i'=h'(\alpha_i-g(\varphi_i))$

Our main result is:

Theorem 1 *Given a classical oval billiard map T, with a C^2 invariant rotational curve $\gamma_0=\{(\varphi,g(\varphi))\}$, consider a compact strip Σ containing γ_0 and a closed interval $I\subset\mathbb{R}$, such that $\alpha-g(\varphi)\in I$ if $(\varphi,\alpha)\in\Sigma$. If $h:I\mapsto\mathbb{R}$ is a C^2 function satisfying $h(0)=0$ and $0\le 1-\underline{l}<h'(0)<1$ (with $\underline{l}$ depending only on γ_0), then there exists a compact strip $S\subset\Sigma$ such that the non elastic billiard map P defined by T, g and h is a C^2-diffeomorphism from S onto $P(S)$. Its limit set $L(P)$ contains γ_0 and has a dominated splitting. Moreover, the non elastic perturbation do not change the dynamics on γ_0.*

Proof: The non elastic billiard $P:\Sigma\to P(\Sigma)\subset[0,2\pi)\times(0,\pi)$ is the composition of the C^2 classical billiard map T with the C^2 perturbation of the identity $(\varphi,\alpha)\mapsto(\varphi,\alpha)-(0,h(\alpha-g(\varphi)))$, where h is a C^2 contraction. Then P is a C^2 diffeomorphism.

Given $\delta>0$, let $u_{(\varphi,\alpha)}=(1,g'(\varphi)-\delta)$ and $v_{(\varphi,\alpha)}=(1,g'(\varphi)+\delta)$ be two linearly independent vector fields defining the cone field $\mathcal{C}(\varphi,\alpha)$ and the associated quadratic form $Q_{(\varphi,\alpha)}$ (as in section 2).

Using the change of bases matrices, we have

$$[DP_{(\varphi_0,\alpha_0)}]_U=\frac{1}{2\delta R_1\sin\alpha_1}\begin{pmatrix}\delta(l_0-\delta L)+(1-h_1')(\delta l_1-l_{01}) & \delta(l_0+\delta L)-(1-h_1')(\delta l_1+l_{01})\\ \delta(l_0-\delta L)-(1-h_1')(\delta l_1-l_{01}) & \delta(l_0+\delta L)+(1-h_1')(\delta l_1+l_{01})\end{pmatrix} \tag{6}$$

where

$$\begin{aligned} l_0 &= L(1+g_0')-R_0\sin\alpha_0 \quad , \quad l_1=L(1-g_1')-R_1\sin\alpha_1\\ l_{01} &= L(1+g_0')(1-g_1')-(1-g_1')R_0\sin\alpha_0-(1+g_0')R_1\sin\alpha_1\ . \end{aligned}$$

Relation (4) implies that for every (φ_0,α_0) and (φ_1,α_1) on γ_0 we have

$$l_{01}=0 \quad , \quad l_0=\frac{1+g_0'}{1-g_1'}R_1\sin\alpha_1 \quad , \quad l_1=\frac{1-g_1'}{1+g_0'}R_0\sin\alpha_0.$$

The billiard boundary Γ is an oval and as it is compact there exist constants a and A and a width D such that $0 < a \leq R(\varphi) \leq A$ and $0 < L \leq D$. As the invariant curve is also compact, for every (φ_0, α_0) and (φ_1, α_1) on γ_0, there exist constants c and C such that $C \geq l_0 \geq c > 0$ and $C \geq l_1 \geq c > 0$. So for points on γ_0 there are constants $0 < \underline{l} \leq 1$ and $0 < \overline{L}$ such that

$$\underline{l} \leq \frac{l_0}{l_1} \qquad \text{and} \qquad 0 < \frac{L}{l_1} \leq \overline{L} \, .$$

By formula (6), each entry of the matrix $[DP_{(\varphi_0,\alpha_0)}]_U$ for $(\varphi_0, \alpha_0) \in \gamma_0$ is of the form

$$\delta(l_0 \pm \delta\, L \pm (1 - h'(0))\, l_1) \geq \delta l_1 (\frac{l_0}{l_1} - \delta \frac{L}{l_1} - (1 - h'(0))) \geq \delta c(\underline{l} - \delta \overline{L} - (1 - h'(0))).$$

Now, if $0 \leq 1 - \underline{l} < h'(0) < 1$, we can choose $\delta > 0$ such that $\delta c(\underline{l} - \delta\overline{L} - (1 - h'(0))) > 0$. As P is a C^2 diffeomorphism and remembering that γ_0 is compact, we can find a strip $S \subset \Sigma$, containing γ_0, where P is well defined and all the entries of $[DP_{(\varphi_0,\alpha_0)}]_U$ are strictly positive.

Then, by lemma 1, $L(P) \subset S$ has a dominated splitting and contains γ_0, since $h(0) = 0$. Moreover, $P|_{\gamma_0} = T|_{\gamma_0}$ and the dynamics under P, on γ_0, is the same as under T. As γ_0 and P are C^2, for our billiard dynamics, γ_0 is either a set of periodic points of same period linked by homo/heteroclinic arcs or supports a rational or an irrational rotation. ■

This result will guide us to construct examples of non elastic billiards on ovals with limit set having a dominated splitting and supporting a rational or an irrational rotation (pieces of type $\mathcal{I}$ or $\mathcal{R}$ of Pujals-Sambarino's Theorem). This will be done in the next section. We will pay attention to the maximal possible size of the strip S and will try to see if there are other attractors on $L(P)$ than γ_0.

5 Examples

5.1 The circle

The simplest example of a classical billiard with invariant rotational curves is the circular one. This billiard map is linear and is given by $T(\varphi_0, \alpha_0) = (\varphi_0 + 2\alpha_0, \alpha_0)$. The phase space $[0, 2\pi) \times (0, \pi)$ is foliated by invariant horizontal curves and the dynamics, on each one of them, is simply a rotation of $2\alpha_0$.

We pick one invariant curve γ_0, defined by $\alpha = g(\varphi) = \overline{\beta}_0$. At γ_0, $g' \equiv 0$, $R_i = R$, the radius of the circle, and $\sin \alpha_i = \sin \overline{\beta}_0$, implying $l_0 = l_1$ and $\underline{l} = 1$.

Fix I, a closed interval with $0 \in I \subset (-\overline{\beta}_0, \pi - \overline{\beta}_0)$ and $h : I \mapsto \mathbb{R}$, any C^2 strictly increasing contraction such that $h(0) = 0$ and $0 < h'(0) < 1$. The non elastic billiard P is defined on the strip $[0, 2\pi) \times \{I + \overline{\beta}_0\}$ and is given by $P(\varphi_0, \alpha_0) = (\varphi_0 + 2\alpha_0, \alpha_0 - h(\alpha_0 - \overline{\beta}_0))$.

By theorem 1, there exists a compact strip S such that $P|_S$ is a C^2 diffeomorphism and $L(P|_S)$ contains γ_0 and has a dominated splitting.

Now, we choose $\overline{\beta}_-$ and $\overline{\beta}_+$ such that $W = [0, 2\pi) \times [\overline{\beta}_-, \overline{\beta}_+]$ is the biggest horizontal straight strip contained in S. As each boundary $\gamma_\pm = \{(\varphi, \overline{\beta}_\pm)\}$ is invariant under T, we have that $P(W) \subset W$. The map P is a horizontal rotation followed by a vertical contraction. Denoting $(\varphi_n, \alpha_n) = P^n(\varphi_0, \alpha_0)$, it is then easy to see that $\alpha_n \to \overline{\beta}_0$, as $n \to +\infty$ and the horizontal circle γ_0 is the unique attractor of P on W.

Moreover, the restricted map $P|_{\gamma_0}$ is just a rotation of angle $2\overline{\beta}_0$. If $\overline{\beta}_0/\pi$ is rational, γ_0 is a normally hyperbolic simple closed curve, composed by periodic points of same period. If $\overline{\beta}_0/\pi$ is irrational γ_0 is a normally hyperbolic closed curve supporting an irrational rotation.

This yields an example of a diffeomorphism, defined on a strip W, whose limit set has dominated splitting and is composed by a unique piece of type $\mathcal{I}$ or $\mathcal{R}$ of Pujals-Sambarino's Theorem.

Clearly, the size of the strip W depends on the choice of the contraction h. Taking for instance $h(x) = \mu x$, with $0 < \mu < 1$, the non elastic billiard is given by $P(\varphi_0, \alpha_0) = (\varphi_0 + 2\alpha_0, \alpha_0 - \mu(\alpha_0 - \overline{\beta}_0))$ and the basin of attraction of γ_0 contains any straight strip $W = [0, 2\pi) \times [\overline{\beta}_-, \overline{\beta}_+]$.

5.2 The ellipse

A similar example is given by the classical elliptical billiard. We consider an ellipse Γ with eccentricity e and minor axis 1. Its radius of curvature R satisfies $\sqrt{1-e^2} \leq R \leq \dfrac{1}{1-e^2}$. The associated classical billiard map is denoted by $T : [0, 2\pi) \times (0, \pi) \mapsto [0, 2\pi) \times (0, \pi)$.

This billiard system is integrable: the function $F(\varphi, \alpha) = \dfrac{\cos^2\alpha - e^2\cos^2\varphi}{1 - e^2\cos^2\varphi}$ is a first integral (see, for instance [3]) and $[0, 2\pi) \times (0, \pi)$ is foliated by the levels of $F = F_0$, with $-\frac{e^2}{1-e^2} < F_0 < 1$. If $0 < F_0 < 1$, the level set consists of two invariant, analytic and symmetric rotational curves, the lower one contained in $[0, 2\pi) \times (0, \frac{\pi}{2})$ and the upper one in $[0, 2\pi) \times (\frac{\pi}{2}, \pi)$. But, unlike the circular case, T has two elliptic islands of period 2, obstructing the rotational invariant curves to foliate the whole phase-space, as can be seen on Figure 1(left). It also has a hyperbolic 2-periodic orbit, with a saddle connection, corresponding to the level $F_0 = 0$.

For a fixed $0 < F_0 < 1$ let γ_0 be the lower invariant rotational curve in $F(\varphi, \alpha) = F_0$ (the upper case is analogous). It is the graph of $\alpha = g(\varphi)$ given implicitly by $\cos\alpha = \sqrt{F_0 + (1 - F_0)e^2\cos^2\varphi}$.

We have then that, for any $(\varphi, \alpha) \in \gamma_0$, $\sqrt{(1 - F_0)(1 - e^2)} \leq \sin\alpha \leq \sqrt{1 - F_0}$. Differentiating twice with respect to φ we get $(1 - F_0)e^2 \sin 2\varphi = g'(\varphi)\sin 2\alpha$ and $2(1 - F_0)e^2\cos 2\varphi = 2g'(\varphi)\cos 2\alpha + g''(\varphi)\sin 2\alpha$ and the extremal points of g' must satisfy $\tan 2\varphi = \tan 2\alpha$ which implies $\max\{g'(\varphi)\} = (1 - F_0)e^2 = -\min\{g'(\varphi)\}$. We can then take

$$\underline{l} = (1 - e^2)^2 \left(\frac{1 - (1 - F_0)e^2}{1 + (1 - F_0)e^2}\right)^2 \leq \frac{(1 + g_0')^2 R_1 \sin\alpha_1}{(1 - g_1')^2 R_0 \sin\alpha_0} = \frac{l_0}{l_1} \tag{7}$$

The associated non elastic billiard map is given by $P(\varphi_0, \alpha_0) = (\varphi_1, \alpha_1 - h(\alpha_1 - g(\varphi_1))$ where the contraction $h : I \mapsto \mathbb{R}$ is an arbitrary C^2 function satisfying $h(0) = 0$ and $0 < 1 - \underline{l} \leq h'(0)$, and I is a closed interval containing 0. Then, by theorem 1, there exists a compact strip S, containing γ_0, such that $P|_S$ is a C^2 diffeomorphism, $L(P|_S)$ has a dominated splitting and contains γ_0.

Let $F_\pm$ be two constants of motion satisfying $1 > F_- > F_0 > F_+ > 0$, and $\gamma_\pm$=graph$(g_\pm)$ be the lowest invariant rotational curves associated to $F_\pm$, respectively. We also suppose that $W = \{(\varphi, \alpha), g_-(\varphi) \leq \alpha \leq g_+(\varphi)\} \subset S$ is the biggest strip of this type contained on S. As $\gamma_\pm$ are invariant under T we have $P(W) \subset W$.

For $(\varphi_0, \alpha_0) \in W$, we denote $(\varphi_n, \alpha_n) = P^n(\varphi_0, \alpha_0)$. Let us suppose, for instance, that $F(\varphi_0, \alpha_0) > F_0$, the other case being analogous. As P is a translation on a T-invariant rotational curve followed by a contraction on the vertical direction, toward γ_0, then $F(\varphi_n, \alpha_n) \to F_0$ and γ_0 is the w-limit of any $(\varphi_0, \alpha_0) \in W$. As for the circular case, γ_0 is the unique attractor of $P|_W$.

Using action-angle variables, Chang and Friedberg [5] have shown how to decide if the rotation number associated to the level $F_0 = F(\varphi_0, \alpha_0)$ of a given initial condition (φ_0, α_0), is rational or irrational and then determine the dynamics on each level curve. As $P|_{\gamma_0} = T|_{\gamma_0}$, this allows us to choose F_0 in order to have, as the unique attractor, a normally hyperbolic closed curve supporting an irrational rotation or a normally hyperbolic closed curve composed by periodic points of same period.

Although theoretically promising as a result, depending on the choice of γ_0 and h, the strip W can be very thin. In particular it will never contain points of the elliptical islands. In trying to go beyond the theoretical predictions and find examples of non elastic elliptical billiards defined on a bigger part of the phase space, we shall remember that weak contractions will never destroy the rotation carried by the linear ellipticity of the 2-periodic orbit. So we can not expect the associated non elastic billiard to be defined on a strip containing the islands. However, it is an interesting question whether this can be achieved by taking a strong contraction.

We present some numerical simulations where this can be done. We choose an ellipse with $e = 0.35$ and fix the invariant curve γ_0 at $F_0 = 0.25$. Our choices of the eccentricity and γ_0 are rather arbitrary. Figure 1(left) below shows the phase space of the classical elliptical billiard. The invariant curve γ_0 is enhanced. The horizontal axis corresponds to $\varphi \in [0, 2\pi)$ and the vertical to $\alpha \in (0, \pi)$, the left bottom corner being the origin.

We will consider linear perturbations $h(x) = \mu x$, with $0 < \mu < 1$. $\mu = 0$ implies that there is no perturbation (classical billiard) and $\mu = 1$ implies that all the points in the phase space land on the invariant curve after one iteration. The non elastic billiard $P(\varphi_0, \alpha_0) = (\varphi_1, \alpha_1 - \mu(\alpha_1 - g(\varphi_1)))$ is a C^2 diffeomorphism on any compact strip contained on $[0, 2\pi) \times (0, \pi)$. Calculating $\underline{l}$ by formula (7) we get that if $\mu > 1 - \underline{l} \approx 0.47$ there is a strip W such that γ_0 is the unique attractor of P on W and has a dominated splitting. Figure 1(right) illustrates the basin of attraction of the curve γ_0 for $\mu = 0.5$: black points correspond to initial conditions which approach γ_0 under iteration. This simulation thus indicates that the basin of attraction of γ_0 is the whole phase space, ie, that W can be any compact strip contained in $[0, 2\pi) \times (0, \pi)$ and γ_0 is the unique attractor of P. Note that, as we have mentioned before, either γ_0 is composed of periodic points of same period or supports an irrational rotation.

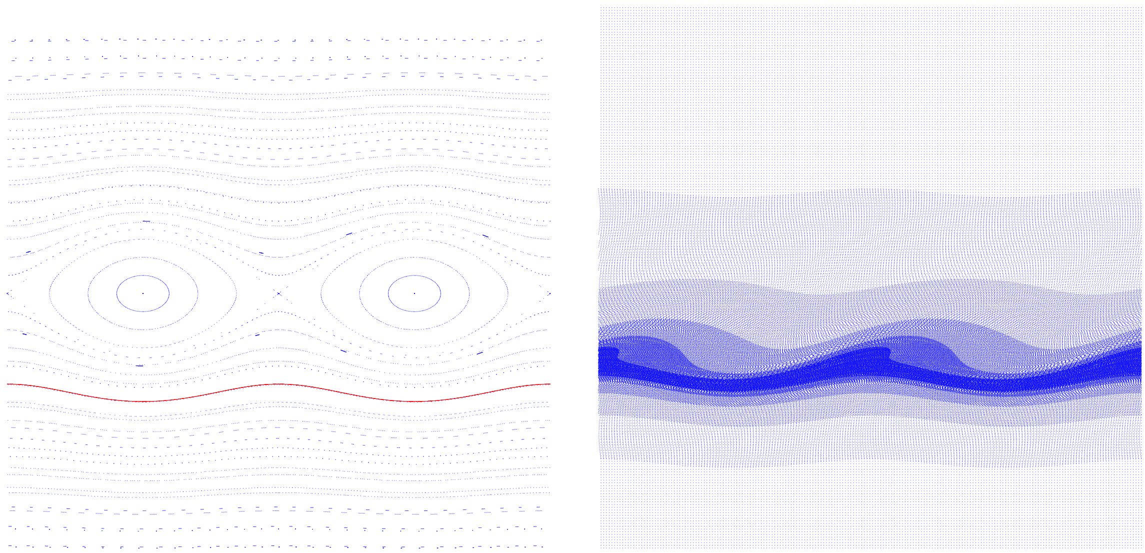

Figure 1: Classical and non elastic elliptical billiards

5.3 Non integrable billiards

There are no other known C^2 ovals such that the classical billiard map is integrable, other than the circle and the ellipse. But if the oval is C^k, $k \geq 5$, Lazutkin Theorem [9] guarantee the existence of a whole family of rotational invariant curves on any neighbourhood of $\alpha = 0$ or $\alpha = \pi$ on the phase-space $[0, 2\pi) \times (0, \pi)$. They are as differentiable as the associated classical billiard map (so at least C^4), are graphs over $[0, 2\pi)$ and the dynamics on each one of them is conjugated to a diophantine irrational rotation.

This does not mean, in any way, that there can not exist other rotational invariant curves. For instance, classical billiards on sufficiently small perturbations of a circle show, in addition to the diophantine invariant curves, uncountably many invariant rotational curves with Liouvillian rotation number [7]. Each one is a continuous graph [4] but may be just a little more regular than Lipschitz [1].

There are also rotational invariant curves with rational rotation number. A good example is the line $\alpha = \pi/2$ in the phase space of a classical billiard on a curve of constant width [8], which has rotation number $1/2$. In this case every point of the curve sits on a diameter and so gives rise to a period 2 trajectory. But, apart from special examples like this one, it is difficult to find billiards with this property because having an invariant rotational curve with rational rotation number is not a generic property for billiards on ovals. The generic dynamics is having, for each rational rotation number, a finite number of periodic orbits with this rotation number and at least one of them hyperbolic, with transverse homoclinic orbits [6].

Analogously to the integrable case, we can build examples of non elastic billiards by taking a sufficiently differentiable, non integrable, classical oval billiard map T and a C^2 invariant rotational curve γ_0=graph(g). As before, we take a contraction h satisfying the hypothesis of theorem 1 and define the non elastic billiard map P. Then there exists a strip S on which P is a C^2 diffeomorphism on S, $L(P)$ contains γ_0 and has a dominated splitting. Depending on the rotation number of γ_0 we can have, as in the previous examples, a normally hyperbolic closed curve supporting an irrational rotation or a normally hyperbolic closed curve composed by periodic points of same period. But we can no longer guarantee that the only attractor of P is γ_0.

5.3.1 Invariant straight line

In order to explore numerically what happens in those more general cases, we have to pick a concrete example. Although we can prove the existence of whole families of invariant curves, it is almost impossible for any fixed one, to write the function g for which it is the graph. In general this can only be done in very specific examples as, for instance, the constant width curves, where $g(\varphi) = \pi/2$. We will deal with a much richer example, given by symmetric perturbations of the circle.

Let Γ_n be the oval parameterized by the angle φ and which radius of curvature is of the form $R(\varphi) = 1 + a\cos n\varphi$, $|a| < 1$ and $n \geq 4$. Tabachnikov [13](Section 2.11) showed that its associated classical billiard map have an invariant straight line given by $g(\varphi) = \beta_0$ if β_0 satisfies $n\tan\beta_0 = \tan n\beta_0$. It is not difficult to show that the dynamics on γ_0 is an irrational rotation.

For $n = 6$, $\beta_0 = \tan^{-1}\sqrt{7 + 4\sqrt{21}/3} \approx 0.41\pi$ satisfies this condition and the line γ_0 given by $\alpha = g(\varphi) = \beta_0$ is invariant. Figure 2 displays the billiard table and the corresponding phase space of Γ_6 defined by $R(\varphi) = 1 + 0.01\cos 6\varphi$. As in the previous section, the horizontal axis corresponds to $\varphi \in [0, 2\pi)$ and the vertical to $\alpha \in (0, \pi)$, the left bottom corner being the origin. Note the invariant straight line γ_0.

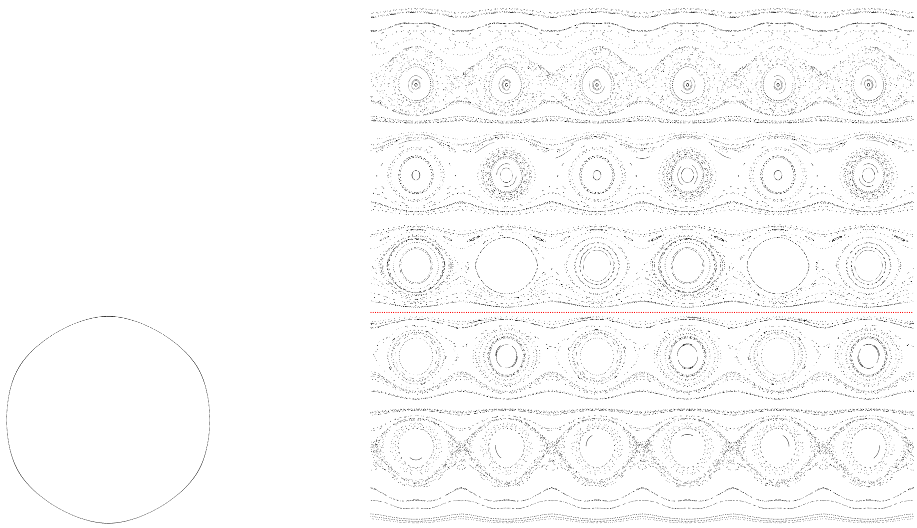

Figure 2: Billiard table and classical phase space for Γ_6

As for the other numerical examples, we consider linear perturbations $h(x) = \mu x$, $0 < \mu < 1$. On γ_0, we have $l_0 = R_1 \sin\beta_0$ and $l_1 = R_0 \sin\beta_0$ and we can take $\underline{l} = \frac{\min R(\varphi)}{\max R(\varphi)} = \frac{0.99}{1.01}$. Then, if $\mu > 1 - \underline{l} \approx 0.02$, there exists a strip S on which P is a C^2 diffeomorphism, $L(P)$ contains γ_0 and has a dominated splitting.

Figure 3 illustrates the basin of attraction of γ_0 for $\mu = 0.1$, 0.35, 0.37 and 0.4: black points correspond to initial conditions which approach γ_0 under iteration. The white tadpoles correspond to points attracted to the 6-periodic orbits, linearly elliptic for the original classical billiard.

We observe that as the contraction factor μ is increased, the basin of attraction of γ_0 grows until it eventually occupies the whole phase space. Thus, for strong contractions, we may have P defined on any compact strip S and having γ_0 as its unique attractor.

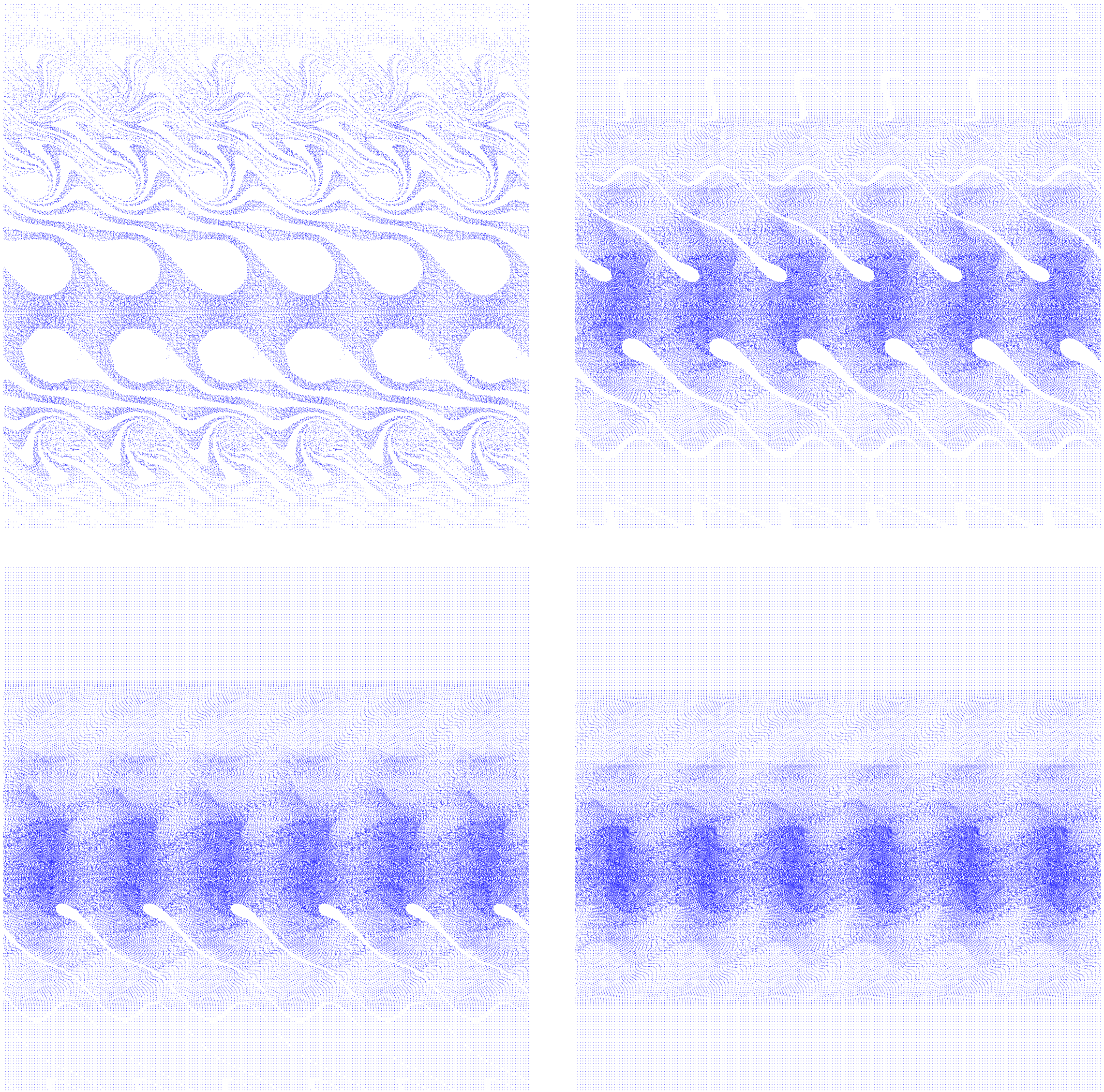

Figure 3: Γ_6: the basin of attraction of γ_0 for $\mu = 0.1, 0.35, 0.37$ and 0.4

5.4 An example that is not one

Any strictly convex classical billiard map is a monotone twist map with rotation interval $(0,1)$. To each $\rho \in (0,1)$ is associated a set $\mathcal{O}_\rho$. If ρ is irrational, $\mathcal{O}_\rho$ is either a rotational invariant curve or an Aubry-Mather set. An Aubry Mather set is a closed, invariant, minimal set, projecting injectively on a Cantor set of $S^1 \equiv [0, 2\pi)$ and such that the dynamics preserves the order of S^1. It is contained in a non invariant graph of a continuous piecewise linear Lipschitz function $\alpha = g(\varphi)$ (see, for instance [7], Section 13.2).

Let us take a classical billiard map T with two rotational invariant curves γ_- and γ_+ and an Aubry-Mather set $\mathcal{A}$ contained in the strip bounded by γ_- and γ_+ and a perturbation h on this strip. The non elastic billiard map can be then defined as before, as $P(\varphi_0, \alpha_0) = (\varphi_1, \alpha_1 - h(\alpha_1 - g(\varphi_1))$, where $(\varphi_1, \alpha_1) = T(\varphi_0, \alpha_0)$.

As h is a contraction, the limit set of P will contain the Aubry-Mather set $\mathcal{A}$. However, we must remark that P is not C^2 because g is not even a differentiable function. Then, we can not apply Pujal-Sambarino's theorem.

Acknowledgment. We thank Mário Jorge Dias Carneiro for his precious ideas. We thank the Brazilian agencies FAPEMIG and CNPq for financial support.

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