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Transients in a time-dependent logistic map

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Abstract

We study the one-dimensional logistic map with parametric perturbation. Using a small periodic function as the perturbation, new attractors may appear. Beside these new attractors, complex attractors exist and are responsible for transients in many trajectories. We observe that each one of these transients is characterized by a power law decay. We find the exponent related to this decay.

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1. Introduction

The one-dimensional logistic map has been extensively studied in the past years [1–3]. This map describes the typical behavior of many dissipative dynamical systems and has applications in physics, biology, electronics and many other different subjects.

Different kinds of the perturbation can be introduced in the logistic map, like the additive perturbation [2], the parametric perturbation [4], or both. Our main goal in this paper is to investigate the effect of a small parametric perturbation in the logistic map and obtain the exponent related to the power law decay of the transient time.

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The logistic map considered in this work is defined by

$$X_{n+1} = F_n(X_n) = R_n X_n(1 - X_n)\,, \tag{1}$$

$$R_n = R + \varepsilon f(n)\,, \tag{2}$$

where $X \in [0,1]$ and ε is a small number and $f(n)$ is a periodic function. For $\varepsilon = 0$, we have the conventional logistic map [5] and for $\varepsilon \neq 0$, we have the so called time-dependent case. Typically, as the control parameter R varies, attractors may appear or change stability. In particular, a complex fixed point may become real at $R = R_c$. For $R < R_c$, the still complex attractors are responsible for transients in many trajectories. We obtain numerically the z exponent which characterizes this transient. This paper is organized as follows: In Section 2, we discuss the logistic map with a control parameter of period 2. We find the values of R where the new attractors become real and describe the mechanism of this creation. We also obtain the z exponent related to the observed transient time. A similar process with a control parameter of period 3 is also described. Our conclusions are presented in the last section.

2. Periodic control parameter

Let us consider here the logistic map with a control parameter of period 2 [4,6]. This situation can be modeled by Eqs. (1) and (2) with $f(n) = \cos(n\pi)$. Since R oscillates between two values $R(1+\varepsilon)$ and $R(1-\varepsilon)$, the logistic map will have fixed points also periodic in time (n). This means that $F_n(X_a) = X_b$ and $F_{n+1}(X_b) = X_a$. We rather call this situation as time dependent, in fact a periodic, fixed point than a cycle-2, as almost all initial conditions will asymptotically oscillate between X_a and X_b. So we have a time dependent fixed point [7].

The fixed points are obtained by solving the equation $X^*_{n+2} = X^*_n$, where iteration is defined by Eqs. (1) and (2). The problem has three non-zero solutions for n even (odd). At least one of them is real and corresponds to a fixed point $\boldsymbol{X}_1^* = (X_a, X_b)$. If we fix ε and start increasing R, this fixed point becomes unstable and a cycle-2 (of period 2) becomes stable. So, as R is varied we have the usual sequence of period doubling until the chaotic behavior is reached. The other two roots of the fixed point equation change from complex to real by increasing R. The particular value of the control parameter where complex fixed points become real is named R_c. One of these new fixed points is unstable and the others are stable. This new stable fixed point $\boldsymbol{X}_2^*$ coexists with the original real fixed point or with one of its stable descendent cycle-m. It has its own basin of attraction. In fact, some of the initial conditions belong to the basin of attraction of $\boldsymbol{X}_2^*$ and others belong to the basin of the stable cycle-m ($m = 1, 2, 3, 4\ldots$), which appears in the bifurcation sequence originated from $\boldsymbol{X}_1^*$. The new fixed point, $\boldsymbol{X}_2^*$, gives rise to a second sequence of bifurcations as R is varied.

The picture described above depends on ε. For example, if $\varepsilon = 0.01$ we have that $R_c \sim 3.144390069$. For $R < R_c$ we have only a fixed point of period two $\boldsymbol{X}_1^*$ that

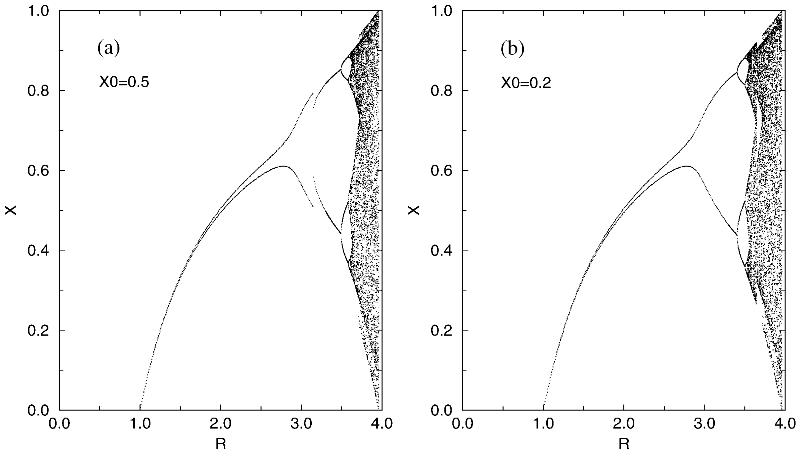

Fig. 1. Orbit diagram for $R_n = R(1 + \varepsilon\cos(n\pi))$. (a) The initial condition is $X_0 = 1/2$. For $R < R_c = 3.1443900\ldots$, X_0 is attracted by the real periodic fixed point; for $R = R_c$, we have a new real periodic fixed point responsible for a second basin of attraction. The discontinuity at R_c is due to the fact that X_0 is then attracted by the new fixed point. (b) The initial condition is $X_0 = 0.2$.

attracts almost all initial conditions. When $R > R_c$ we have two stable fixed points of period two, $\boldsymbol{X}_1^*$ and $\boldsymbol{X}_2^*$, each one with its own basin of attraction. The sudden birth of a new basin of attraction can be observed in the orbit diagram as a discontinuity. In Fig. 1 the orbit diagram for this value of ε and two different initial conditions is shown: (a) $X_0 = 0.5$ and (b) $X_0 = 0.2$. The discontinuity at R_c in Fig. 1(a) means that $X_0 = 0.5$ has changed basin of attraction. This does not occur for $X_0 = 0.2$ (see Fig. 1(b)).

We have then a description of the mechanism in which the new attraction basins appear. The transition from one basin to another may be characterized by a transient time. For $R < R_c$ an initial condition X_0 is attracted asymptotically to the fixed point $\boldsymbol{X}_1^*$, but before *reaching* this fixed point the system may spend a long time near the real part of the (still) complex fixed point $\boldsymbol{X}_2^*$. This time is the transient time τ. As long as R approaches R_c, τ grows up and for $R = R_c$ it diverges, meaning that now the system spends all the time in the neighborhood of the new real fixed point. The divergence of the transient time at R_c is given by $\tau \sim \mu^{-z}$, where $\mu = |R - R_c|$. In Fig. 2(a) it is shown the time evolution of $X_0 = 0.5$. Observe that the transition from one fixed point to the other one is very sharp. It is then easy to evaluate the transient time. In Fig. 2(b) the evaluated transient time is given for different values of μ. The exponent $z = 0.501(1)$ was obtained by a least squares fitting.

A similar situation may occur when a complex fixed point becomes real and stable when a cycle-m is already stable. This is the case for $\varepsilon = 0.038$. The fixed point $\boldsymbol{X}_2^*$ becomes real and stable at $R = R_c \sim 3.372405139$. When $R < R_c =$, $\boldsymbol{X}_2^*$ is complex and a cycle-2 of period 2 attracts all the initial conditions. In this case we have a transient time that is characterized by the exponent $z = 0.506(2)$.

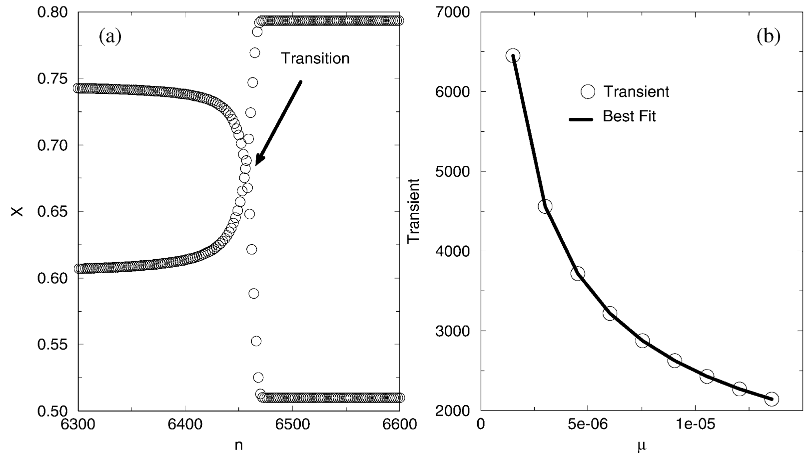

Fig. 2. The control parameter is given by $R_n = R(1 + \varepsilon\cos(n\pi))$. (a) Orbit of $X_0 = 1/2$ when $R < R_c = 3.1443900\ldots$. In this situation the new attractor is complex and responsible for a transient. The orbit spends a long time near the real part of the complex attractor ($n < 6450$) before reaching the real periodic fixed point. The arrow shows the transition defining the transient time. (b) Plot of transient time versus μ. A least squares fitting furnishes $z = 0.501(1)$.

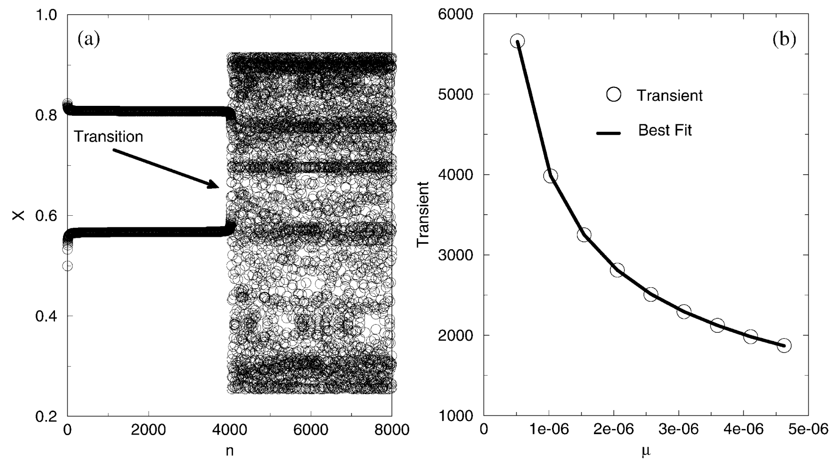

Fig. 3. (a) Orbit of the initial condition $X_0 = 1/2$ for $R_n = R(1 + \varepsilon\cos(n\pi))$ with $R < R_c = 3.47726581\ldots$. There are a real chaotic attractor and a complex periodic fixed point. The sharp transition characterizes the time transient. (b) Transient time by μ. A least squares fitting give us the exponent $z = 0.502(2)$.

For $\varepsilon = 0.053$ and $R < R_c \sim 3.477265810$ we have a chaotic attractor and a transient occurs between it and the real part of the complex fixed point. This is shown in Fig. 3(a). The transient times for several values μ are shown in Fig. 3(b). A least squares fitting furnishes $z = 0.502(2)$.

The transients described above may be compared to the intermittence phenomenon. In the intermittent behavior we have the coexistence between an established chaotic regime and a cycle-m which would be the attractor for a slight change in the control parameter. There is a channel between the graph of the map $F^{(m)}$ and the straight line $y=x$. As an initial condition is iterated it can enter into this channel and thus have a cycle-like behavior. When it gets out of the channel we have the so-called burst of chaotic behavior. Due to a reinjection mechanism the system enters the channel and gets out frequently. A characteristic time τ is defined as the average time of bursts and is characterized by $z = 0.5$. In our case, we also have a channel related to a tangent bifurcation but there is no reinjection mechanism. In fact, the system enters the channel only once and spends a long time there, which also seems to be characterized by $z=0.5$.

A similar process is observed when we have a time dependent control parameter with period three. For example, when $R_n=R(1+\varepsilon\cos(2/3n\pi)$, we have 3 possible values for R_n, namely $R_a=R(1-\varepsilon/2)$, $R_b=R(1-\varepsilon/2)$ and $R_c=R(1+\varepsilon)$. In order to obtain the fixed points, we need to solve the equation $X^*_{n+3}=X^*_n$, which gives us 7 non-zero roots for each value of $n=1,2,3$. Using $\varepsilon=0.01$, one of the 7 roots is at first real and the other ones are complex. These complex solutions become real in pairs for specific values of R. In this case, they became real for $R_1 = 3.783544316\ldots$, $R_2 = 3.850718107\ldots$ and $R_3=3.851414845\ldots$. We observe again a similar transient and similar mechanism of creation of the new attractors. The z exponents can be obtained numerically and we have found a value around 0.5 for all transients.

3. Conclusions

We studied the logistic map with a parametric perturbation. We described how a complex solution becomes real at $R=R_c$ implying the creation of a new attractor. For $R<R_c$, we obtained the z exponent related to a transient time. For $R_n=R(1+\varepsilon\cos(n\pi))$ we obtained that R_c depends on ε. Also depending on ε, the new attractor may arise when the old attractor was a periodic fixed point, a periodic cycle-2, and so on, until the chaotic attractor. For each situation, the z exponent was found to be approximately 0.5. The same results were observed for R_n with period 3, i.e., $R_n=R(1+\varepsilon\cos(2/3n\pi))$.

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